Sixth conference of the
International Network for
Didactic Research in University Mathematics

INDRUM 2026

June
15-19

DUBROVNIK

University of Dubrovnik

PRE-PROCEEDINGS

https://indrum2026.sciencesconf.org/

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Table of contents

TWG1A: The teaching and learning of calculus and real analysis **1**

Students' use of the multivariable chain rule in non-routine problems, Gabel Mika [et al.] 1

The Archimedean analysis as a didactic obstacle and a condition of possibility to nonstandard analysis, Espín Buendía José Ginés [et al.] 12

Concept Inventory for Real Analysis (RACI) - Theoretical foundation and item analysis, Bauer Thomas [et al.] 22

Investigating the second-order differential equation with engineering students, Praprotnik Selena [et al.] 32

Applying the Mathematical Idea Analysis Technique to Interpret Pre-service Teachers' Root Metaphors of Infinity, Tchoshanov Mourat [et al.] 42

Introducing differential equations in the first year of university: a case study, Grenier-Boley Nicolas [et al.] 53

Students' understanding of triple integrals in spherical coordinates, Borji Vahid [et al.] 63

Design of variational tasks: the case of continuous exponential covariation, Amador Silva Luis Miguel [et al.] 73

Conflict identification from an Ontosemiotical Analysis: the case of improper integrals, Igolnikov Nicolas Elias Daniel [et al.] 83

Beyond Procedures: A Look at Students' Understanding of Applied Optimization Problems in Introductory Calculus, Abadir Rasha [et al.] 93

TWG1B: The teaching and learning of proof and algebra **104**

Learning to think and teach functorially: categorical covariation and effective noticing in Advanced Mathematics Education, Miranda Annamaria 104

Teaching the concept of the root of a polynomial at the beginning of higher education in France: a study of practices, Veuillez–Mainard Juliette [et al.] . . . 115

On phylogenetics of matrices in finite mathematics textbooks, Gilroy Haile [et al.] 125

Supporting students' conceptual understanding in linear algebra through peer discussion and dual-mode exercises: an APOS-based approach, Kentaro Yoshitomi 127

Referential Linking and Clarifying Descriptive Equivalence, Reed Zackery [et al.] 129

Which understanding of the scalar product do preservice teachers intend for pupils at upper secondary level?, Feudel Frank [et al.] 139

Investigating De-ritualisation in Students' Proof Evaluation Discourse on Mathematical Induction, Lestyanto Latifah Mustofa [et al.] 149

Design research on a card-based learning environment for exploring congruence classes and modular arithmetic, Hoffmann Max 159

Exploring the features of linear algebra Peer Instruction questions, Docherty Pamela [et al.] 161

Instrumental and relational understanding of the diagonalization of a matrix, Pavlopoulou Kalliopi [et al.] 171

Ostensive Nature of Matrices: Toward an Epistemological Analysis of Linear Algebra, Hakamata Ryoto 173

Symbolizing in an Inquiry-Oriented Sequence for Least Squares, Zandieh Michelle [et al.] 175

Students' proof styles as individualizations of the genre of proof, Kontorovich Igor'185

Teaching Logic to Support Abstraction of Logical Structure, Dawkins Paul [et al.] 195

A Cross-Disciplinary Didactic Transposition from the Literature Domain to the Context of Proof, Melhuish Kathleen [et al.] 197

An intersection of linear algebra, complex numbers, and quantum computing: The case of the Bloch sphere, Wawro Megan [et al.] 207

TWG2: The teaching and learning of university mathematics across different

disciplines (including mathematical modelling) 209

From Mathematics to Economics: The Student Mentor Perspective on Relevance, Landgärds-Tarvoll Ida [et al.] 209

Students' Difficulties when Sketching Trajectories of Differential Equation Models, Bennoun Steve [et al.] 220

Differences between integrals in thermodynamics and mathematics, Matsumoto Yu [et al.] 222

Reconstructing the Mathematical Organizations of Modeling Life, Czocher Jennifer [et al.] 232

Linking deductive reasoning and concept images Rolle's Theorem with undergraduate computer science engineering, González De Rábago Miriam [et al.] . . . 242

Teaching the Laplace transform: a mathematician's and an engineer's perspective, Marković Valentino [et al.] 244

Didactic strategies for engineering mathematics: bridging the theory-application gap, Salinas Reyna Idalia María [et al.] 246

A taxonomy of parameter-related task types for analysing university students' modelling praxeologies, Strømskag Heidi 249

Comparative study of modelling tasks in post-reform mathematics courses in engineering education in Spain and Sweden, Asami-Johansson Yukiko 259

Promoting graph understanding through a short intervention: an eye tracking study, Sajka Miroslawa [et al.] 269

Mathematics in Two Engineering Industrial Contexts: What Didactic Implications for Mathematics Teaching, Perez-Sarmiento Maleni [et al.] 271

Mathematical modelling for future teachers: Case of sea tides, Cindrić Tomislav [et al.] 281

The dialectics of systems and models at the interface between mathematics and advanced physics, Lombard Nathan 283

Computational thinking and programming in first year STEM courses at bachelor level, Lie Johan [et al.] 294

TWG3: Transitions to, across, and beyond university mathematics 296

Revisiting group axioms for prospective teacher training, Rolland Agathe 296

Understanding students' affective experiences in the transition from high school to university mathematics, Svendsen Maiken Westen Holm [et al.] 307

Learning strategies and affective responses at the transition to university mathematics, Brignell Chris [et al.] 318

Prestructuralist and Structuralist Praxeologies: The Case of Connectedness and the Intermediate Value Theorem, Hochmuth Reinhard [et al.] 328

Discrepancies between teachers and students and negotiation of the didactic contract in the first year of university, Gueudet Ghislaine [et al.] 338

Pre-service Secondary School Teachers' Collective Productive Struggle in a University Mathematics Content Course, Fortune Nicholas [et al.] 348

From mathematics to pedagogy and back again: Secondary teachers learning real analysis through co-tutoring, Pinto Alon 358

Klein's second discontinuity: a case from basic probability, Carl Winsløw [et al.] 368

Designing inquiry-based mathematics tasks to support institutional transitions between secondary and university levels, Hamanaka Hiroaki [et al.] 378

Prospective teachers' knowledge of prisms: Institutional perspective, Cizmesija Aleksandra [et al.] 380

Initiating dialogue between the ATD and Commognition in the context of university mathematics education research, Viirman Olov [et al.] 382

Pedagogically productive talk across the school-university divide, Hagin-Metzer Tsofnat [et al.] 392

Interweaving Mathematics – a discursive view on pre-service teachers navigating between university mathematics and school teaching, Weith Lukas [et al.] 402

A Reference Epistemological Model for Swedish high school Calculus, Viirman Olov412

TWG4: Digital technologies in the teaching and learning of university mathematics, including artificial intelligence 414

Mathematical Discourse of Generative AI Tools on Improper Integrals: The Case of ChatGPT, Radmehr Farzad [et al.] 414

Touching the Imaginary: A Multi-Touch Interface for Complex Numbers, Madaparambu Rajan Navaneetha [et al.] 425

The effect of using CalcPlot3D to teach partial derivatives in different educational settings, Martínez-Planell Rafael [et al.] 427

Insights into Mathematics by In-Depth Studies on Mathematicians with the Digital Interactive Mathematical Map for Calculus, Brandl Matthias [et al.] 437

Integration of generative AI tools in linear algebra learning: Effects on the Personal Mathematical Working Space, Solar Yasna 439

Supporting Proper and Spotting Improper Uses of AI in Math Teacher Seminar Papers, Segre Sabine 441

Didactic transposition through generative AI: Exploring the structure, functioning, and utility of the mean value theorem, Hanke Erik [et al.] 443

Designing targeted exercises and a scaffolded learning environment for learning to work with the theorem prover Lean, Garnelo Irene [et al.] 453

Integrating programming and mathematical proof perspectives: towards a framework for conceptual knowledge in Lean, Canclini Chiara [et al.] 455

Connections Across Mathematical Discourses in a Digital Dynamic Task on the Archimedean Property, Vovos Marios [et al.] 465

Cultivating undergraduate students' agentive participation in mathematical discourse when ChatGPT is involved, Biza Irene 475

Digital Tasks and automated feedback - a scoping review, Reich Birte [et al.] . . 485

Construction of the vector space concept through programming, Trigueros Gaisman María [et al.] 487

GenAI in Mathematics Teacher Education: In-Service Teachers' Experiences with AI-Generated Problem Variation, Liljekvist Yvonne [et al.] 497

Epistemic Archetypes and the Teaching–Research Gap in Mathematicians' Use of Programming, Olsen Jan-Fredrik [et al.] 507

Exploring how first-year students evaluate the AI task-solving process, Albano Giovannina [et al.] 509

A Preliminary Survey Focusing on Calculus Students' Practices and Beliefs Regarding Their Use of Generative Artificial Intelligence, Watford Mark [et al.] . . 519

Can AI Support the Development of Resilience, Perseverance and Metacognition in Learning Mathematics?, Melkonian Houry 529

Large Language Models as Co-designers in the Creation of 'Sci-problems': An Analysis of Interactions, Rizzo Ottavio [et al.] 532

TWG5: Teaching university mathematics: practices, innovation, and professional development 534

Influence of Initial Didactics Training on the Teaching of Quadratic Functions and Equations: A Case Study in French-Speaking Belgium, Dumonceau Pauline [et al.]534

Building the Definition of Limit A Discursive Analysis of Classroom Practice, Piñeros Rodríguez Manuela 537

Towards beliefs on proof in mathematics: a survey of future elementary school teachers in Switzerland, Da Ronch Mickael [et al.] 547

Gaining Syntactic Understanding through Semantic Reasoning in Symbolic Logic, Schwell Rachel 557

The role of analysing reasoning tasks in promoting critical thinking, Breen Sinead [et al.] 559

Sustainability and impact of study and research paths on university teaching practice, Vásquez Susana [et al.] 569

A Sequence of Active Learning Tasks to Facilitate De-Ritualisation in a First Year Transition Course: The Case of Injective Function, Hooper Owen [et al.] 579

A university mathematics lecturer's experience designing and implementing active-learning teaching sessions with digital resources, Witty George [et al.] 589

Modelling mathematical thinking in university mathematics lectures: An analysis of pronoun use, Kamaledaki Stella [et al.] 591

Supporting undergraduate tutors to notice students' mathematical thinking in first-year mathematics tutorials, Meehan Maria 601

Collaborative problem solving in calculus exercise classes: Undergraduate teaching assistants' perspectives, Del Chicca Lucia [et al.] 603

The Impact of Professional Development for Mathematics Professors Teaching Mathematics Courses for Future Teachers, Cawley Anne [et al.] 605

Aspects of Approximations of Practice that Support Calculus Instructors' Enactment of Equitable Active Learning Teaching Practices, Serbin Kaitlyn [et al.] . . 615

Principles for effective collaboration between undergraduate mathematics education researchers and mathematicians, Weber Keith [et al.] 625

How proofs are organized: a Theme analysis, Küchle Valentin [et al.] 627

Teachers' reflections on the causes of students' errors in university mathematics: epistemological obstacle as a potential analysis tool, Ramírez Bernal Henry Alexander 637

Domain of Validity framework: A new instructional theory for addressing students' preconceptions, Boukhatem Elias [et al.] 647

Exploring Peer-Led Team Learning from a Vygotskian Perspective with Mathematics Pre-service Teachers, Erione Elisabetta Giuseppina [et al.] 650

TWG6: Equity and affect in university mathematics **660**

Students' motivation in a course design with a focus on formative assessment, Wlassak Felix [et al.] 660

The role of learning strategies and performance feedback for dropout intention in mathematics-intensive study programmes, Göller Robin 671

University students' attitude towards proof, Rämö Johanna [et al.] 681

The role of basic needs satisfaction in mathematics tutorials for university exam performance, Göller Robin [et al.] 683

"He ended up being against the stereotype of a White Man. Not sexist, not racist.": A Narrative Exploration of Mathematical Creativity and Identity, Tang Gail [et al.] 693

Postsecondary Privileging of Pure over Applied Mathematics and Sense of Belonging for "Nontraditional" Mathematics Students, Amman Kristen [et al.] . . . 704

Creating bilingual resources to teach magnitudes through a master's course based on the Questioning the World Paradigm, Dimey Armande [et al.] 714

Students' Emotion Graphs during Proof Construction, Satyam V. Rani 716

Mathematical Identities of Women in Doctoral Mathematics: The Case of Ida, Latona-Tequida Talia 718

TWG1A: The teaching and learning of calculus and real analysis

Students' use of the multivariable chain rule in non-routine problems

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This study examines students' ability to solve non-routine (NR) problems in multivariable calculus. A participatory action-research intervention was conducted with digital quizzes, student surveys, and metacognitive instruction and examined its effects focusing on an NR exam problem that requires recognizing implicitly presented function composition and the chain rule. Data included quiz responses, surveys, and written exam work. On the exam, most students failed to recognize composition, hence did not use the chain rule; some differentiated directly to produce a gradient without cognitive conflict, indicating procedural understandings and difficulty with NR tasks. Both experimental groups outperformed the control on NR problems despite inconsistent results in overall exam achievement, prompting further research.

Keywords: Teaching tertiary level calculus, Students' difficulties in multivariable calculus, Composition and chain rule, Non-routine problems, Action research.

This project is part of larger program to study effects of small interventions on students' metacognition and problem solving in a multivariable calculus course at an engineering college. It focuses on students' ability to solve text-based mathematical problems that implicitly requires the use of multiple concepts and theorems, referred to as *non-routine (NR) problems* following Harel (2021). The following Problem, Q3, is an NR problem, that implicitly relates function composition, the chain rule, and the gradient, and explicitly to the graph and the tangent plane:

> Q3: Two paths are drawn on the graph of a function $z = f(x,y)$, $f \in C^1(\mathbb{R}^2)$. It is given that the projection of path-1 onto the *xy*-plane is the parabola $y_1(x) = x^2 + 3x - 3$ and along path-1 is $z(x) = x^3$; also, the projection of path-2 onto the *xy*-plane is the parabola $y_2(x) = x^2 + 2x - 1$ and along path-2 is $z(x) = x^3 + x - 2$. The two paths intersect at a point $\mathbf{P} = (x_0, y_0, z_0)$ which lies on the graph of the function f. Find the intersection point $\mathbf{P}$ and the tangent plane to the graph of the function f at $\mathbf{P}$.

Similar problems are regularly given in the final exam of the course, and it was apparent students consistently struggle with them and avoid choosing them (Figure 1). This suggested both their ability and their self-efficacy for answering these types of questions are low. Therefore, a pedagogical intervention was designed using digital quizzes and adapted class discussions to improve students' ability and self-efficacy to solve such problems.

This context is well adapted to the methodological framework of Participatory Action Research (PAR). The PAR framework is suited for close-to-practice research in

education, as it involves teachers and academic researchers collaborating to address real problems in the classroom (Wright, 2021). This collaborative approach leverages the teachers' intimate knowledge of the classroom environment to generate practical and relevant knowledge that accounts for their perspectives, challenges, and opportunities (Costello 2011, Wright 2021). The process is guided by the action research model, which involves cycles of planning a pedagogical change, performing an intervention (action), observing the outcomes, and critically reflecting upon them to refine the plan for the next cycle. (Costello, 2011).

This paper describes the intervention, its effects, characterization of students' difficulties with Q3 and implications for future research.

LITERATURE REVIEW

Students face deep and interconnected conceptual challenges in mastering multivariable calculus, which are often rooted in prior learning difficulties. Carlson (1998) suggests that a foundational issue is a general lack of understanding of the function concept, and that students "do not possess helpful mathematical habits. They do not view their mathematics as useful to solving problems, do not expect mathematics problems to 'make sense, and have little confidence in their abilities to solve unfamiliar problems… High performing students do not demonstrate belief in their mathematical knowledge as useful when required to solve an unfamiliar problem, even when it is solvable using the mathematics they know (p. 142)".

This weak foundation is exacerbated in the transition to multivariable functions, and instruction on multivariable differential calculus concepts like the chain rule and tangent plane often falls short (Harel, 2021), with a particular lack of insight into how students grasp these basic ideas, including partial derivatives and directional derivatives (Martinez-Planell et al., 2015). Specifically concerning the chain rule, students frequently fail to recognize when it is necessary due to the absence of a clear cognitive schema (in the sense of APOS theoretical framework) for function composition and decomposition, which is essential for correctly setting up word problems (Clark et al., 1997).

Harel (2021) contends that student difficulties in multivariable calculus stem less from cognitive limits and more from pedagogical choices, among them the lack of repeated reasoning necessary to internalize and organize concepts. Typical instruction often skips the total derivative and jumps to component formulas for the chain rule, preventing students from constructing the underlying structure of composition and leading to inflexible application. Similarly, the gradient is introduced as a ready-made tool, which leaves it disconnected from linearization, tangent planes, and the implicit-function framework. The result is fragile knowledge, where students treat structural links (composition, gradient, normal, tangent planes) as isolated formulas. Harel therefore recommends using non-routine (NR) problems that repeatedly force students to compose functions until the essential cognitive structure stabilizes.

Alcock and Simpson (2004) found that first-year students who rely on visual imagery to reason about sequences and series may gain a false sense of understanding, which ultimately hinders their engagement with complex formal material. This tension is amplified in multivariable calculus: unlike the single-variable setting where graphs provide strong intuition, most multivariable surfaces are difficult to visualize. Consequently, students' intuition often diminishes, forcing them to rely more heavily on algebraic and formal techniques without the support of a clear mental image of the object being analyzed.

Martínez-Planell et al. (2015) and Gabel et al. (2025) conclude that the transition from single-variable to multivariable calculus is *not* a seamless generalization, as students face distinct conceptual demands involving 3D geometry, vector ideas, and coordinating multiple representations. Common difficulties include locating surfaces, distinguishing open statements from their geometric context, generalizing domain/range, and graphically interpreting directional derivatives; students also struggle with direction vectors. Consequently, Martínez-Planell et al. (2015) urge instructors to explicitly link partial derivatives, directional derivatives, and the tangent plane, and to clarify vector representations.

The ability to solve NR problems is related to an aspect of metacognition called *metacognitive regulation*: the active control over one's thought processes, including planning, monitoring, and evaluating progress (Stanton et al., 2021). Research suggests that these critical skills can be developed through explicit, systematic metacognitive instruction (Schoenfeld, 1992).

INSTRUCTIONAL CONTEXT AND RESEARCH QUESTIONS

Traditional frontal lecturing remains dominant in tertiary mathematics and STEM courses (Vroom et al., 2022). This persistence is attributed to factors such as instructors' beliefs that lecturing is the best way to model problem-solving and disciplinary practices, as well as institutional constraints, for example strict schedule and content in coordinated courses (Dawkins & Weber, 2023; Vroom et al., 2022). To bridge the gap between reform advocates and teaching practice, scholars suggest two main strategies: first, encouraging small, practical modifications to traditional lectures that require minimal effort; and second, increasing the involvement of mathematicians in the design of teaching reforms (Dawkins & Weber, 2023; Kensington-Miller et al., 2013). Studies on minor interventions (e.g., Leitner & Gabel, 2024; Kensington-Miller et al., 2013) have shown positive effects on student engagement, self-efficacy and achievement.

The study described here adheres to these two recommendations and is conducted within PAR methodological framework. It was prompted by an analysis of students' exams, which revealed difficulties with NR-type problems similar to Q3 (see Figure 1 below). Thus, a small pedagogical intervention was conducted to improve students' ability and self-efficacy to solve NR problems, i.e. students' ability to read a textual mathematical problem, understand implicit concepts, choose appropriate techniques and theorems to solve the problem and plan the solution process. By moving NR

problems from the isolation of homework/exams into the social sphere of the classroom, we are attempting to remove the gatekeeping aspect of NR problems by: validating the students' experience of the material's difficulty, reducing the high-stakes anxiety that usually kills creative problem-solving, and co-constructing a "toolkit" with our students that works for their specific hurdles.

The following research questions are answered:

RQ1) What were the prevailing students' difficulties reflected in students' written exam answers to Q3?
RQ2) What were the effects of the intervention on: (a) students' achievements in Q3 in the exam, and (b) students' self-efficacy concerning NR problems?

METHOD

This is part of a larger study investigating the effects of a small pedagogical intervention focusing on students' metacognitive regulation in a large, coordinated multivariable calculus course at an engineering college. There were 566 enrolled students taught across multiple sections. The intervention was implemented by Lecturer A (pure mathematician, 51 enrolled) and Lecturer B (mathematics education researcher, 87 enrolled) in their respective sections, which served as the experimental groups. The remaining sections formed the Control group (428 enrolled). The intervention included the administration of six digital online quizzes via the online MathMatize app. Not all quizzes included NR problems. After students answered the online quizzes, an in-class discussion was conducted, focusing on metacognitive regulation. All students have lecture four hours per week, and a two-hour recitation section with TAs, who also give homework.

This paper analyses Quizzes 4-5 which focussed on solving NR problems. These digital quizzes used a 'drag and drop' format where students selected the most appropriate concept from an answer bank without fully solving the problem, promoting deliberate choices and exposing them to multiple potential approaches. Following anonymous submission, lecturers guided a class discussion to compare strategies and support metacognitive regulation. Quiz-4 included an NR problem very similar to Q3, which included, for example, the following statement: "…given that the projection of a path on the graph of a function onto the *xy*-plane is $(x, y) = (x, x^2)$ and along the path $z(x, x^2) = x^3 - 3x$ [Blank 1]". Students were expected to identify that the most appropriate concept is *composition* and drag it into the blank. Quiz-5 was an NR problem on function optimization over a close and bounded domain in $\mathbb{R}^2$. See the online resources for a mathematical analysis of Q3 and the in class quizzes.

Due to Group A exhibiting substantially higher academic achievements (Table 1), the analysis treats Groups A and B separately against the common Control group. Student agreement with statements regarding metacognitive regulation was assessed using two online surveys administered before Quizzes 4-5 and at the end of the semester. Lecturers A and B consciously amplified their focus on metacognitive regulation in

their lectures in the following ways. Both lecturers made their planning phase visible by narrating their thought process aloud and asking students to justify their chosen steps (e.g., "Why does this sentence translate to the chain rule?"). Lecturer A explicitly introduced the concept of metacognition early on, framing the course around understanding *when* and *why* to apply concepts. Lecturer B emphasized subjectivity of strategy choice and worked to help students construct their own coherent "mathematical narrative" mainly through guided class discussions to create "organizing maps" of concepts and strategies.

The findings are based on students' answers to the quizzes and surveys, students' grades in the final exam and analysis of students' answers to Q3 on the final exam.

FINDINGS

General exam achievements

A total of 493 attended the final exam, as presented in Table 1.

Group	A	B	Control	All
N (students)	42	76	375	493
Average Exam Grade	65.95	56.11	62.38	61.72
Failure Rate (%)	30.95	53.95	42.13	43

Table 1: Breakdown of groups and general exam achievements

As shown in Table 1, the average grade of Group A is substantially higher than that of Group B, while the failure rate in Group A is considerably lower. These findings suggest that Groups A and B exhibit distinct performance characteristics and therefore should not be considered a single cohort.

Exam data (Figure 1) reveals that Question 3 (Q3) was the most avoided, indicating it was perceived as the highest-difficulty NR problem. While Group A showed higher opt-out rates than Group B, a five-exam longitudinal review confirms a consistent student trend of avoiding NR problems similar to Q3. A detailed discussion follows in a separate analysis.

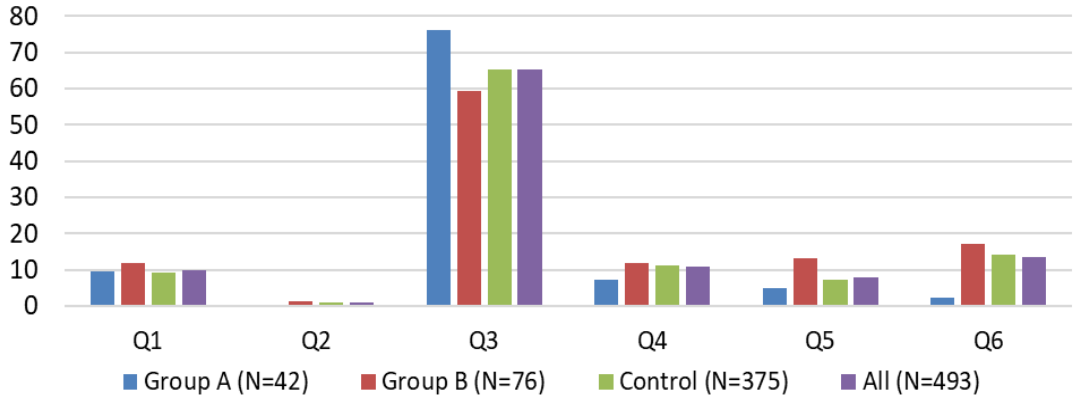

Figure 1: Percentage of students that did *not* choose exam question Q3

Students' answers to Q3 in the final exam

Table 2 presents the number and percentage of students who succeeded in finding the intersection point (in $\mathbb{R}^2$ and in $\mathbb{R}^3$) and the average grade on Q3.

	N	$\mathbb{R}^2$ Intersection	$\mathbb{R}^3$ Intersection	Average Grade of Q3 (0f 100)
Group A	10	9 (90%)	9 (90%)	61.47
Group B	31	28 (90.32%)	27 (87.1%)	62.27
Control	130	121 (90%)	117 (90%)	56.33
All	171	158 (92.39%)	153 (89.47%)	66.67

Table 2: Percentage of students that found intersection point and average grade on Q3

A total of 153 students succeeded in finding the intersection point **P** in $\mathbb{R}^3$, the majority of students in all groups. In the next analyses consideration is limited to these students. Table 3 presents the percentage of students who recognized the projection of paths on the *xy*-plane as a composition $z(x) = f(x, y(x))$ correctly, partly, or not at all.

		Composition		
	N	Not at all	Partial/Wrong	Correct
Group A	9	7 (77.77)	0 (0%)	2 (22.22%)
Group B	27	25 (92.59%)	1 (3.7%)	1 (3.7%)
Control	117	95 (81.19%)	6 (5.12%)	16 (13.67%)
All	153	127 (83.00%)	7 (4.57%)	19 (12.41%)

Table 3: Students who found *P* addressing composition

Table 3 shows that across all groups, most students did not even consider composition, with Group A performing better than Group B, aligning with their respective overall achievement levels.

Interestingly, although 127 students did not formally recognize composition, some continued to incorrectly calculate $\vec{\nabla} f(P)$ (e.g.: $z(x) = x^3 \Rightarrow z_x = 3x^2, z_y = 0 \Rightarrow \vec{\nabla} f(P) = (12, 0)$) and wrote the equation of the tangent plane. Table 4 presents the number and percentage of students who did so and shows that ~32% somehow "skipped" the chain rule and found an incorrect gradient $\vec{\nabla} f(P)$, without encountering any conflict.

	N	Calculated $\vec{\nabla} f(P)$ (incorrectly)
Group A	7	3 (42.85%)
Group B	25	5 (20%)
Control	95	33 (34.73%)
All	127	41 (32.28%)

Table 4: Students who found *P* but did not recognize composition

A total of 19 students identified composition correctly (Table 3, right column). Table 5 presents how many of them applied the chain rule correctly. Table 5 shows that if

students identified composition correctly, then they most likely used the chain rule correctly. A small percentage of students experienced difficulties in applying the chain rule after correctly identifying the composition.

	N	None	Wrong	Correct
Group A	2	0 (0%)	0 (0%)	2 (100%)
Group B	1	0 (50%)	0 (0%)	1 (50%)
Control	16	2 (12.5%)	2 (12.5%)	12 (75%)
All	19	2 (10.5%)	2 (10.5%)	15 (79%)

Table 5: Chain rule application of students who identified composition correctly

Digital quizzes

An analysis of students' answers to Quizzes 4-5 is presented, in which students were asked to solve NR problems (See Online Resources for full problems). Table 6 presents the percentage of students who chose appropriate concepts in Quizzes 4-5.

	Expected concept in [ ]	Group A	Group B
Quiz-4 (Similar to Q3) Group A - N=22 Group B - N=69	Blank 1 [composition]	27.3	29
	Blank 2 [2D Gradient]	4.5	1.4
	Blank 3 [3D Gradient]	31.8	11.6
	Blank 4 [Tangent plane]	68.1	56.5
Quiz-5 (Optimization) Group A - N=21 Group B - N=70	Blank 1 [Function to optimize]	42.8	30
	Blank 2 [Domain]	42.8	41.4
	Blank 3 [Boundary/Constraint]	76.2	70
	Blank 4 [Circle]	28.6	38.6

Table 6: Percentage correct answers for Quiz-4 and Quiz-5

Group A appears to be a bit stronger, and both groups struggled in the same places. The low success rate in this problem may explain students' self-efficacy regarding this problem and their avoidance from choosing Q3 in the exam, as was shown in Figure 1.

Student surveys

Two student surveys were given, one during the semester (approximately week 5) before Quiz-4, and the other together with the final review after the semester ended (therefore, very few students attended). Students were asked to rate their agreement with two statements on a 5-point Likert scale (1=low, 5=high).

Statement 1: When I read a problem, I can easily recognize which concepts and techniques I need to use to solve the problem. (included in both surveys)

Statement 2: The quizzes help me learn how to recognize the concepts and techniques I will need to use to solve a problem. (only on final survey)

Table 7 presents the average Likert score for agreement with each statement in the experimental groups in both surveys.

	Statement 1 – Midterm	Statement 1 – Final survey	Statement 2 – Final survey
Group A	3.1 (N=25)	3.1 (N=10)	3.8 (N=10)
Group B	2.9 (N=64)	3.3 (N=26)	4.2 (N=26)

Table 7: Average Likert score for agreement with each statement

Group B (the weaker group) showed greater improvement in their self-efficacy, and thought the discussion helped them more overall.

DISCUSSION AND IMPLICATIONS

To answer RQ1, students' written answers to Q3 in the exam were analysed. The primary obstacle is that approximately 83% of the students who found the intersection point in $\mathbb{R}^3$, i.e. had some understanding of the 3D structure of the problem, failed to identify either composition, or that the projection of the a path $y = y(x)$ onto the *xy*-plane results in a composition $z(x) = f(x, y(x))$ (Table 3). Although only a small number of students correctly identified composition, those students were efficient in applying the chain rule correctly (Table 5). Note, that in Q3, the expression $z(x) = f(x, y(x))$ was intentionally not explicitly written, so the source of difficulty may be students' poor understanding of text (mathematical reading comprehension), difficulty in connecting the geometric description with function composition, or difficulties with graphs of functions $z = f(x, y), f : \mathbb{R}^2 \to \mathbb{R}$. Either way, the findings point to a major obstacle in student understanding of composition – and not of chain rule.

The second notable finding is the absence of cognitive conflict when students failed to recognize composition: about 32% did not identify function composition and therefore skipped the chain rule yet proceeded to compute an (incorrect) gradient $\vec{\nabla} f(P)$ without sensing a problem. Many simply derived: $z(x) = x^3 \Rightarrow z_x = 3x^2, z_y = 0$ and did not consider what might have occurred if they had also used $z(x) = x^3 + x - 2$ (path-2) and found a different $\vec{\nabla} f(P)$ (Table 4). This points to a procedural understanding of differential ideas and difficulty with NR tasks where key concepts and techniques are implicit among multiple givens. Why a potential contradiction did not trigger strategy re-evaluation remains an open question. Planning involves task-based interviews to probe this phenomenon and to examine whether some concepts are intrinsically harder to recognize, particularly since both experimental groups struggled at the same conceptual places.

Research question RQ2 was answered though the analysis of digital Quizzes 4-5, two student surveys (midterm, final) and final exam achievements. On the final exam, Group A outperformed the control, and Group B underperformed (Table 1). However, on Q3 both groups outperformed the control (Table 2). So, this suggests that for both experimental Groups the direct instruction on that subject was beneficial.

However, during the semester both groups struggled with the quizzes (Table 6), as supported by preliminary instructor reflections (lecturer A noted an "abysmally" performed Quiz-4). Both lecturers reported that the use of the quizzes actively supports

students in developing strategy selection and conceptual understanding. Because the quizzes require students to begin formulating reasoning and solution approaches before the lecturer presents a solution, students have more substance to their answers. Consequently, the discussions that follow are more engaged and richer in mathematical argumentation and content. Yet, although student surveys demonstrate a slight improvement in self-efficacy (Table 7), the reality of the exam was that most students did not feel confident to choose Q3 (Figure 1).

Finally, the findings suggest two other directions for future research. First, future research should focus on tracing the genesis of students' difficulty with function composition, specifically investigating whether the deficit originates in single-variable calculus or earlier. A key area of inquiry is students' ability to recognize and apply the chain rule in non-routine contexts, such as related rates problems, to determine where procedural understanding breaks down. This investigation will seek to address whether the failure is rooted in students' inability to perceive known mathematics as useful (Carlson, 1998) or whether pedagogical reforms like those proposed by Harel (2021) and Martinez-Planell (2015) are sufficient to remediate these conceptual gaps. To this end, an initial study is designed for the coming academic year that probes students' performance on NR composition and chain-rule tasks in single-variable functions.

Second, a deeper analysis is forthcoming of student difficulties with multivariable composition and the chain rule using the APOS framework. APOS theory describes how students construct mathematical understanding through stages: Action, Process, Object, and Schema (Arnon et al., 2014). These structures arise through instances of reflective abstraction (e.g., interiorization and encapsulation). APOS serves as both an analytical evaluative tool (for example, Clark et al. (1997) used APOS to investigate students' chain-rule understanding) and a guide for instructional design (Arnon et al 2014). The initial findings suggest students' difficulty may stem from keeping function composition at the action level, hindering correct chain rule application, a topic for further study. A preliminary genetic decomposition will be articulated, an empirically testable account of the required mental constructions, for the multivariable chain rule. This genetic decomposition will be refined through task-based interviews and used to design instruction that scaffolds students' progress from actions to a coherent schema.

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The Archimedean analysis as a didactic obstacle and a condition of possibility to nonstandard analysis

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To the memory of Guy Brousseau

Starting from a reinterpretation of Guy Brousseau's notion of didactic obstacle, we show in what sense Archimedean mathematical analysis can be considered both a didactic obstacle and a condition of possibility for the emergence and development of nonstandard analysis. To this end, we analyse the modality of study of infinitesimal calculus supported by the Archimedean paradigm, which prevails in the didactic institutions of various engineering programs and university degrees in natural sciences. We identify a set of limitations or shortcomings that are considered undesirable and avoidable from the perspective of nonstandard analysis. In parallel, we also examine the role of the Archimedean paradigm in the historical development of mathematical analysis.

Keywords: didactic obstacle, modality of study, didactic paradigm, Archimedean mathematical analysis, nonstandard analysis.

INTRODUCTION

Based on the work of Guy Brousseau, Spagnolo (2006) proposed a new definition of *epistemological obstacle* from a linguistic perspective, locating such obstacles within the evolution of the semantic fields of mathematical language. He considers that epistemological obstacles can be characterized by *rather deep* modifications of the language, together with changes in the axiomatic systems. He argued that the main source of obstacles lies in the change of postulates, axioms accepted as self-evident and unavoidable for too long and too universally. In this sense, he claimed that the *Eudoxus–Archimedes postulate* could constitute an epistemological obstacle to the introduction of hyperreal numbers and to the understanding of nonstandard analysis.

In this work, we will use the tools of the *Anthropological Theory of the Didactic* (ATD) to reinterpret Guy Brousseau's notion of didactic obstacle in a direction different from that proposed by Spagnolo. Consequently, our way of interpreting *Archimedean mathematical analysis*[1] as a didactic obstacle to *nonstandard analysis* will differ from

[1] We will distinguish between *mathematical analysis* and *infinitesimal calculus* as is usually done, understanding that the former formalizes and extends the methods of the latter. Being informal here, we could say that infinitesimal calculus is the mathematical analysis studied in engineering programmes and natural science university degrees.

his. Nevertheless, we share a basic intuition with Spagnolo regarding the relationship between Archimedean analysis and nonstandard analysis, which we will express in terms of modalities of study and didactic paradigms.

THE NOTION OF EPISTEMOLOGICAL OBSTACLE

The notion of *epistemological obstacle* was introduced by Gaston Bachelard in *The Formation of the Scientific Mind* (Bachelard, 1938) to explain the psychological conditions underlying the progress of the natural sciences. Bachelard argues that the problem of knowledge must be formulated in terms of obstacles, and that it is in the very act of knowing that the hindrances and confusions arising from such obstacles appear. He considers that epistemological obstacles can be studied both in the historical development of scientific thought and in educational practice, but adds that, in education, the notion of a pedagogical obstacle is unknown.

The obstacles considered by Bachelard concern the experimental sciences, and he explicitly rejects the possibility of applying the notion of epistemological obstacle to mathematics. In short, for Gaston Bachelard, the notion of epistemological obstacle refers to forms of *knowledge* about nature based on opinion, developed through superficial observation in everyday life, and resulting from an immediate empiricism (of first experience), a kind of sensory knowledge incapable of understanding physical phenomena and, more generally, the phenomena studied in the experimental sciences.

For his part, Brousseau takes up the basic intuition underlying this notion to reinterpret it profoundly and to apply it, contrary to Bachelard's view, to the case of mathematics. In a lecture delivered at the 28th CIEAEM meeting in 1976, Brousseau presented his first ideas, later published in (Brousseau, 1983). He subsequently developed these ideas in several works, among others, (Brousseau, 1989, 1998, 2007).

First of all, Brousseau observes that Bachelard's concept does not fit well with the case of mathematics, and that the search for epistemological obstacles in mathematics requires a considerable effort of invention. Considering the *teacher–student–knowledge* system, he classifies the obstacles that arise within the teaching system (that is, the obstacles to teaching) according to the subsystem in which they are presumed to originate. He identifies obstacles: (a) *of ontogenic origin*, which arise as a consequence of the student's limitations (neurophysiological, among others); (b) *of didactic origin*, which depend on a specific project of the educational system; and (c) *of epistemological origin*, which are those that neither can nor should be avoided, since they are constitutive of the very knowledge being sought (that is, of the knowledge they themselves obstruct). The latter can be found in the history of mathematics.

An obstacle manifests itself through errors that are not due to chance but are, on the contrary, reproducible, recurrent, and persistent. These errors are connected by a common source, a particular way of knowing, a coherent and locally valid conception. But, what are the characteristics or properties that a piece of knowledge or a conception must satisfy to constitute an obstacle? Duroux (1983) specified some of these

conditions and, based on that description, Brousseau formulates the following characterization of obstacles:

> - An obstacle is "knowledge" in the sense that we have given it, meaning "a regular way of dealing with a set of situations".
>
> - This knowledge yields correct results or appreciable advantages in a given field, but proves to be false or completely unsuitable in a new or broader field.
>
> - New, true, or valid knowledge about a broader field is not established "based on" previous knowledge, but rather in opposition to it: it uses other points of view, other methods, etc.
>
> - This knowledge [the obstacles] is not a variable personal construct. It is a universal response in specific areas. It therefore appears almost inevitably in the genesis of knowledge, whether historical or didactic. (Brousseau, 2007, p. 45).

In short, Brousseau modifies the notion of *epistemological obstacle* introduced by Bachelard and generalizes it through a broader notion of *obstacle* that refers to a *conception*, that is, a way of knowing and of using a particular field of mathematics. He concludes that if learning is necessarily organized through a succession of different conceptions, each of which constitutes an obstacle for the following, then classical pedagogical practices, based on the progressive accumulation of knowledge, must be rejected. If we accept that a form of knowledge (a conception) is established in opposition to another, one that it builds upon and ultimately replaces, then we must agree that organizing the overcoming of a conception–obstacle consists in proposing a situation capable of evolving. It is not a matter of transmitting the knowledge to be taught, but of designing a didactic situation in which the student confronts the limits of their initial conception and moves toward a more comprehensive one. In short, Brousseau considers that the main objective of didactics is precisely to study the conditions that the situations presented to students must satisfy to foster the emergence, functioning, and rejection of these successive conceptions (Brousseau, 1998).

MODALITIES OF STUDY AND DIDACTIC PARADIGMS

Before proposing our own way of conceptualizing obstacles and interpreting the role they play in education and in the history of mathematics, it is necessary to introduce some notions that will be used in what follows.

The ATD identifies didactics as the *science of the conditions that make learning possible*, or, more precisely, the science of the conditions that make possible (or hinder) the genesis and development of *institutionalized modalities of study* related to the learning of all kinds of knowledge. Consequently, for the ATD, a central problem of the didactic science consists in investigating the nature of the existing (or possible) modalities of study in didactic institutions, that is, the rules and principles that govern their structure and functioning (their *economy*)[2]. Complementarily, didactics studies the set of conditions of all kinds that have influenced and continue to influence these modalities of

[2] The economy of a system refers to the coordination of the components involved in its functioning (Moliner, 2007).

study, why the prevailing modalities have come to be as they are, and how they could be modified in a given direction (their *ecology*)[3].

We model each modality of study through a *didactic paradigm* (DP) and to analyse the economy and ecology of the current (or possible) didactic paradigms (CDP) in didactic institutions, the researcher uses, as a reference, a DP that is a theoretical construct developed by research, what we call a *reference didactic paradigm* (RDP) (Gascón, 2024; Gascón & Nicolás, 2021).

A DP constitutes a complex system composed of three components: an *epistemological model* (EM) of the knowledge at stake (which answers the question *what is studied*); the *didactic ends* (DE) pursued (which answer *why it is studied*); and the *didactic means* (DM) employed (which answer *how it is studied*).

Each DP is supported by an epistemological model (EM) that provides a representation (from a given perspective) of the knowledge at stake, to which a raison d'être is assigned, that is, certain types of tasks, together with their associated questions, whose resolution requires the use of that knowledge. Moreover, the DP assigns to the *study* of such knowledge (when carried out through the didactic means, DM) a set of didactic ends (DE), or study ends. Among these ends is that of avoiding certain *didactic phenomena*[4] (Dφ), to which the construction[5] of the paradigm itself responds. It could be said that the construction of this modality of study is, in fact, a response to such phenomena. In addition to providing the notions necessary to formulate didactic ends, the EM also constrains the kinds of DM that may be effective in achieving them (Gascón, 2001). This reveals the functional unity of a DP, which must therefore be regarded as a complex system in the sense proposed by Rolando García (2006).

A NEW INTERPRETATION OF DIDACTIC OBSTACLES

By *didactic obstacles* we do not refer to epistemological obstacles of didactic origin, but rather to a new way of interpreting obstacles which, as we shall see, does not aim to distinguish different types of obstacles according to their origin. To begin with, we raise two basic questions regarding didactic obstacles: (1) what kind of *object* is a didactic obstacle?, and (2) what is the *thing* being obstructed (by a didactic obstacle)?

Using the tools provided by the ATD, we will not say that a didactic obstacle is simply a piece of knowledge or a conception (of a certain field of knowledge), nor that it can be characterized by modifications of language together with changes in axiomatic systems (Spagnolo, 2006). Neither will we say that the thing being obstructed is *the learning of a notion or the effective solving of a certain type of problem* (Artigue, 1990). We identify a didactic obstacle with a specific way of conceptualizing, using,

[3] The ecology of a system is determined by the set of conditions and constraints that affect or have affected it. These conditions and constraints make it possible to explain, at least partly, its past evolution, its current situation (its economy) and its possible developments.

[4] Without attempting to provide a formal definition of a didactic phenomenon, we will simply say that it manifests itself through an observable set of events or processes related to study. These can be typified, recur regularly under certain circumstances and in different institutions, and are surprising (from a given perspective), thus requiring an explanation.

[5] Whether it is a *cultural construct* that has crystallized over time or a *theoretical artifact* created by research.

and representing a field of knowledge $\mathcal{F}_1$, together with a particular way of studying it in an institution I, and with the didactic ends associated with that study, that is, with a $DP_1(\mathcal{F}_1)$. Meanwhile, we conceptualize the thing being obstructed through an alternative didactic paradigm, $DP_2(\mathcal{F}_2)$, with $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$ (and, in many cases, $\mathcal{F}_1 \subseteq \mathcal{F}_2$), which constitutes a *response* and a reaction to $DP_1(\mathcal{F}_1)$, in the sense that among the didactic ends of $DP_2(\mathcal{F}_2)$ is the avoidance of certain phenomena that necessarily occur when $DP_1(\mathcal{F}_1)$ is prevailing. Since $DP_2(\mathcal{F}_2)$ is a reaction to $DP_1(\mathcal{F}_1)$, we can say that $DP_1(\mathcal{F}_1)$ generates the *conditions of possibility*[6] for the emergence of $DP_2(\mathcal{F}_2)$.

In summary, we say that $DP_1(\mathcal{F}_1)$ is a *didactic obstacle* to $DP_2(\mathcal{F}_2)$, and, conversely, that $DP_2(\mathcal{F}_2)$ is obstructed by $DP_1(\mathcal{F}_1)$, if among the didactic ends of $DP_2(\mathcal{F}_2)$ is the avoidance of certain phenomena that necessarily occur when $DP_1(\mathcal{F}_1)$ is prevailing, and that are regarded as *undesirable*[7] from the perspective of $DP_2(\mathcal{F}_2)$ because their occurrence is incompatible with the achievement of the ends of this paradigm.

Given a DP-obstacle prevailing during a certain historical period, different potentially obstructed DPs may exist, since when a DP-obstacle is prevailing, various didactic phenomena may occur that are regarded as undesirable and avoidable from different perspectives. Thus, a certain *relative nature of didactic obstacles* emerges.

Considering our redefinition of didactic obstacles as DP-obstacles, and the functional unity of DPs, it becomes clear that epistemological, didactic, and sociocultural factors are all involved to a greater or lesser extent, making it practically impossible to separate them. Consequently, we will speak of *didactic obstacles* without attempting to distinguish between those of epistemological, didactic, or sociocultural origin. Brousseau himself, on whose notion of *obstacle* our interpretation is based, notes, when referring to the introduction of decimals in French elementary education, that for students these are perceived as natural numbers with a change of unit. This conception, reinforced by the students' mechanization, constitutes an obstacle whose origin is at once didactic and sociocultural (Brousseau, 1983), and we might add, epistemological.

THE ARCHIMEDEAN PARADIGM AS OBSTACLE IN THE DIDACTIC GENESIS OF NONSTANDARD CALCULUS

We can use the notion of obstacle to analyse the modality of study of infinitesimal calculus prevailing in didactic institutions within the various fields of engineering and the natural sciences. This modality of study is supported by a didactic paradigm that we may label the *Archimedean analysis paradigm* (AAP). Our aim is to justify that this AAP constitutes an obstacle to the *nonstandard analysis paradigm* (NAP).

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[6] A *condition of possibility* is an element, structure, or principle that allows a certain type of experience, knowledge, or phenomenon to occur. The expression *condition of possibility* was first used systematically and philosophically decisively by Immanuel Kant, within the framework of his critical philosophy, where it designates the a priori foundations that make both human experience and knowledge possible.

[7] In general, we shall say that a didactic phenomenon Dφ is *undesirable from the perspective of a DP* if the occurrence of Dφ is incompatible with the ends of the DP.

For reasons of space, it is not possible to describe these paradigms in depth. In particular, we cannot examine in detail the epistemological model underlying the NAP; we will simply note that it can be based on Robinson (1966) or, more conveniently, on one of its axiomatic reformulations, such as the one recently developed by Hrbacek, Lessman, and O'Donovan (2014). In any of its versions[8], nonstandard analysis succeeds in formalizing the intuitive theory grounded in the use of infinitesimals that Leibniz employed to develop calculus. These ideas appear more natural than the modes of reasoning that ultimately dominated modern calculus, based on the ε–δ language.

In order to regard the AAP as both a didactic obstacle and a condition of possibility for the NAP, we should show three things: (1) that in the modality of study governed by the AAP, when analysed by taking the NAP as the reference didactic paradigm, certain didactic phenomena necessarily occur; (2) that these phenomena are undesirable from the perspective of the NAP, because their occurrence is incompatible with the achievement of certain didactic ends of the NAP; and (3) that if the NAP were prevailing, these phenomena would be avoided.

Let us proceed step by step:

(1) When the prevailing modality of study of infinitesimal calculus in the institutions responsible for training the various types of engineers or graduates in the natural sciences is examined from the perspective of the NAP, a set of facts related to study can be identified that constitute a didactic phenomenon (see footnote 4). Among these we may mention: the use of a *language* (in which symbols such as dx, df, df/dx, … appear) with an *intuitive interpretation* that lacks formal grounding; the *definition* of certain notions (for example, the notion of continuity) in a manner that is unintuitive and technically complex; the *use* of techniques without any justification for computing primitives and, more generally, for formulating and solving differential equations; and the *avoidance* of certain fundamental proofs on the grounds that they are technically complex or unnecessary. In what follows, we unpack some of these facts.

The textbooks (Stewart, 2006) and (Hass, Weir, & Thomas, 2020) are well-known references for the teaching of calculus at the university level in various engineering and natural science programs. In both books, the use of infinitesimals is explicitly ruled out. Rigorous definitions of the key concepts of calculus are presented in ε–δ terms, and these are used to state and prove some elementary results. However, more fundamental results are stated without proof. For example, in (Stewart, 2006), the statement of the *Intermediate Value Theorem* is accompanied by the remark, "…whose proof is found in more advanced books of calculus" (p. 126), and in (Hass et al., 2020)

[8] Robinson (1966) constructed a non-Archimedean ordered field, the field of hyperreal numbers, $^*\mathbb{R}$, which extends $\mathbb{R}$. This extension contains an infinity of nonzero infinitesimals. Constructing $^*\mathbb{R}$ is not simple: it requires advanced knowledge of mathematical logic. However, several axiomatic approaches to Robinson's theory, more accessible with elementary resources, have been proposed from the 1970s to the present, e.g., (Nelson, 1977) or (Hrbacek, 2009). These are theories in which, instead of constructing $^*\mathbb{R}$ from $\mathbb{R}$ and the axioms of Zermelo–Fraenkel set theory with the axiom of choice (ZFC), one starts from a language richer than that of ZFC and from a series of axioms added to ZFC that allow deducing the existence of infinitesimals (within $\mathbb{R}$). In (Hrbacek et al., 2014), based on the approach of (Hrbacek, 2009), a course in calculus is presented, covering the usual contents of such a course at the university level.

one can read that "the proof of the Extreme Value Theorem requires a detailed knowledge of the real number system and will not give it here" (p. 213).

The use, even if only partially decorative, of ε–δ methods in textbooks such as those mentioned above contrasts sharply with their complete absence from natural science and engineering texts in which calculus is applied. For example, a well-known reference for teaching the rudiments of calculus specifically designed for chemistry instruction is (Steiner, 2008). The text alludes to the AAP but, in practice, opts for a more intuitive exposition (in particular, it avoids presenting definitions using ε–δ). At the same time, it cautions that "we note that the symbol dy/dx does not mean the quantity 'dy' divided by the quantity 'dx'; the symbol represents the limit" (p. 96).

However, the use of infinitesimals (represented by symbols such as dx or dt) is widespread in the applications of calculus within the natural sciences. Richard Feynman himself, in his famous *Lectures on Physics*, makes extensive use of a kind of heuristic method based on the frequent manipulation of differentials, treating them as genuine quantities. The same occurs in reference texts such as *Physics* by Alonso and Finn (1992): "the instantaneous acceleration is found by observing the small change of velocity dv that takes place in a very small time interval dt." (p. 38). Throughout the text, differentials are often manipulated as if they were actual numbers, in order to deduce equations that model various physical phenomena. To solve these (differential) equations, the *method of separated variables* stands out, for example, which is used continuously without justification.

(2) From the perspective of the NAP, these are limitations or shortcomings of the aforementioned modality of study. Taken as a whole, they can be interpreted as a didactic phenomenon that necessarily arises when the AAP is prevailing and that could be characterized by the label *decorative use of rigor*. This is an undesirable phenomenon from the viewpoint of the NAP, since its occurrence is incompatible with the achievement of one of the didactic ends of the NAP: to simplify and make intuitive, while maintaining rigorous foundations, certain notions and basic techniques of infinitesimal calculus, as well as proofs of fundamental theorems.

(3) This phenomenon is considered avoidable from the perspective of the NAP, since its avoidance is among its didactic ends and, in a certain sense, the construction of the NAP constitutes a response to the aforementioned limitations of the AAP. Nevertheless, the fact that this phenomenon could indeed be avoided within a modality of study governed by the NAP requires empirical verification.

In summary, it can be stated that the AAP constitutes a didactic obstacle, and also a condition of possibility, for the emergence of the NAP in the mentioned didactic institutions. The foregoing does not deny the possibility that the hypothetical implementation of the NAP in those institutions might give rise to didactic phenomena regarded as undesirable from other perspectives and, consequently, that the NAP itself might appear as a didactic obstacle to other alternative paradigms.

THE ARCHIMEDEAN PARADIGM AS OBSTACLE IN THE HISTORY GENESIS OF NONSTANDARD ANALYSIS

In the ATD, the term *didactic* refers to *study* in a broad sense that encompasses not only *teaching and learning* but also the *creation*, *development*, and *application* of knowledge. In this sense, we may speak of *didactic obstacles* (or *obstacles to study*), interpreting them as obstacles not only to teaching and learning but also to the creation, development, and application of mathematics. Thus, our way of interpreting obstacles, and the corresponding conditions of possibility they generate, applies both to teaching and learning and to the historical development of mathematics. As Brousseau points out, didactic obstacles almost necessarily arise in the genesis of knowledge; they are *constitutive* of it, whether in its historical or didactic genesis.

In the case of mathematical analysis, it is possible to raise, from the perspective of the NAP, the following question: which parts of mathematical analysis are independent of the Archimedean axiom? The answer to this question gives rise to what may be called *neutral or absolute mathematical analysis*, that is, the body of results that can be proved without assuming the Archimedean axiom. We shall see that the AAP can be regarded, analogously to the case of the geometries (though with some important differences), as both a didactic obstacle and a condition of possibility for the emergence of the NAP and of neutral or absolute mathematical analysis.

The AAP, which prevails in the institution of mathematical research, is the result of the process of *arithmetization of analysis* (Espín, Gascón, & Nicolás, 2025), consolidated in the second half of the 19th century. This process was characterized by a complete rejection of infinitesimals, the establishment of a new canon of rigor for mathematical analysis, and the suppression of geometric and dynamic intuition. Consequently, the AAP constituted, for more than a century, a didactic obstacle to the NAP and, therefore, to neutral mathematical analysis, since its basic postulates were incompatible with the principal ends of the NAP. But, on the other hand, the AAP, through the rigorous formalization of the notions of limit, continuity, and derivative, provided the conceptual and technical framework that, together with the development of mathematical logic in modern terms (Reck, 2013), enabled Robinson (1966) to precisely formulate the foundations of the NAP, drawing on model theory and first-order logic. In this sense, it is clear that the AAP was not only a didactic obstacle but also a condition of possibility for the emergence of the NAP and of neutral analysis.

IN SYNTHESIS AND FUTURE WORK

This communication is part of a recently initiated line of research focused on the notion of didactic paradigm (DP), whose heuristic potential for addressing various problems in didactics research had already been demonstrated in several works (Espín et al., 2024, 2025; Gascón, 2024; Gascón & Nicolás, 2021). In this paper, we use the notion of DP to reinterpret and extend the concept of obstacle introduced in didactics by Guy Brousseau. Moreover, by considering obstacles as DP-obstacles, we have been able to interpret them simultaneously as conditions of possibility for the obstructed DPs. In

this sense, we provide an interpretation of Brousseau's intuition that epistemological obstacles are *constitutive* of the obstructed knowledge.

Our application of this new notion of didactic obstacle to the case of Archimedean analysis, viewed as obstacle to nonstandard analysis, both in its didactic and historical genesis, will require broader empirical verification. This is one of the studies we intend to carry out in the future. More generally, in an upcoming paper, we aim to develop the ideas that we have only been able to present here in a very synthetic way. In particular, we will apply the notion of didactic obstacle to the case of Euclidean geometry, as both an obstacle to and a condition of possibility for non-Euclidean geometries.

ACKNOWLEDGMENT

This work has been funded by projects PID2021-126717NB-C31, PID2021-126717NB-C32 and PID2021-126717NB-C33.

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Concept Inventory for Real Analysis (RACI) - Theoretical foundation and item analysis

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The paper resumes the analysis of the Concept Inventory for Real Analysis (RACI) test. A new methodology is introduced to analyze test items by "used knowledge elements", which can be definitions, explicit and implicit theorems, and properties of examples. Moreover, logical and algebraic knowledge elements could be identified. The totality of knowledge elements is checked against three standard textbooks to determine which parts of a standard course are covered by the test. Finally, the knowledge elements can be used to construct an overall concept map that shows the conceptual relations relevant for the test. This methodology can be applied to other concept inventory tests in proof-oriented mathematics courses, too.

Keywords: Real Analysis, concept inventory, conceptual knowledge.

INTRODUCTION: THE RACI TEST

In Bauer et al. (2024), we presented the RACI (Real Analysis Concept Inventory), including the design background, detailed item analyses, and empirical results. The RACI Test focuses on conceptual knowledge (see below) related to essential concepts of German Analysis I courses: Such courses aim at a systematic theory introduction based on the epsilon-delta definition of limits and continuity, and rigorous proof. For example, theorems such as the Intermediate Value Theorem or the Mean Value Theorem are rigorously proved in these courses based on the completeness axiom of real numbers. In terms of mathematical style, these courses are similar to Real Analysis courses in Anglo-Saxon universities. In Bauer et al. (2024, p. 84), we characterized three design principles for the test: requiring application of definitions and theorems regarding one-dimensional real analysis to particular situations or examples or connecting different concepts (DP1), solvability without complex mental computations or multi-step reasoning (DP2) and distractors based on beliefs about typical students' intuitive misconceptions (DP3).

The topics of the two tests were: Test 1: (A) completeness of the real numbers, (B) convergence of sequences of real numbers, (C) convergence of series of real numbers; Test 2: (D) limits and continuity of functions of one real variable, and (E) differentiability of functions of one real variable. Both tests (30 minutes each) were given to students just when the lecturer had finished the topics in their lecture. In total, we tried to cover the most important topics, based on the authors' teaching experience. The suitability of the selection was confirmed by expert validation (Bauer et al, 2024). A further form of validation is the comparison of the knowledge required for the test with that presented in standard textbooks, which we undertake in this paper. Task 7

(Figure 1) exemplifies our design principles DP1 and DP2. The example is an eventually constant sequence, which might be seen as non-convergent due to the misconception that sequences never reach their limit. The (wrong) claim "the sequence has two accumulation points" (4) touches on a misconception of the definition of accumulation point (a finite number in the neighborhood suffices). In this regard, DP3 is taken into account.

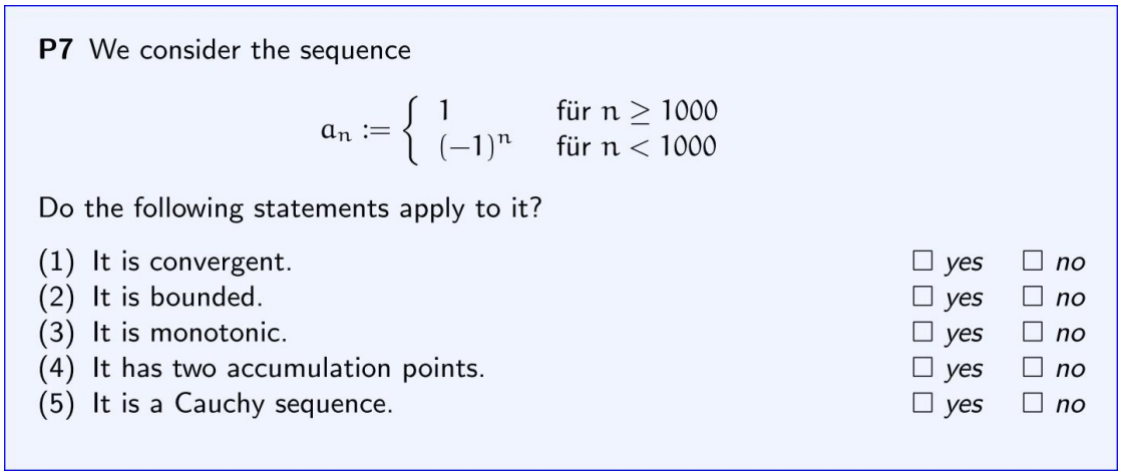


Figure 1: Task 7 in the RACI Test

Here, theorems on the properties of sequences must be known and used. The conceptual relations between convergent and Cauchy sequences, and between limits and accumulation points, have to be known and used.

We first conducted an a priori analysis of all items, identifying the lecture-based knowledge required for their solution and formulating hypotheses about intuitive correct or incorrect responses. These analyses were local, focusing on individual items. Then, we develop a systematic approach to analyzing a test regarding conceptual knowledge in proof-oriented mathematics courses with a special focus on relationships between concepts, and we present our RACI test's content regarding sequences using this systematic approach.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Conceptual knowledge. Hiebert and Lefevre defined conceptual knowledge as "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & Lefevre, 1986, p. 3-4). In this view, there is a strong emphasis on connections: A piece of information can be conceptual knowledge only if the holder is aware of its relationship to other pieces of information. The taxonomy of Anderson and Krathwohl (2001) adopts this view in their characterization of conceptual knowledge to "interrelationships among the basic elements within a larger structure that enable them to function together" (Anderson & Krathwohl, 2001, p. 29). In this view, conceptual knowledge is distinguished from procedural knowledge, which, in the conception of Hiebert and Lefevre (1986, p. 6), encompasses two types of knowledge: the formal language of mathematics on the one hand, and knowledge of rules, algorithms, or procedures on the other hand.

Concept maps. Novak (1998) introduced concept maps as an educational tool that provides a graphical representation of concepts and their relationships. The particular focus on relationships is motivated by Ausubel's theory of meaningful learning (Ausubel, 1962). In a concept map, nodes represent concepts and edges represent relationships between them. Each edge is labeled with a "linking word" that describes the specific relationship between the two concepts (see Figure 4).

Implicit theorems. Lectures and textbooks present examples, definitions, theorems, and proofs that should ideally be referenced when solving problems or introducing new knowledge elements. However, they often rely on results that follow directly from these elements without being explicitly stated as theorems—implicit assumptions highlighted by Lakatos (1976). We refer to such elements as *implicit theorems*. Authors may omit them as they appear straightforward, and stating all of them explicitly could obscure the overall structure. Accordingly, texts vary in how much is stated as a theorem, with implicit theorems sometimes appearing in remarks or comments.

Testing conceptual knowledge. Concept inventories such as the well-known "Force concept inventory test" (FCI) by Hestenes et al. (1992) put intuitive knowledge and mental models in the foreground. This idea was adapted for Calculus in a "Calculus concept inventory" (Epstein, 2007; 2013) and the "Revised calculus concept inventory" (RCCI) (Lee et al., 2025), focusing on "most basic principles of differential calculus" (Epstein, 2013, p. 1080). Our test items, in contrast, aim at using definitions and theorems to describe relationships between concepts, and are similar in nature to Bauer's (2019) peer instruction tasks that can be used in lectures or tutorials to stimulate peer discussion. We were also inspired by the test on the convergence of sequences developed by Ostsieker (2020), which also aimed at conceptual knowledge.

Research questions:

(RQ1) What kinds of knowledge elements can we distinguish in the RACI test (definitions, examples, and theorems, possibly more)? What totality of knowledge elements are used in the items for a certain topic, possibly used in more than one item?

(RQ2) Does this totality cover all "important" elements of "usual textbooks" or are some missing?

(RQ3) How can knowledge elements be represented as relations between concepts? Which types of relations can be identified, and can their structure be visualized to capture the conceptual knowledge relevant for the test?

METHODOLOGY

Identification of used knowledge elements and their kinds (RQ1)

We create model solutions for each item that justify the correct answer based on the content of standard Real Analysis courses (as documented in common textbooks). Content can be definitions, theorems, and examples, and we are open to other elements. The authors' expertise is used in this step in two ways: first, we do not consider

solutions that could only be expected from a higher perspective of a research mathematician. Second, we formulate model solutions using available results (especially theorems) rather than reproving them, since any theorem application could be "unpacked" into a proof without changing the underlying knowledge. We compile the identified knowledge elements into a list, merging similar elements across items to obtain a minimal set. We distinguish types of elements, such as theorems, definitions, and examples, and others that are needed.

Comparing the totality of knowledge elements with three standard textbooks (RQ2)

We analyzed three widely used German Analysis textbooks, Forster and Lindemann (2023), Königsberger (2004), and Walter (2004), focusing on their coverage of sequences. In our analysis of the texts, we looked at all passages except examples and exercises. Our approach involved two steps: first, we coded whether and where each of our knowledge elements appears in each book; second, we identified topics, theorems, and definitions that appear in all three books but not in our test.

Representation of the system of knowledge elements as a concept map (RQ3)

As a contribution corresponding to RQ3, we developed a method that represents a set of knowledge elements as a web of knowledge. The set might belong to a single task or to an entire knowledge domain. Our idea is to view a mathematical knowledge element, such as "every convergent sequence is bounded," as a relationship between two concepts, namely "convergent sequence" and "bounded sequence". The specific relationship in this example case is an "*is a*" relation. In general, we proceed in two steps: Given a set of knowledge elements, we first analyze them in terms of concepts and their relationships, producing a "conceptual relations table" as an intermediate result. As a second step, the content of the table is then transformed to a concept map. The map provides a structural view of the given body of knowledge. If concepts appear several times in the table, then the graph will have more connections between concepts – it thus shows how strongly intertwined the body of knowledge is.

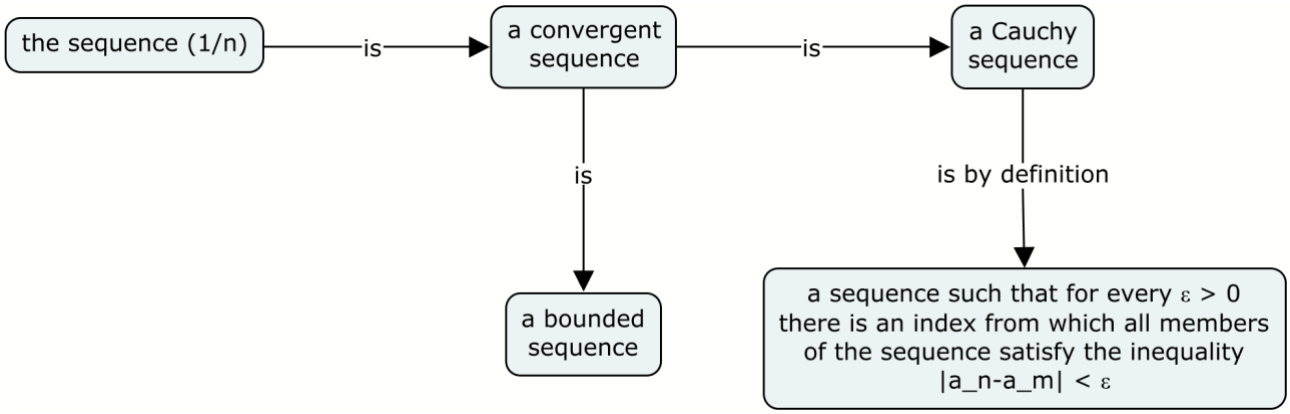


Figure 2: An example of a concept map as we obtain it through conceptual task analysis

As an example, Figure 2 shows the concept map obtained for a very small set of knowledge elements. It illustrates two adaptations to the use of concept maps that we

made to achieve the desired granularity of analysis. First, a node such as "Cauchy sequence" represents the term only, whereas the defining condition appears as a separate node (their relationship being "*is by definition*"). Second, our concept maps include concrete example objects as nodes: The concrete sequence (1/n) is related to "convergent sequence" through an "*is a*" relationship.

RESULTS

Knowledge elements of the model solutions (RQ1)

As we described in the methodology section, producing the model solution for each item and identifying the knowledge elements and their types were the first steps, followed by developing the list of elements. We present the kinds of elements first, then the list, and, based on this, we present an exemplary model solution for task 6.

Kinds of knowledge elements

We found that knowledge elements can be definitions, theorems, properties of examples, and what we call "implicit theorems". The "implicit" theorems are statements that are not usually formulated separately in books or lectures. They are direct conclusions from definitions or statements, or they follow from the existence of counterexamples, such as "constant sequences that are not constantly 0 are not a null sequence". Properties of examples could have been covered in the lecture or in the assigned homework tasks; sometimes they are formulated as theorems. For the analysis, we also needed what we called "knowledge elements from logic," such as "A single counterexample is sufficient for refuting an all-statement." We also included a category "knowledge elements from algebra," such as "formula sense," "calculating with inequalities," and "calculating with absolute values." Our aim was not to compile a complete list of logical or algebraic elements but to identify those that are relatively new in Analysis I compared to school mathematics, such as working with inequalities.

List of knowledge elements on sequences and convergence in the RACI Test

The following definitions were used in the test: Monotonicity (Q1), convergence (Q2), boundedness (Q3) of a sequence, Cauchy sequence (Q4), subsequence (Q5), accumulation point defined by subsequences (Q6) and defined by epsilon interval (Q7).

Q21	Every convergent sequence is bounded.
Q22	Every monotonically increasing and bounded-above sequence is convergent.
Q23	The limit of a convergent, monotonically increasing sequence is the supremum of its terms.
Q26	Constant sequences are convergent; specifically, $(a_n) = (a)$ converges to a.
Q28	If a sequence (a_n) converges to a, then a is its only accumulation point.
Q29	A sequence is convergent if and only if it is a Cauchy sequence.
Q30	Convergence of products of convergent sequences.
Q31	Convergence of subsequences of convergent sequences.
Q32	Monotonicity of the limit.

Q33	(Bolzano–Weierstrass.) Every bounded sequence has an accumulation point.
Q34	Convergence of sums and differences of convergent sequences
Q105	A sequence (a_n) converges to a if and only if for all $\varepsilon > 0$ all but at most a finite number of sequence elements lie in the interval $[a - \varepsilon, a + \varepsilon]$.

Table 1: Knowledge elements in the sequences test: Used theorems

Q100	Constant sequences that are not constantly zero are not null sequences.
Q101	If, from some index onward, all terms of a sequence are equal to a constant different from zero, then the sequence is not a null sequence.
Q102	If infinitely many terms of a sequence are equal to 1, then it is not a null sequence.
Q103	A sequence that is constant from some index onward is convergent.
Q104	Convergent sequences (and also their absolute values) need not be monotonic.
Q106	If a subsequence of a sequence converges, the sequence itself need not be convergent.
Q107	If a sequence (a_n) converges to a, then the sequence $(a_n - a)$ is a null sequence.
Q108	(Special case of Q32.) If (a_n) converges and $a_n \leq b$ holds for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq b$.
Q24	A sequence made by alternately taking terms from two convergent sequences with the same limit is convergent to the same limit.
Q25	If (a_n) converges to zero, then $(\lvert a_n \rvert)$ too.
Q27	Sequences that are constant (equal to a) from some index onward are convergent (to a).

Table 2: Knowledge elements in the sequences test: Used implicit theorems

In addition, there were four knowledge elements regarding logic (L1–L4) and four elements regarding algebra (A1–A4), as well as five knowledge elements regarding properties of examples (e.g., $\lim_{n\to\infty} \frac{1}{n} = 0$).

Model solution of task 6 (see figure 3)

6.1. (Correct solution: yes). Solution 1. The statement (1) is equivalent to the boundedness of the sequence (Q3, A1). From the assumed convergence, boundedness follows (Q21).

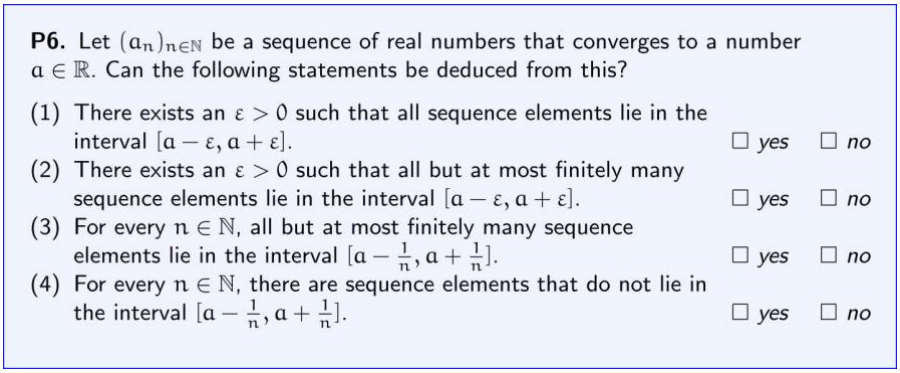
6.1 Solution 2. Since (a_n) is assumed to be convergent, it is bounded (Q21). This means that all terms of the sequence lie within a finite interval [c, d] (Q3). For sufficiently large ε, this interval is contained in $[a - \varepsilon, a + \varepsilon]$ (A1).

6.2 (correct solution: yes) Solution 1. Q105 (equivalent definition of convergence with "except finitely many"). For all ε implies one ε exists (L3).

6.2 Solution 2. It follows from the previous item 6.1, because "all but at most finitely many" is a consequence of "all" (L2).

6.3 (correct solution: yes) Solution. Q105. For all ε implies for all $\varepsilon = \frac{1}{n}$ (A1).

6.4 (correct solution: no) Solution. A counterexample is the constant sequence $(a_n) = (a)$, because: it is convergent (Q26), but all sequence terms lie in every interval $[a - \frac{1}{n},\ a + \frac{1}{n}]$ (A1).



P6. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that converges to a number $a \in \mathbb{R}$. Can the following statements be deduced from this?

(1) There exists an $\varepsilon > 0$ such that all sequence elements lie in the interval $[a - \varepsilon, a + \varepsilon]$. ☐ *yes* ☐ *no*

(2) There exists an $\varepsilon > 0$ such that all but at most finitely many sequence elements lie in the interval $[a - \varepsilon, a + \varepsilon]$. ☐ *yes* ☐ *no*

(3) For every $n \in \mathbb{N}$, all but at most finitely many sequence elements lie in the interval $[a - \frac{1}{n}, a + \frac{1}{n}]$. ☐ *yes* ☐ *no*

(4) For every $n \in \mathbb{N}$, there are sequence elements that do not lie in the interval $[a - \frac{1}{n}, a + \frac{1}{n}]$. ☐ *yes* ☐ *no*

Figure 3. Task 6 in the RACI- Test

The totality of items in comparison to standard textbooks (RQ 2)

Based on our comparison with the textbooks, we reclassified our items, distinguishing "theorems" as those referenced in at least two of the three books, and "implicit theorems" otherwise. This classification was previously based on intuitive judgment informed by our experience with analysis lectures. The comparison led to the reclassification of four items: three previously identified as "theorems" (Q24, Q25, Q27) were absent from all books and were reclassified accordingly, while Q105, initially an "implicit theorem," appeared in all three and is now categorized as a "theorem." Additionally, we identified the following topics and theorems present in all three textbooks but absent from our test: infinite limits, uniqueness of limits, convergence of quotient sequences, convergence of monotonically decreasing and bounded-below sequences (our list only covered monotonic increasing and bounded-above sequences (Q22)), Heron's algorithm, and *liminf* and *limsup*. We consider not including every possible topic a natural aspect of the RACI test rather than a limitation. Overall, most typical topics are included, with explicit and implicit references to key theorems.

Representation of conceptual relations (RQ3) - exemplified by task 6

As an analytical step, we conceived each knowledge element as a relationship between two concepts. We will illustrate this below with the knowledge elements in the solution of Task 6. There we had two definitions Q2, Q3, three theorems Q21, Q26, Q 105. In the RACI items on the topic of sequences, we found the following kinds of relationships between concepts:

- *concept definition:* A term is connected with a defining condition.
- *concept inclusion* and *concept non-inclusion*: Two concepts are (or are not, resp.) connected by an "is a" relation.
- *concept equivalence:* Two concepts are connected via the relation "is, equivalently, a".

- *concept membership:* A concrete object belongs to a concept.

Knowledge Element	Concept 1	Link	Concept 2	Link type
Q2	A sequence converging to a	is (by def.) a	sequence with for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\lvert a_n - a \rvert < \varepsilon$ for all $n \geq N$	Concept definition
Q3	A bounded sequence	is (by def.) a	sequence where there is a finite interval [c, d] in which all sequence elements lie: $c \leq a_n \leq d\ \forall n \in \mathbb{N}$.	Concept definition
Q21	A convergent sequence	is (in particular) a	bounded sequence	Concept inclusion
Q26	a constant sequence $(a_n) = (a)$	is (in particular) a	sequence converging to a	Concept inclusion
Q 105	A sequence converging to a	is, equivalently, a	sequence in which, for every $\varepsilon > 0$, all but at most a finite number of sequence elements lie in the interval $[a - \varepsilon, a + \varepsilon]$.	Concept equivalence

Table 3: The conceptual relations table for the knowledge elements obtained from the analysis of model solutions for Task 6

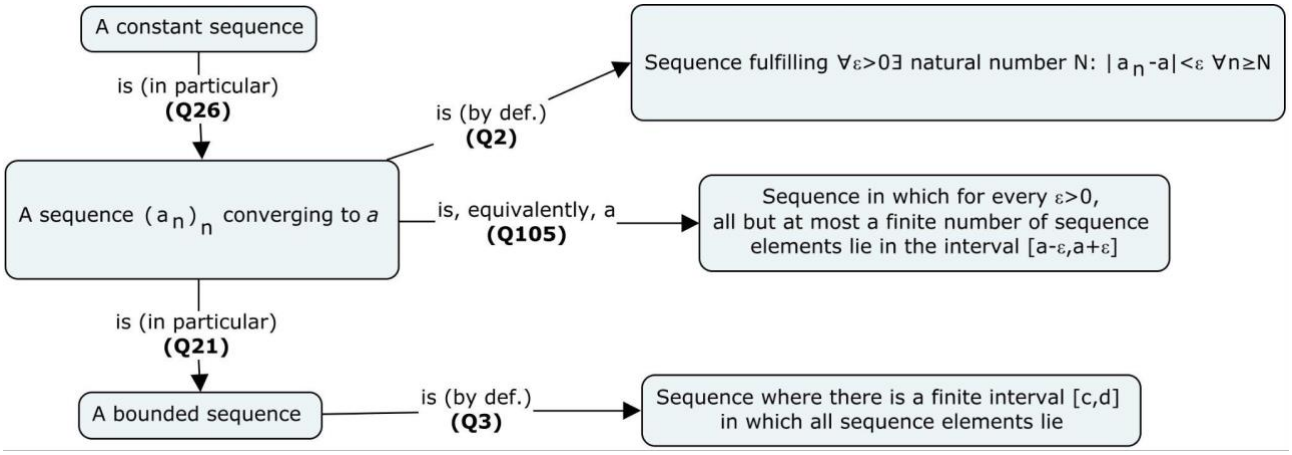


Figure 4: The concept map obtained through analysis of the model solution of Task 6

Figure 4 shows the concept map built from the conceptual relations table. At the surface level, the concept map is a graphical representation of the information encoded in the conceptual relations table. On a theoretical level, it exhibits the knowledge underlying our model solutions as a "web of knowledge". It is in this way that we interpret Hiebert and Lefevre's characterization of conceptual knowledge as a "web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (Hiebert & Lefevre, 1986, p. 3).

DISCUSSION AND OUTLOOK

In this paper, we propose a methodology for a conceptual analysis of concept tests for proof-oriented mathematics courses. We presented in detail the analysis of the RACI test using this approach, which also enabled comparison with the knowledge elements in standard textbooks. In contrast to the recent literature on Calculus Concept Inventory Tests, for instance, Lee et al. (2025), we base our test on different theoretical foundations regarding conceptual knowledge relevant to proof-oriented Real Analysis courses. A detailed conceptual comparison of our test to the most recent RCCI might be helpful for the future. Our aim in this paper was to highlight the different theoretical bases of our tests, where theorems, definitions, and formal properties of examples play an essential role as compared to studies that focus on bridging intuitive knowledge of convergence with more formal knowledge of convergence in Real Analysis courses (Roh & Lee, 2017). Another approach for developing a test for conceptual understanding of the convergence of sequences is based on the German mental model (Grundvorstellungen) discourse (Umgelter & Geissler, 2024). We made the underlying nature of conceptual knowledge explicit and showed how a conceptual analysis of items from the perspective of knowledge elements can show what the test is supposed to test and what it is not. We hypothesize that this method is suitable for tests focusing on conceptual knowledge for proof-oriented courses, regardless of the specific topics. For future work, it would be valuable to relate and compare the different theoretical backgrounds of the three quoted papers. This would require a similar conceptual analysis of items, as suggested in this paper. We suggest that the method of conceptual analysis introduced in our paper might serve two purposes: First, it can help in implementing and validating tests, and second, students and lecturers might benefit from explicating and visualizing conceptual relationships in their teaching and learning.

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Investigating the second-order differential equation with engineering students

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This study reports on the design and implementation of an inquiry-based mathematics lesson for first-year university students of electrical engineering. The lesson builds on earlier high-school teaching experience from the educational research project MERIA. Students were presented with a challenge in a realistic context and encouraged to investigate it independently using prior mathematical knowledge, aiming to construct a solution to an unfamiliar problem. Student strategies and ideas were documented during the lesson, and teacher guidance was used to structure the investigative process without diminishing student ownership. After the class, students' perceptions of this learning approach were collected, and their behaviour in subsequent classes was observed. The findings indicate that students perceive mathematics as more meaningful and are less anxious to engage with unfamiliar mathematical problems.

Keywords: ordinary differential equations, conceptual understanding, inquiry-based mathematics teaching, realistic mathematics education, case study.

BACKGROUND AND RATIONALE

In recent years, mathematics education research has emphasized teaching approaches in which students construct their own solution strategies. Research in mathematics education developed methods and ideas that are now called Inquiry-Based Mathematics Teaching (IBMT). Two of the more prominent methods are Realistic Mathematics Education – RME (Freudenthal, 1973, 1991) and Theory of Didactical Situations – TDS (Brousseau, 1997). In our design, we pursued both concepts, but this investigation study is grounded more in the principles of RME.

RME was developed in the Netherlands and is actively researched at the Freudenthal Institute in Utrecht. Freudenthal noted that the usual way of teaching is opposite to the human learning experience. The students are given the result and try to understand its meaning. In this way, motivation and context are taken away from the teaching. Students learn for the exams and find no significance in the content. He argued that mathematics should be viewed as a human activity and that students should learn by constructing knowledge through meaningful situations.

Within the RME framework, learning begins with meaningful contextual problems that allow students to explore mathematical ideas and gradually develop formal concepts. The students learn while they explore how mathematical thought objects may help to organize and structure real phenomena. This process is known as *mathematization*. It emphasizes the importance of the process that happens between getting an idea and writing a finalized theorem. Mathematization is happening in two forms: horizontal

mathematization, in which students translate a realistic situation into a mathematical model, and vertical mathematization, in which the model is progressively generalized and formalized (Treffers, 1987). Guided reinvention plays a central role in this process, as students are supported in reconstructing mathematical concepts using models that evolve from informal to more formal and more general representations (Gravemeijer, 1994). More recent developments of RME principles are discussed in (Van den Heuvel-Panhuizen, 2020a, 2020b, 2020c).

The teacher's role in RME-based lessons is to find contexts and develop tasks that invite students to reinvent the intended mathematical concepts, to facilitate and coordinate the knowledge reinvention, to guide the students in their investigation, leaving enough room and time for independent investigative work of the students, to coordinate discussions, and to combine individual findings into a collective knowledge. At the end of the lesson, results are organized to form an institutionalized, conventional form of mathematical knowledge. In this role, the teacher acts primarily as a learning mentor and facilitator of learning rather than a knowledge presenter.

Previous research highlights the importance of students' own realizations that are the outcome of investigation (Artigue, 2012, 2013, 2020). These findings motivate a change of teaching styles, classroom instructions, and task design. There is a growing community of mathematicians who develop and test investigative lesson content. As part of the MERIA project, we designed and implemented RME and TDS inspired materials for high-school classes. Studies arising from this work show that both principles can be successfully merged, cultivating ownership and meaning (Bos, 2019, 2020; Jessen, 2023).

At the university level, mathematics is formalised to a very high extent. Historically, mathematics was taught from axioms and definitions to lemmas and theorems. This tradition enables precise communication, but can limit opportunities for exploration and conceptual engagement, especially for non-mathematicians. There is therefore a need to design university-level tasks that remain demanding while being accessible to university students, supportive of reasoning, generalization, and multiple solution approaches (Canogullari, 2026; Watson, 2015).

In the specific context of ordinary differential equations (ODEs), the traditional methods of instruction are mostly algebraic or analytical procedural techniques. Students learn to classify equations and apply standard solution algorithms, yet still struggle with the interpretation of solutions and their connection to real-life phenomena. Recent reviews of engineering mathematics education emphasize the need for curriculum redesign with relevant didactic tools and authentic engineering problems (Pepin, 2021). Newer methodologies are based on information and communication technology (ICT), active learning approaches, interdisciplinary task designs, and inquiry-based methodologies (Lozada, 2021; Biehler, 2024; Branchetti, 2025, 2026).

Some attempts have been made to connect the traditional theory of differential equations to realistic contexts and to use software modelling tools for teaching (Rodríguez, 2022; Vargas, 2023).

Building on the previous work, the present study explores the feasibility and impact of an RME-based investigation-oriented lesson on second-order linear ordinary differential equations for first-year engineering students.

METHOD

The intention of this study was to design and implement an investigation-oriented university level mathematics lesson aimed at helping students develop a deeper understanding of ODEs and increasing the motivation for learning. We developed instructional materials for an inquiry-based classroom session, based on the principles of RME.

The selected topics had to satisfy the following criteria: they needed to be part of the curriculum of the subject we taught, connected to a real-life context or students' prior experience, and the lesson execution should not interfere with the standard semester time plan. Two topics were selected: trigonometric Fourier series and homogeneous second-order linear ordinary differential equations with constant coefficients. In this article, we focus on the latter.

The lesson materials were designed so that the students are active for most of the class time. Tasks were sequenced to gradually increase complexity while still maintaining real-world context. In addition to student handouts, a teacher lesson plan was developed, including expected students' actions, anticipated difficulties, and timeline guidelines. All materials are publicly available (Praprotnik, 2026a, 2026b).

The design was based on our experience with traditional teaching of ODEs, where a frequent misconception is the students' (not) understanding of what constitutes a solution. More precisely, students expect that the solution to a differential equation is a number, as was the case in all equations in prior mathematics classes. The lesson was structured to support the notion of a solution as a function throughout the student activity without emphasizing or explicitly stating this idea to the students.

A further design goal was to identify a real-life context that students would recognize and associate with a specific class of functions, in accordance with the RME framework. The chosen context was the car suspension system that the students are familiar with through everyday experience. In particular, bouncing cars in older models provide an intuitive association with oscillations and damped motion.

Our research questions are:

- Can an RME-based lesson plan be created such that university engineering students find its context interesting and meaningful?
- Does the lesson context invite mathematization?
- Does the investigation-oriented lesson increase the motivation of students?

— Do the students' strategies show a better conceptual understanding of the concept of the solution as a function?

CLASS SCENARIO AND EXECUTION

The investigative lesson was implemented in a first-year undergraduate course at the Faculty of Electrical Engineering, University of Ljubljana. The students came from very different backgrounds and therefore had varied levels of prior knowledge, some students were also foreign and had minor difficulties with language proficiency.

We conducted the ODE lesson scenario in three separate classes of approximately 90 students in total. After each class, we refined the class scenario because we, as teachers, learned a lot from implementing each lesson and also from the students' responses.

The lesson began with a video of a bouncing car that is funny and includes uplifting music. It is intended to capture the students' attention and activate the experiential prior knowledge. The energy in the classroom changed immediately. The students became curious and excited. When they were asked about the mathematical form of the motion, the students associated bouncing with sine and cosine functions. The initial discussion identified damping as a key factor in a suspension system and provided a short physics background on damped oscillations that explains the differential equation model and the meaning of the involved constants.

Students worked in small groups of three to four during the investigative phases, followed by a whole-class discussion in which different groups would compare strategies, refine their solutions, and agree upon formalizing a solution before continuing to a more difficult problem.

The first investigation phase is to solve an undamped system. In this phase the students are mathematizing. The teacher observes and notes the students' thinking process. It is imperative that the teacher does not interfere with the students in reinventing knowledge during the investigative phases of the class.

The students were at first a little lost and tried to understand what they needed to do. It was clear to us that they were not accustomed to mathematizing as they struggled with how to begin. When they grasped the concept that the solution is a function, progress was easier. This realization was students' group work and not prompted or pointed out by the teacher. The students clearly understood that the solution is a function they insert into the given differential equation, as we can see from Figure 1.

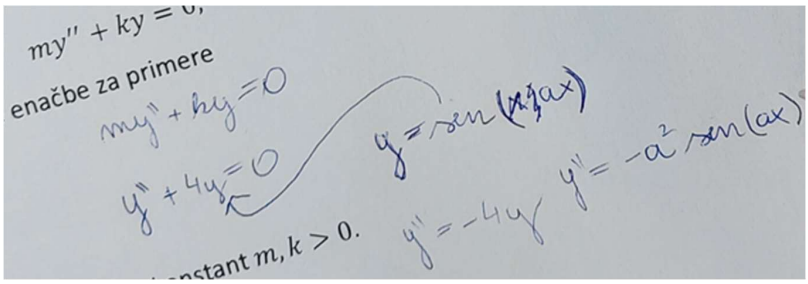

Figure 1: Students understand that the solution to the differential equation is a function.

At first the students tried to guess the factor in the solution function argument. When they started with the second case, they understood what needed to be done. We see in Figure 2 that they used corrections in the first example. In the second example, they generalized, introduced parameters, and solved the problem without difficulties.

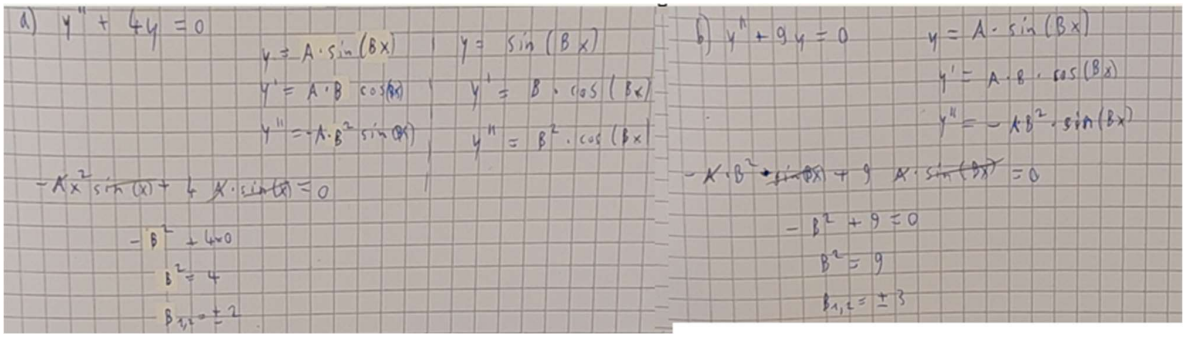

Figure 2: The students find a general way to solve simple differential equations.

All student groups got a sine or a cosine solution at the end of the first investigative phase. During the discussion they agreed that both solutions are good. We pointed out that there are always two different solutions to a second-order ordinary differential equation and that their linear combination is also a solution.

In the second investigation phase, students must solve a strongly damped system. The students searched for two solutions and introduced parameters as they learned in the first phase. From experience they knew they were searching for a function which vanishes with time. Some groups of students wanted to choose a rational function. They quickly encountered difficulties and discussed other options, at the end choosing an exponential function. They chose an exponential function with a negative exponent and inserted it into the equation (Figure 3, left). Some groups had difficulties deriving the solution from this. Other groups computed the zeros of the characteristic quadratic polynomial, depending on the parameter, and formed a general solution (Figure 3, right).

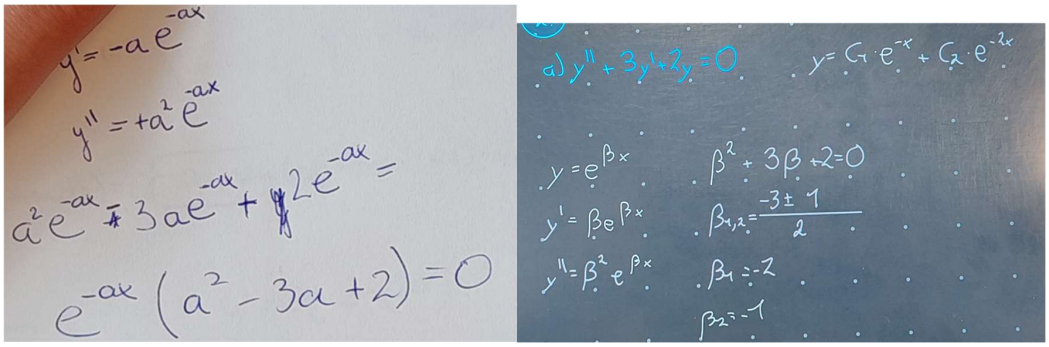

Figure 3: Solution that vanishes leads to an equation with a quadratic polynomial (left), general solution (right).

In the following discussion, we first called the groups with the least progress to present their findings, so that each group added something to the collective knowledge. During

the discussion, the students hypothesized that there exists a quadratic polynomial that is essential for constructing the solutions of the second-order ordinary differential equations. We pointed out that the characteristic polynomial is key also in the first case, the difference is in the form of its zeros. Some students then asked what happens when the zeros of the polynomial are of general form and they noted that the unexplored cases are when the polynomial has a double real zero or general complex zeros.

The more advanced students came to this realization beforehand on their own and were given additional instructions during the second phase to investigate what happens in the general case. They were very excited when they got a solution that explained their experience of suspension, i.e., there is oscillation in which the amplitude is decreasing with time. We were also very proud when some students arrived at this solution on their own with almost no help from us (Figure 4).

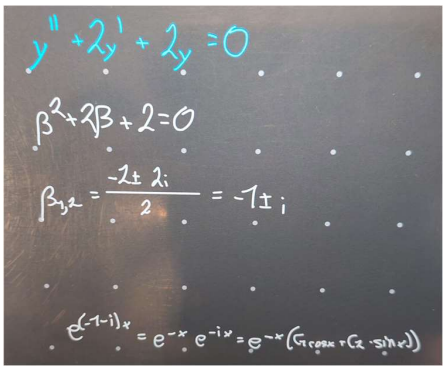

Figure 4: Arriving at the general solution of a second-order ODE.

The teacher can then decide to continue the class with a third phase of students' investigation, proposing a general damped system which is the hardest to tackle. The decision is made regarding the students' level of engagement and the remaining time. We suggest posing the general problem individually to the students/groups that finish with the second problem faster than most and try to find a solution for the general case.

At the conclusion of the lesson, students' collective findings were organized and written in a formalized, institutionalized way, consistent with the conventional presentation of second-order linear ordinary differential equations.

Throughout the lesson, the teachers' role is more of a discussion moderator and an observer. The majority of the lesson time is allotted for independent students' group work where the teacher does not interfere with the process and lets the students solve the problems; even if (or more importantly also when) the student direction is not what they would expect, e.g., choosing a rational function as a possible solution. The teacher intervenes only in case the students become stuck for a longer period of time and stop inventing. The intervention should be very short and should be made only to the group in question.

FINDINGS

Data was collected through classroom observations and a post-lesson student survey. Students worked in small groups and their discussions, strategies, and difficulties were informally documented by the teachers.

The survey consisted of Likert-scale items ranging from 1 (low agreement) to 5 (strong agreement) and included open-ended questions about qualitative feedback and their impressions. The percentage of responses rated 4 or 5 indicated that:

— the class encouraged independent thinking (90 %),
— students were more curious and motivated than in a traditional class (70 %),
— the lesson encouraged them to test their own ideas (75 %), and
— they found mathematics more relevant for an engineer (80 %).

Qualitative observations supported these results.

The students consistently drew on real-life experience to anticipate the form of solutions. More precisely, they distinguished that the undamped system will have oscillations, pointing to a sine/cosine solution, and the overdamped system will have a solution that vanishes with time. During the group discussions, students from different groups collaborated so that they cleared the misconceptions collectively.

The students appreciated the openness of the investigative format and that there is not one *correct* way to solve the problem. They were free to explore in a way that they saw fit and liked that different approaches were all accepted as possible paths to arrive at the final solution. They noted that in their experience with mathematics that was not the case. All students managed some level of mathematization with various degrees of generalizing. We noted that the more advanced students used, of their own volition, computer software after the lesson to graph and further explore the solution for the general case. It was very rewarding for us to see the students engage in mathematics even during breaks and lunch hours, talking excitedly about what is usually seen as a boring textbook problem they need to learn.

The students commented that the class was *interesting* and that they had a sense of achievement when they managed to independently solve the problem. Some students commented that the class was more fun. For most students the knowledge they gained was more relevant than what they usually learnt in a traditional way. They commented that they would like to engage in more such classes.

They also got to know each other better because they had to collaborate with colleagues that they usually had little contact with. We noticed that they were a more homogeneous group after this class and that some groups were studying for the final exams together, helping each other after their regular classes.

Finally, the students obviously grasped the concept of solution as a function. Our observation in the subsequent traditional lessons was that it was easier for them to guess at the form and construct particular solutions of non-homogeneous equations.

CONCLUSIONS

The investigation-oriented lesson on ODEs was well received by students, suggesting that the proposed lesson plan is feasible in a university engineering context. Additionally, the approach proved to be a valuable learning experience for the teachers. We learned a lot about the way students think and how they interact with others.

Initially, many students were uncertain what to do when they were asked to independently work on a problem for the first time because this way of learning was completely unfamiliar. They had no idea how to begin and needed encouragement to try anything they thought could help. A lot of times they were hesitant to make mathematical statements in risk of making a mistake. This was shaped by prior experience of mathematics lessons. After a while such fears subsided and the students became more confident. We saw later in traditional classes that the students attempted to solve exercises where they did not see the whole procedure of solving the exercise beforehand. They gained confidence in the process and in solving parts that would connect later. For a large majority of students, their motivation increased.

Based on these observations, we recommend implementing at least two investigative lessons within a course to allow the students to start trusting the process of trial and error in the inquiry process. This was a second inquiry-oriented lesson in our course and it was implemented more smoothly and with greater student engagement than the first one.

Future research should investigate how the RME-based classes influence knowledge retention and transferability of gained knowledge to other disciplines.

Finally, inquiry-based instruction requires specific pedagogical skills and preparation, therefore such lessons may serve as a form of teacher professional development tool. An important area for further development is also adapting assessment practices so that they include an investigative component.

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Applying the Mathematical Idea Analysis Technique to Interpret Pre-service Teachers' Root Metaphors of Infinity

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This paper examines undergraduate students' conceptual metaphors of infinity while they completed the standard sequence of collegiate calculus coursework. We leveraged conceptual metaphor theory along with the mathematical idea analysis technique as a tool to investigate the metaphorical arcs of N=188 college students across multiple levels (precalculus, single variable calculus, multivariable calculus) in two open-ended prompts: 1) an open-ended concept-definition prompt and 2) a concept-image drawing prompt. Using qualitative analysis based on the Delphi method among three independent expert raters, we established a framework for student metaphors as recessive singular to dominant dual conceiving at four levels. Our analysis identified three overarching root metaphors for infinity: "something," "number," and "endless." The results reveal that the traditional calculus sequence seems to perpetuate a singular, process-oriented metaphor. The proportion of students exhibiting duality conception dropped significantly from 41% in Pre-Calculus to 5% in Calculus III, as would be expected, singularity conception increased accordingly from 59% to 95%. Chi-square results revealed the trend to be significant ($\chi^2=9.686$, $df=3$, $p<.05$). These results suggest that current practices might, in effect, be limited instead of opening learning opportunities to develop a cohesive conception of infinity.

Keywords: Calculus sequence, pre-service teachers, process-object duality of infinity, mixed methods research.

INTRODUCTION: RESEARCH FOCUS

The concept of infinity is one of the most important and yet challenging links in the college mathematics sequence for undergraduate science, technology, engineering, and mathematics (STEM) students. Studies have confirmed that most students have extensive difficulty with the notions of infinity (Gray and Tall, 1994; Monaghan, 2001; Sfard, 1991).

Most of the prior research on college students' concept of infinity focused on the development of the concept within a particular course (Falk, 2009; Fischbein, 2001; Kolar & Hodnik Čadež, 2012; Singer & Voica, 2008). This study attempts to contribute to the field by closely examining the development of college students' metaphors of infinity across the entire sequence of courses, from Pre-Calculus to Calculus III.

Conceptual metaphor (Lakoff & Johnson, 1980) could be used as a way to examine students' understanding of infinity. Glucksberg (2008) posits that metaphor is a categorical assertion, whereby the attributes of one concept are used to describe the attributes of another. Studying the development of college students' conceptual metaphors is critically important to students' success in college mathematics. Therefore, the study focused on the following *research questions*:

1. How do college students externalize conceptual metaphors of infinity in each course in the Calculus sequence?
2. How does the Calculus coursework sequence contribute to the development of college students' metaphors of infinity through the lens of duality conception?

THEORETICAL PERSPECTIVES

The study is grounded on the conceptual metaphor and blending theories (Lakoff & Johnson, 1980; Fauconnier & Turner, 2002). The study also uses the mathematical idea analysis technique (Lakoff & Nunez, 2000) as a method to examine college students' conceptual metaphors of infinity.

Nunez (2005) proposes a basic mapping of infinity (BMI) as an approach to address the most challenging aspect of the metaphor of infinity - its duality. Duality conception is the ability to conceive infinity as a process (potential infinity) as well as an object (actual infinity). Nunez (2005) claims that the BMI, as "a double-scope conceptual blend" (p. 1729), could be used to address duality through a new inferential structure integrating the finite (completed iterative process with end) and infinite (endless iterative process with no end) into blended space (an endless process with an end - actual infinity).

Aside from the BMI (Nunez, 2005), several other theories have addressed process-object duality as a model of infinity concept development: procept theory (Gray & Tall, 1994), APOS theory (Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros & Weller, 2014), and reification theory (Sfard, 1991). These theories were built on Piagetian ideas of genetic epistemology, reflective abstraction, and encapsulation to conceptualize the construct of the process–object duality (Sfard, 1991; Gray and Tall, 1994).

METHODOLOGY

Participants

In order to allow for a variety of conceptions from different stages of students' learning trajectory, we selected N=188 college students enrolled in the Calculus sequence. We used cluster random sampling (Onwuegbuzie & Collins, 2007) to select participants who were willing to participate in the study, 69 of whom were

enrolled in Pre-Calculus, 74 in Calculus I, 26 in Calculus II, and 19 in Calculus III. The majority (54%) of the participants were male, and 44% were female. Three of the participants (2%) did not disclose their gender.

Data Collection

Data were collected using two open-ended prompts: (1) concept-definition prompt: when you think of infinity, what comes to your mind; (2) concept-image prompt: draw infinity in the space provided. These prompts were purposefully selected to engage students in externalization of their conceptual metaphors because they relate to an individual's cognitive structure associated with the concept and have the potential to reveal associated misconceptions of infinity that college students may hold (Tall & Vinner, 1981).

Data Analysis

The open-ended responses were coded by three independent expert raters using metaphor analysis and drawing methodology techniques (Cameron & Maslen, 2010; Theron, Mitchell, Smith, & Stuart, 2011). NVivo 10 software was utilized in examining students' responses to the prompts provided (Table 1).

Data source	Question	Data analysis
Concept-definition prompt	When you think of infinity, what comes to your mind?	Metaphor analysis
Concept-image prompt	Draw infinity in the space provided.	Visual metaphor analysis

Table 1: Data sources and data analysis

Additionally, the Delphi method was employed during the process of coding because "[t]he Delphi method is well suited as a research instrument when there is incomplete knowledge about a problem or phenomenon" (Skulmoski, Hartman & Krahn, 2007, p. 1). The number of Delphi iterations can vary from three to five (Skulmoski, Hartman & Krahn, 2007). Throughout the Delphi iterations, consensus was reached among three raters, with a Fleiss's Kappa of *.7165,* which is considered strong agreement (Landis & Koch, 1977).

We employed the mathematical idea analysis technique (Lakoff & Nunez, 2000) to operationalize evolved sub-categories of *singularity and duality,* along with *dominance and recessiveness* (Figure 1) in the categorization of students' conceptual metaphors of duality based on a double-scope conceptual blend approach (Nunez, 2005).

Singular	Dual
Recessive	Dominant

Figure 1: Operationalization of the process-object duality framework.

Using the operationalization frame above, students' metaphors were categorized into the following levels:

- Level 1 - recessive singular. In this case, only one view is displayed by a student, which could either be the recessive process view or the recessive object view with a weak conviction ('p' or 'o').
- Level 2 - dominant singular. In this case, students tend to display one view in which the intensity of conviction is strong ('O' or 'P').
- Level 3 - recessive dual. This level represents a blended space where students demonstrate process and object views that are both recessive or one of the views is recessive (i.e., not strong or convincing). We believe this case to be an indication of emergent duality conception ('Po' or 'pO' or 'po').
- Level 4 - dominant dual. This blended space indicated the strong and convincing duality of students' process-object view ('P' and 'O' denoted as 'PO').

Finally, the non-parametric technique (Chi-square statistic) was selected to measure the variance between independent groups of participants across the Calculus sequence. The selection of the Chi-square was also based on the categorical nature of the data used for determining students' levels of the duality conception.

FINDINGS

In this section, we present the main results of the study by prompt. Results of students' responses to Prompt 1 are displayed in Table 2.

		Pre-Cal	Cal 1	Cal 2	Cal 3	Total
Conception	Levels	N/%	N/%	N/%	N/%	N/%
Not determinable	ND	3/5	4/5	2/8	0/0	9/5
Singularity Conception	Level 1	35/51	47/63	12/48	13/69	107/57
	Level 2	28/40	16/22	4/14	6/31	54/29
Duality Conception	Level 3	2/3	7/10	7/26	0/0	16/8
	Level 4	1/1	0/0	1/4	0/0	2/1

Table 2: Levels of students' duality conception in response to the concept-definition prompt

The majority (86%) of the responses were categorized as singularity conception at Levels 1 and 2, while 9% were categorized as duality conception at Levels 3 and 4. The remaining 5% of students' responses could not be determined and were categorized as ND because either the response was blank or there was insufficient

information to determine the view. More than half of the participants (58%) in Level 1 used process metaphors to describe infinity, while the remaining participants (42%) used object metaphors.

The ability to conceive of infinity as "an endless process with an end" (Nunez, 2005) such as a "*number*", an "*amount*", a "*set*", a "*cardinality*", "*something*", a "*limit*", etc. were considered as object metaphors (actual infinity) and the ability to conceive of infinity as "process with no end" (Nunez, 2005) such as "*continuing*", "*going on and on*", "*forever*", "*never ending", "over and over",* etc. were considered as process metaphors of infinity (potential infinity). Selected examples of students' conceptual metaphors are illustrated below.

Metaphor 1 - infinity as "something".

Analysis using NVivo (Figure 2) showed that "*something*" appeared most often (58 times) in students' responses to Prompt 1 across the Calculus sequence. Participants predominantly used a recessive dual metaphor to describe infinity (e.g., "it is something that continues forever," where 'something' indicates an object, whereas "continues forever" – a process).

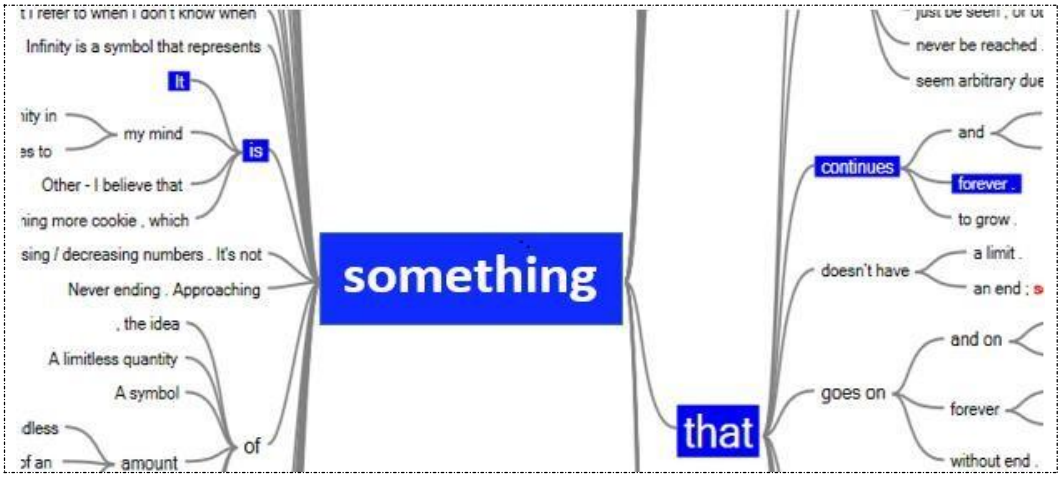

Figure 2: Example of students' use of the metaphor "something" in response to Prompt 1.

Metaphor 2 - infinity as a "number".

Another most frequent metaphor in students' definition of infinity is "*number,*" and it appeared 35 times. Based on the experts' rating, the student's ability to conceive of infinity as "*a large number sequence*" is considered to be an indication of the object conception; however, the phrase "*that does not end*" highlights the metaphor of the process (Figure 3), reflecting the notion of a "blended space" (Nunez, 2005).

Metaphor 3 - infinity as "endless".

The NVivo-generated word tree in Figure 4 establishes how students used the metaphor "*endless,*" which was one of the most frequent words used to define infinity, and it appeared 17 times.

The second prompt asked students, "Draw infinity in the space provided. Explain your drawing." Table 3 aggregates students' visual metaphors of infinity, including graphs, circles, arrows, number lines, and blank spaces, to name a few. Some students used multiple ways to represent infinity. It was not surprising that a majority of the students (68%) drew an infinity symbol (∞) with predominantly (53%) process metaphor (e.g., *"It is a symbol that has no starting point and no ending point")*.

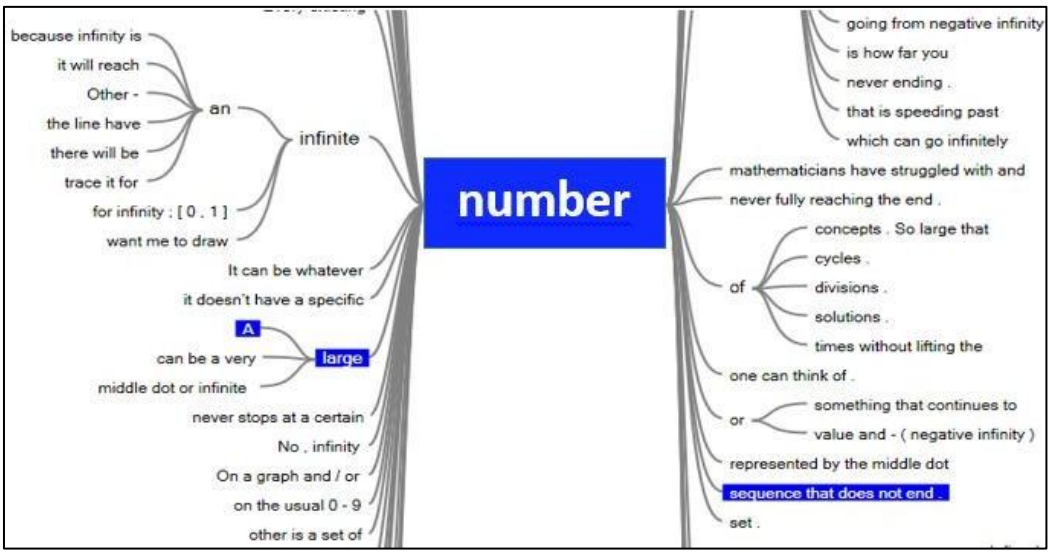

Figure 3: Example of students' use of the metaphor "number" in response to Prompt 1.

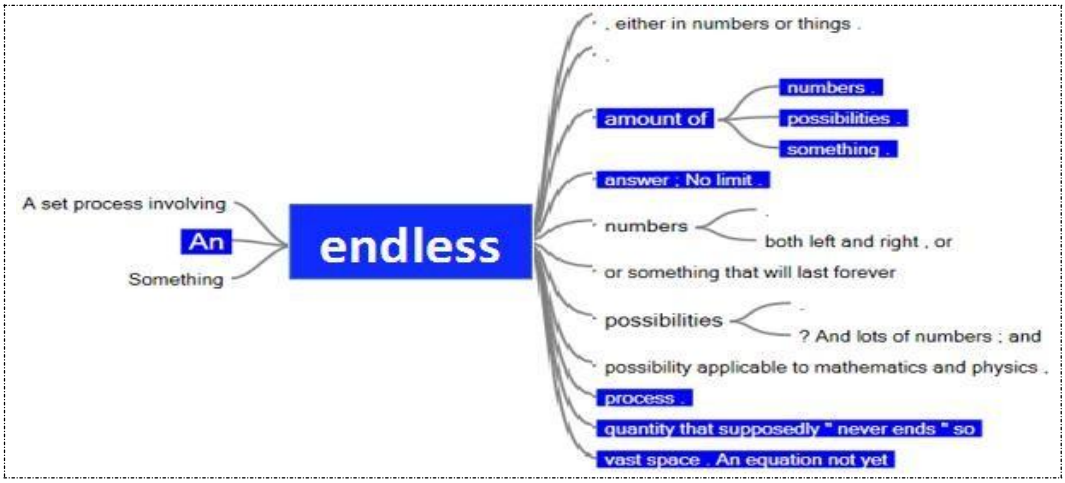

Figure 4: Examples of students' use of the metaphor "endless" in response to Prompt 1.

Experts' rating of the responses to Prompt 2 (Table 4) indicates that 5% of responses were categorized as duality conception at Levels 3 and 4. The most widely held conception of the draw infinity task is the singularity conception (63%), with a predominantly process view of infinity (71% of responses within the singularity conception category). The conception of 32% of participants could not be determined (ND) because either the participants did not explain their drawings or the explanation was not sufficient to determine the participant's view. Considerably fewer students (10%) drew the infinity sign as an indication of an object view. Below (figure 5), we present an example of a student using an object metaphor - "*an infinite number or value*". A few students (6%) used blank space to represent infinity.

Representation	N/%
Symbol of infinity	128/68
Graph	26/14
Number line	19/10
Blank space	11/6
Arrow	8/4
Circle	6/3
Numbers	2/1
Ellipsis	2 /1

Table 3: Students' metaphoric representations of infinity in response to Prompt 2.

Conception	Views	Pre-Cal	Cal 1	Cal 2	Cal 3	Total
		N/%	N/%	N/%	N/%	N/%
Not determinable	ND	18/26	26/35	11/41	4/21	59/31
Singularity conception	'p' or 'P'	34/50	27/36	12/48	13/68	86/46
	'o' or 'O'	7/10	9/12	0/0	0/0	16/8.5
Duality conception	'Po' or 'pO' or 'po'	8/11	11/15	3/11	2/11	24/13
	'PO'	2/3	1/2	0/0	0/0	3/1.5

Table 4: Results of students' responses to Prompt 2

Finally, we report aggregated findings across the Calculus coursework sequence by conceptions and levels in a contingency table. Table 4 clearly illustrates that every next course in the Calculus sequence has more students possessing a singularity

conception: it gradually increases from Pre-Calculus (41 out of 69 or 59%) to Calculus 1 (45 out of 74 or 61%) and further to Calculus 2 (19 out of 26 or 73%) and Calculus 3 (18 out of 19 or 95%). Consequently, the percentage of students having duality conception decreases from 41% (28 out of 69) in Pre-Calculus and 39% (29 out of 74) in Calculus 1 to 27% (7 out of 26) in Calculus 2 and, finally, 5% (1 out of 19) in Calculus 3. Chi-square statistics were applied to analyse the data presented in the contingency table 5. Chi-square result ($\chi^2 = 9.686$, $df = 3$, $p < .05$) confirmed that the variation in students' conceptions across the Calculus coursework sequence is significant.

Overall, this finding is an indicator of the trend observed throughout the current study that the traditional Calculus sequence promotes *a process-dominated metaphor* of infinity.

n	Pre-Cal	Cal 1	Cal 2	Cal 3	Total
Singularity	41	45	19	18	123
Duality	28	29	7	1	65
Total	69	74	26	19	188

Table 5: The 2x4 contingency table of college students' conceptual metaphors of infinity across the Calculus sequence.

DISCUSSION AND CONCLUSIONS

The current study contributes methodologically to the field of semiotics in mathematics education research by its ways of addressing the double-scope conceptual blend framework (Nunez, 2005) to address the metaphoric process-object nature of infinity (Arnon, et al., 2014; Gray & Tall, 1994; Sfard, 1991). This study proposed an operationalized framework that allowed for a granular level of examination and interpretation of students' conceptual metaphors of infinity through a *duality* lens. Furthermore, the proposed framework enabled us to elucidate a better strategy that could guide researchers in categorizing students' metaphors of infinity into different levels to assess students' spontaneous *singularity and/or duality* conception of infinity.

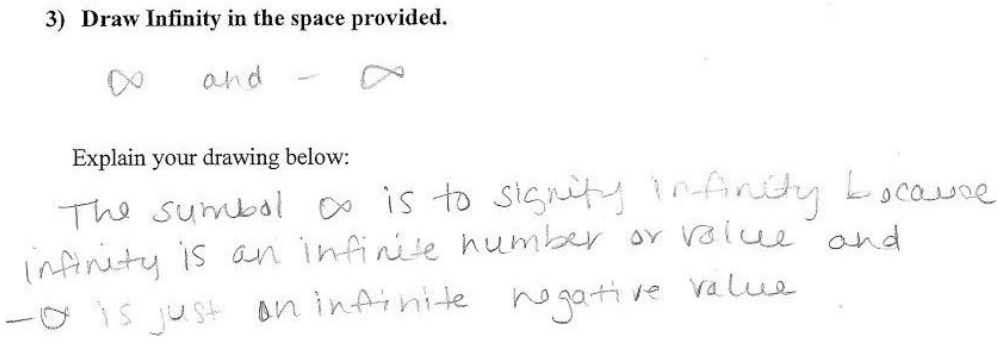

Figure 5: Example of a student's visual metaphor of infinity and its explanation.

This study examined the duality conception of the infinity of the students enrolled in different Calculus sequence courses. In the light of the presented outcomes of this study, the following major findings addressing the research questions could be suggested: a) students' conceptual metaphors of infinity across the Calculus coursework sequence is predominantly based on the process (potential infinity) perspective; b) students' conceptual metaphors of duality is not well developed across the Calculus sequence; c) moreover, the traditional Calculus coursework sequence promotes a singularity conception of infinity.

Among the major *limitations* of the study are the cluster sampling and an uneven number of participants representing different courses in the Calculus sequence. The result might have been different if the same sample sizes of students in the Calculus sequence were selected and if the students were studied from Pre-Calculus through Calculus III to determine if the coursework actually supports the development of the duality conception as students progress through the Calculus sequence.

Overall, the results of our study reveal that college students' experiences in the traditional Calculus coursework sequence are not supportive of the development of a duality conception of infinity. On the contrary, the study suggests that it strengthens students' singularity process-oriented perspective on infinity. Therefore, it is critically important to provide college students with relevant experiences to build the concept of duality, which will help them further to succeed in advanced mathematical courses. Furthermore, the results of this study could serve as a facilitating instrument to analyze cognitive obstacles in college students' understanding of the dual nature of infinity.

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Introducing Differential Equations in the First Year of University: a Case Study

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We are interested in the teaching of Differential Equations in the first year of university. More specifically, we compare a lecture, a video lesson, and a textbook in order to understand the possible introductory choices and infer information about how students conceptualize this notion. To study these examples of knowledge implementation, we establish the 'relief' on the concepts to be taught, which enables us to identify discursive proximities in these courses. The results show that the proximities anticipated in the relief are often absent from the studied discourses.

Keywords: differential equations, relief, discursive proximities, conceptualisation.

CONTEXT OF THE WORK

In this study, we examine the teaching practices of university lecturers. Although the effectiveness of lectures in improving student comprehension has been widely questioned (see Lew et al., 2016, for a review of the literature on this subject), they remain the predominant teaching method at university level (Paoletti et al., 2018). In this context, we study the instances when teachers present knowledge that constitutes "the course" (Bridoux et al., 2016). We refer to these as 'moments of knowledge presentation'.

These moments offer teachers opportunities, for example, to build on students' prior knowledge to introduce new knowledge or to familiarize them with (new) concepts using words, drawings, symbols, and so on. This work can then be generalized to formulate initial definitions. The course may also contain examples and prepare for subsequent activities (proofs, exercises, etc.). As there is little interaction with students during a lecture, the effectiveness of moments of knowledge presentation depends on the discourse used by the teacher to advance students' knowledge.

Our work focuses on the introduction of the definition of a differential equation (DE for short) and on the choices teachers make when constructing it. Similar work has been carried out on the concept of function limit (Bridoux & Grenier-Boley, 2024). These content choices are motivated by the fact that they are taught in the first year of university in many countries. To assess the discourse used when introducing DEs, we compare three types of courses: a lecture, a video lesson, and a textbook.

After describing our discourse analysis tools, we show how they enable us to develop our research question and associated methodology. Next, we present a comparison between the lecture, the video lesson and the textbook.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Our inscription in Activity Theory adapted to mathematics education (Vandebrouck, 2013) leads us to consider students' activities as an indicator of their learning (Bridoux et al., 2016). Following Vygotsky, we hypothesise that student learning is partly the result of teacher's choices regarding teaching scenarios, content and classroom activities (working methods, teacher assistance, etc.). Therefore, one way to advance students' knowledge is to keep the discourse as close as possible to their existing knowledge and experience, which we operationalise here using the concept of the Zone of Proximal Development (Vandebrouck, 2013). In this context, the effectiveness of lectures depends heavily on the teacher's discourse. For this reason, we study the discourse by focusing on particular connections called "discursive proximities" (Robert and Vandebrouck, 2014). Recall that (Bridoux & Grenier-Boley, 2024, p.792):

> Three types of proximities are distinguished. *Bottom-up proximities* lie between what students have already done and the introduction of a new object or property. In this case, the teacher's discourse therefore aims to move from the contextualized to the decontextualized by generalizing the particular case. *Top-down proximities* are situated between what has been explained and examples or exercises. The teacher can then explain how the particular case fits into the general case, moving from the decontextualized to the contextualized. Finally, *horizontal proximities* do not lead to any change between contextualized and decontextualized. They consist of reformulations, explanations of the links between concepts, comments on the structure of the course, etc.

To study the discourse of a teacher or a textbook, our reference is the 'relief' of the notion of differential equation, which consists of a cross-study of epistemological, didactic and curricular studies (Bridoux et al, 2016). This relief enables us to identify several specificities of the concept. We then determine *proximity opportunities* (those which could be identified *a priori* given the determination of the relief). These are compared to the *attempted proximities* (those which are effectively attempted in the three courses). In this regard, our research questions can be formulated as follows: Which specificities of the concept of DE determined in the relief are taken into account in the courses? What are the attempted proximities associated with these specificities?

RELIEF OF THE NOTION OF DIFFERENTIAL EQUATION

We take into account elements of the relief identified in the study by Bridoux and Grenier-Boley (2025). A DE of order *n* is a relationship between the real variable t, an unknown function $t \longrightarrow x(t)$ and its derivatives $x', x'', \dots, x^{(n)}$ at point t defined by $F(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0$ (Demailly, 2006). It should be noted that this formalism is difficult to introduce to students. This is also their first encounter with equations where the unknown is a function rather than a number, as in high school. However, numerous studies in mathematics education have shown the difficulties

university students have with the concept of "function" (Vandebrouck, 2011), and Rasmussen's work (2001) indicates that this impedes their understanding of DEs. DE also have many applications, for example in physics (e.g. radioactivity) and biology (e.g. population dynamics). Thus, DE are both an object whose properties can be studied by mathematicians and a modeling tool. Moreno Gordello (2006) explains that conceptualising DEs requires students to master different registers of semiotic representation (natural language, algebraic, graphical, in particular) in the sense of Duval (2006), the conversion between them, and different frameworks (geometry, physics, analysis, algebra) in the sense of Douady (1986). Nevertheless, Artigue and Rogalski (1990) argue that teaching DEs can quickly become overly algorithmic and operational, limiting it to the algebraic framework. Habre's work (2000) also shows that, for many students, solving a differential equation often involves only a purely algebraic and procedural activity with no connection to the geometric framework.

This non-exhaustive overview shows that, when introducing differential equations, teachers can draw on examples from different fields, particularly before defining them. Such an approach requires sufficient and available knowledge in the different fields to model the phenomena being studied. Teachers can then explain to students how the form of the particular differential equations related to the studied phenomena can be generalised, thereby introducing the formalism of the definition. This approach provides opportunities for bottom-up proximities, such as mentioning that the order of a DE can be any order and is not limited to orders 1 or 2, or explaining what the function F in the general definition corresponds to in specific cases. Examples can also be given after the definition to decontextualize it and adapt the formalism to specific cases. This would provide an opportunity for top-down proximities in the teacher's discourse, showing how this is a specific case that fits in the definition. Explanatory comments on the notion of solution, particularly emphasising that this object is a function, can give rise to horizontal proximities.

What follows is a supplement to the study conducted by Bridoux and Grenier-Boley (2025), in which we compare three independent courses: a lecture, a video lesson and a textbook. This deliberate choice opens up a range of possibilities for introducing the concept of DE. Our methodology consists of determining the relief of DE, and studying, on the basis of this relief, the discursive proximities that are attempted in each course and those that are missed.

COMPARATIVE ANALYSIS

The relief has shown the possibility of working in different frameworks to introduce DEs, and we have emphasised the diversity of registers associated with this approach. Our focus is therefore on the chosen approach to introducing DEs and how the definition is constructed. We also pay particular attention to the notion of solution.

The lecture in question is given to first-year university students in Belgium, studying mathematics and computer science. The video lesson comes from the Exo 7 website[1], and the chapter on differential equations comes from the Fluoresciences textbook (Heu & Teyssier, 2022).

The approach used to introduce the concept of DEs

Both the lecture and the video lesson use examples from physics to introduce DEs. In the lecture, the teacher studies the motion of a spring. The following excerpt is taken from the lecture. This is followed, in Figure 1, by the notes taken by a good student (these are what the teacher writes on the board, our translation):

> So, we are looking for... the way... so the change in position x as a function of time, and what does that mean? The position x will depend on time, and therefore, it will be a function of time. So, we are looking for a function that goes from $\mathbb{R}$ to $\mathbb{R}$, and which, at each time, will give me the corresponding position. You see, that gives us... the unknown in the equation is x there, but it's not a number, it's a function. And now we have to try to write the equation that this function satisfies. It (the spring) will exert a force that will be k times x, and we've stretched it to the right here, so we'll have a minus sign, since the force in this case will be directed to the left. If we do the opposite, if we compress the spring, the x will be negative, but now the spring will exert a force to expand again, so the minus x will be positive, and therefore the force will be in the other direction. If we have compressed the spring, . . . we'll draw a picture, given the look on your faces...

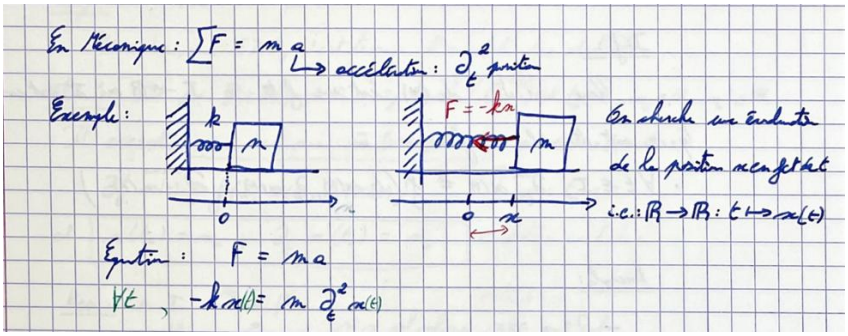

Figure 1: Lecture – a student's notes about the study of springs

The mathematical and physical details are given quickly and verbally. The student's notes show that the teacher's written notes on the board consist mostly of mathematical symbols and graphical representations, with little use of the natural language. The functional aspect of the unknown in the equation is not explained, nor is the dependence on time. However, our findings have revealed that the nature of objects that are solutions to a DE is a difficulty identified in research. The teacher does not elaborate on the transition from the equation $F = ma$ to the corresponding DE. Therefore, the transition from physics-related notation to mathematical notation is left to the students. As we had anticipated in the relief, we did not find any horizontal proximities here.

[1] https://www.youtube.com/watch?v=dkjXofPNMDo&list=PL024XGD7WCIENmP2GOx7B93ezN7JzIuTU&index=1

In the video lesson, the teacher begins by considering the example of free fall and the case of a parachutist (see Figure 2). He discusses the different forces involved in describing the motion (weight and friction) and provides formulas for calculating them. Using the fundamental principle of mechanics, he obtains the equation: $\frac{dv(t)}{dt} = g - fv(t)$. He then states that this is a DE and that 'it is not easy to find the appropriate function v'. He then explains that the objective of the chapter is to find $v(t)$. As in the lecture, the physics knowledge required to construct the equation is not reviewed and is assumed to be available. The teacher also uses physical notation and does not rewrite the equation using mathematical symbols (the function sought is denoted $v(t)$). As in the lecture, the teacher's discourse contains no horizontal proximities.

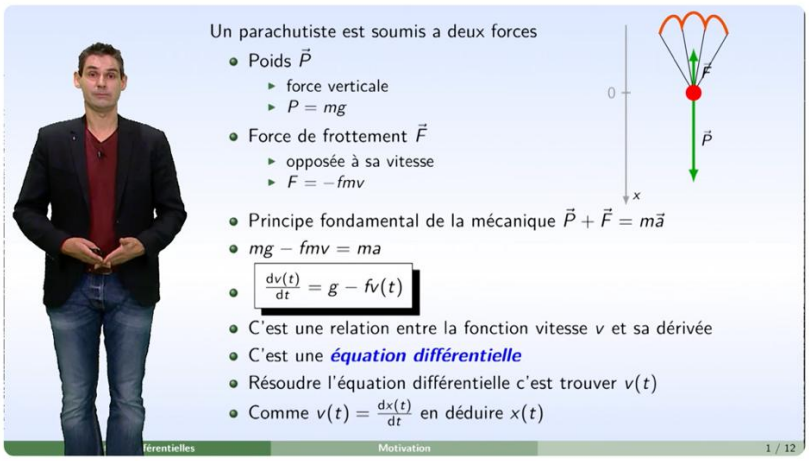

Figure 2: Video lesson – the fall of the parachutist

Rather than relying on other frameworks to introduce DEs, the textbook starts immediately with a mathematical presentation. After giving an intuitive characterisation of a DE, it deals with specific cases of first- and second-order equations. These aspects are detailed in the following section.

The construction of definitions of DEs and solutions

In the lecture, after discussing examples in physics, the teacher wants to motivate the need for a definition (our translation):

> We don't want to limit ourselves in terms of order. So I'm going to assign an arbitrary order. Earlier, I saw order one and order two, which occur less frequently than we go beyond order two, especially in classical mechanics... So what do we want here? There is an equation. I wrote the equality, and the right-hand side is not specifically prescribed, so it is simply a function that can depend on time as well as all the lower-order derivatives, i.e., the derivative of x , the second derivative of x , etc. up to the derivative m minus one of x. So it will be this kind of thing, which means that here you can think of an expression that can mix the derivatives as you wish.

The teacher mentions the possibility of generalising the order of the DEs encountered in the examples, and introduces the idea of a general function that depends on derivatives in order to derive the general form of a DE. We believe that this type of

comment is an attempt of a bottom-up proximity, but this point, which the teacher mentions briefly, remains once again oral and not written. Generalisation is therefore likely to be difficult for students. He then states the definition:

> An ODE for ordinary differential equation, as in my title, of order m , is simply the data. What do I need to do to give you that? Well, give me F. So F, what does it go from to? So here, I'm going to simplify things so as not to complicate the details of the domain, etc. I don't really want to talk about it, so I'm going to take $\mathbb{R}$, which is defined everywhere, so what is it going to do? (inaudible) time, and then it's going to accept derivatives, so how many derivatives is it going to accept? Each derivative will be a real number, and I have m, because I'm going from order 0 to order m minus 1, so I have m derivatives. But what should it return? It should return something compatible with the derivative m here, so a real number. As I said, you can see here that each of these things, a priori, is a real number, and so the function has just enough arguments to accept them all, and since the derivative of order m will also be a real number, it must return something compatible.

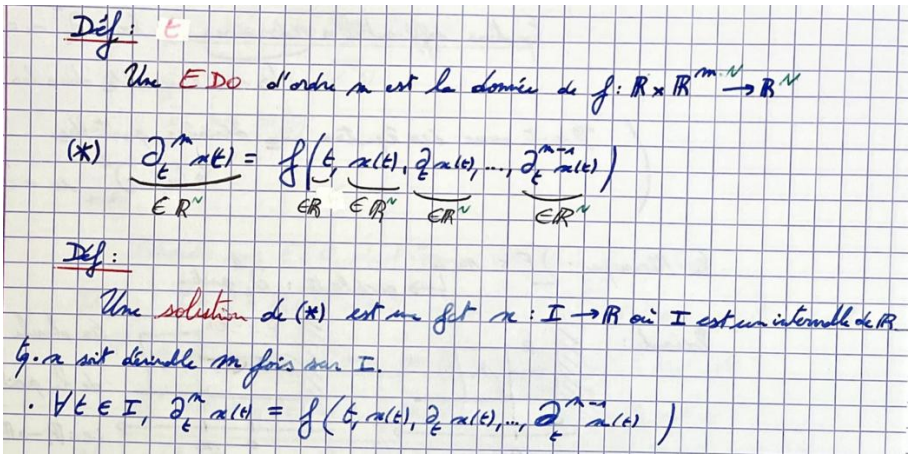

Figure 3: Lecture – definition of an ODE (student's notes)

Once again, the explanations of how this definition is constructed remain oral and are not written down. The teacher also fails to make the connection with the previous examples by showing how the equations encountered in the examples are special cases of ODEs. We therefore found no top-down proximity in this lecture. In the video lesson, the teacher first provides an intuitive characterisation of the concept of a differential equation. We transcribe what is written on the slide and read by the teacher:

> A differential equation is an equation:
>
> - whose unknown is a function (usually denoted by $y(x)$ or simply y)
> - in which some of the derivatives of the function appear (first derivative y', or higher-order derivatives y'', $y^{(3)}$, etc.).

The teacher does not make any connection with the example of the parachutist, nor with the notation used here. Consequently, there are no horizontal proximities in his discourse. He then gives examples of DEs, saying that they are easy to solve, such as $y' = \sin x$ or $y'' = y$. However, he does not explain why these are DEs by making connections with the previous characterisation. Instead, he specifies that "you need a

function $y(x)$ that satisfies each of the equations." He then asks the following question about the first equation: "For example, what function $y(x)$ has a derivative equal to sine of x?" He then says that the function $-\cos x$ is a solution to the equation and sets the other equations as exercises. In this way, the teacher explains what they are trying to achieve. It should be noted, however, that he refers to a solution here, even though this word does not appear in the previous description. There are therefore some reformulations that we consider to be horizontal proximities, but we cannot measure their impact on students.

The teacher then proposes the inverse mathematical task, explaining that for DE $y' = 2xy + 4x$, we can verify that the function $y(x) = k\exp(x^2) - 2$ is a solution on $\mathbb{R}$, for all $k \in \mathbb{R}$: "to do this, you just have to derive this y." The teacher explains the replacement operation in relation to the concept of solution, which can be associated with horizontal proximity here.

The teacher then defines a DE based on the algebraic register and the concept of a solution (see Figure 4), but does not establish any connections with previous examples. The teacher reads it as it is written on the slide. Therefore, we do not identify any bottom-up or top-down proximities as anticipated in the relief since the teacher does not revisit the previous examples to demonstrate how the definition applies to the equations encountered previously.

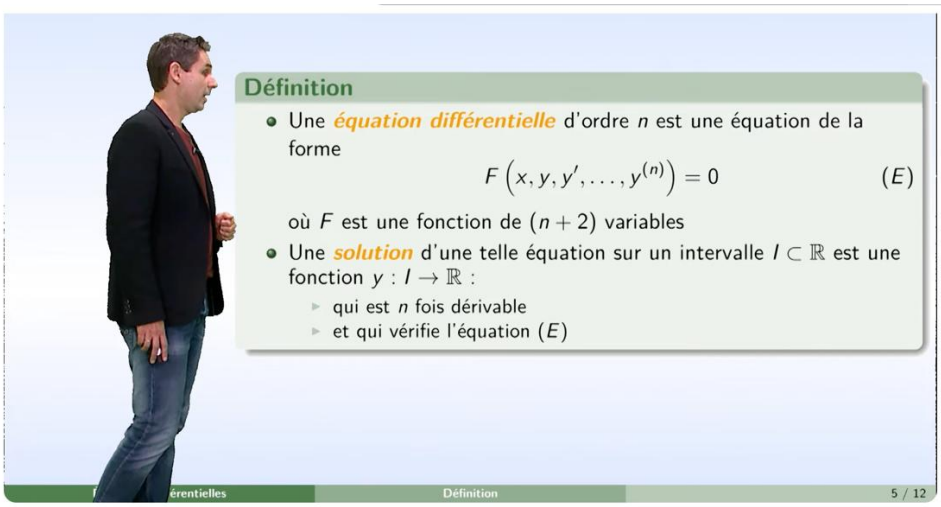

Figure 4: Video lesson – definition

As previously explained, the textbook begins by providing an intuitive characterisation of what a DE is in natural language, and then goes on to give examples of first-order equations while introducing the concept of solutions (Figure 5).

The textbook does not comment on the fact that the given equations are indeed first-order, so there are no top-down proximities to explain that these are special cases in relation to the given characterisation. The textbook then explains how to replace the function f in the equation with the given function g to verify whether it is a solution. Therefore, there is an attempt at top-down proximity here. A similar task is set for second-order equations.

1.1 Équation différentielle du premier ordre

Une équation différentielle du premier ordre (ou d'ordre 1) est une équation faisant intervenir une fonction inconnue et sa dérivée. Résoudre une telle équation sur un intervalle réel I signifie déterminer l'ensemble des fonctions définies et dérivables sur I satisfaisant l'équation.

Exemple 9.1

- Soit $(E_1) : xf'(x) + f(x) = 9x^2$.

Il s'agit d'une équation différentielle d'ordre 1. La fonction $g : x \mapsto 3x^2$ est une solution de (E_1) sur $\mathbb{R}$ puisque : $\forall x \in \mathbb{R}, xg'(x) + g(x) = x \times 6x + 3x^2 = 9x^2$. Le lecteur peut vérifier que la fonction $h : x \mapsto \frac{1}{x} + 3x^2$ est une solution de (E_1) définie sur l'intervalle $\mathbb{R}_{>0}$.

- Soit $(E_2) : y'(x) = \sqrt{y(x)}$.

Il s'agit encore d'une équation différentielle d'ordre 1. Notons que l'inconnue est désormais notée y, c'est l'usage le plus courant dans le monde des équations différentielles. Remarquons ensuite que pour être solution de (E_2), la fonction y doit être positive. Et d'après l'équation, sa dérivée y' sera également positive. Les solutions de (E_2) sont donc (si elles existent) positives et croissantes.

La fonction nulle est une solution de (E_2) sur $\mathbb{R}$. Le lecteur peut vérifier que la fonction $h : x \mapsto \frac{x^2}{4}$ est aussi une solution de (E_2) sur $\mathbb{R}_{\geq 0}$, mais pas sur $\mathbb{R}_{<0}$.

Figure 5: Textbook – introduction

Further on, the textbook provides a graphical interpretation of the solutions to a DE. It is based on the equation $y'(x) = y(x) - x$ and asks the reader to imagine a solution y that satisfies $y(-2) = -1$. The relief had anticipated a possible exercise in the graphical register. The textbook explains that, by substituting into the equation, we then have $y'(-2) = 1$ and that "we can represent this by a tangent with slope 1 at the point $(-2, -1)$". Since this can be done at any point on the plane, "the differential equation allows us to know the slope of the solutions at each point" (pp. 190-191). The graph given in Figure 6 accompanies this passage. Therefore, the textbook draws on prior knowledge and graphical representation to interpret the concept of a solution, which has previously been seen in algebraic terms. We can therefore identify horizontal proximities here.

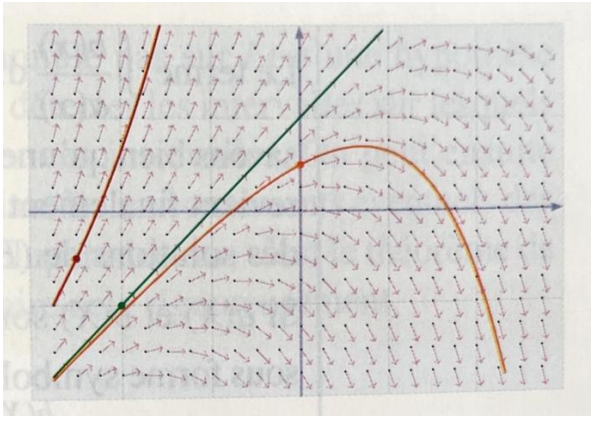

Figure 6: Textbook – graphical interpretation of the solutions of a DE

The textbook defines first- and second-order DEs, but does not cover *nth-order* equations as in the two courses.

DISCUSSION

Both the lecture and the video lesson introduce DEs using examples from physics. The transition to mathematical notation can then be the subject of horizontal proximities to explain the change in frameworks and the different registers of representation highlighted in the relief. However, in both courses, knowledge of physics is assumed to be available and the anticipated horizontal proximities are not attempted. In the lecture, the teacher remains closely tied to mathematical formalism, working primarily in the algebraic register. In contrast, the video lesson first provides an intuitive characterisation of DEs and then moves on to mathematical work on the notion of solution, based on horizontal and top-down proximities. Ultimately, we found few proximities in these two courses, and when they are attempted, they remain oral and not written. The textbook begins with specific cases of DE of orders 1 and 2, without changing the framework. The proposed work is based on a link between various registers of representation (natural language, algebraic, graphical). We identified additional anticipated proximities which are written in the textbook, such as explanations of the functional aspect of solutions or the process of proving that a function is a solution to a DE. The nature of these proximities is also richer, such as the graphical interpretation of the solutions of a first-order DEs. In this sense, the textbook provides a more detailed explanation of several specificities of the concepts to be taught and, in our opinion, contains proximities that are based on more readily available knowledge and will help students to develop their understanding more effectively than in the two courses.

This case study therefore demonstrates that lectures may contain many implicit elements that students do not necessarily share. The proximity tool enables us to identifty these elements. Further research on a larger scale is required, which is being conducted by the ‘Higher Education Teachers’" group at the LDAR (Université Paris-Cité, France). The aim of this research is to compare teaching practices in different disciplines, namely biology, chemistry, mathematics, and physics.

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Students' understanding of triple integrals in spherical coordinates

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This study investigates constructions of undergraduate students when learning triple integrals in spherical coordinates (TISC). Drawing on APOS theory, the research aims to develop a Genetic Decomposition (GD) to model how students might construct an understanding of TISC. Semi-structured interviews were conducted with 11 students from a multivariable calculus course. Data analysis led to the development of a Schema-based GD comprising four key components and the relations between them. The findings revealed widespread difficulties, particularly in visualizing the angular variables ϕ and θ, interpreting the volume element $\rho^2 \sin\phi d\rho d\theta d\phi$ geometrically, and meaningfully connecting integration limits to 3D regions. We conclude the paper by discussing the main findings and offering directions for future research.

Keywords: Triple integrals, spherical coordinates, multivariable calculus, APOS theory, genetic decomposition.

INTRODUCTION

Understanding triple integrals in spherical coordinates (TISC) is a crucial aspect of university-level mathematics, especially in multivariable calculus, engineering, and physics, as it helps students model three-dimensional problems with spherical symmetry and apply mathematical reasoning to real-world contexts. However, despite its importance, students often struggle with visualizing and setting up these integrals due to their geometric and algebraic complexity. While existing research has extensively explored students' understanding of differential concepts in multivariable calculus (Martínez-Planell & Trigueros, 2021), there remains a significant gap regarding how students learn multiple integration, particularly in non-Cartesian systems like spherical coordinates. This study helps to fill this gap by examining how students conceptualize and work with TISC.

RESEARCH LITERATURE

Research on multiple integrals has primarily focused on rectangular coordinates, with fewer studies addressing spherical coordinates. The literature centers on two main themes: geometric reasoning and representation, and students' procedural and conceptual challenges, especially regarding technology and instruction. Bašić and Milin Šipuš (2022) found that students struggle to connect symbolic, graphical, and technological representations and that curricula often neglect geometric and representational development. Theoretical models such as semiotic chains and genetic decomposition (McGee & Martínez-Planell, 2014; Borji et al., 2025) seek to improve coherence across representations. Other research (Jones & Dorko, 2015) studied students' symbolic forms for multiple integrals, while Ramírez Santamaría and Arias

Navarro (2024) showed that context-rich, experiment-based teaching enhances understanding and attitudes. The authors indicate that technological tools like GeoGebra aid visualization only when integrated collaboratively into instruction rather than having students use it in isolation.

Dalton et al. (2024) noted that calculus textbooks heavily favor Cartesian representations, with spherical coordinates appearing in only a small fraction of the material and often being presented superficially. This limited exposure leaves students unprepared for advanced applications in physics, where spherical symmetry is crucial. Montiel et al. (2009) further revealed that students struggle to visualize and interpret spherical surfaces, such as $\rho=\rho_0$, $\theta=\theta_0$, and $\phi=\phi_0$, showing a broader difficulty in connecting algebraic expressions to three-dimensional geometric concepts. Subsequent studies have confirmed similar difficulties. Bašić and Milin Šipuš (2022) found that students have trouble visualizing regions and determining integration bounds in spherical coordinates, often failing to use symmetry as a guide in choosing coordinate systems. Schermerhorn and Thompson (2019) also discovered that many students rely on memorized formulas instead of understanding the geometric meaning behind differential elements, often omitting essential scaling factors like $\rho^2 sin\phi$ in volume integrals. Collectively, these findings indicate weak conceptual foundations in students' understanding of spherical systems and their associated integrals, and emphasize the need for improved instructional strategies and more comprehensive textbook treatment of non-Cartesian coordinates.

THEORETICAL FRAMEWORK

In this paper, we will rely on the Schema structure of Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014). A Schema is a coherent collection (component structures) of Actions, Processes, Objects, and other previously constructed Schemas dealing with a specific mathematical notion. The Schema is coherent in the sense that its different components are interrelated in such a way that the individual can decide when a problem situation falls within the scope of the schema. This definition suggests that to model Schemas, one may list their component structures and describe the relations between the components that students may construct to understand the mathematical notion. Schemas are dynamic structures constantly under construction as students' mathematical horizons expand. To discuss their evolution, the Intra-, Inter-, Trans- triad of Piaget and García (1983) is interpreted as stages of Schema development in APOS. To classify a Schema development into one of the three stages, the notion of types of relations between Schema components is useful, as the triad stages can be described in terms of the preponderance of the type of relations between components (Trigueros, 2019). A *correspondence* relation is established out of habit or repetition, but the student cannot justify it. It can result from repeatedly seeing two or more components used in tandem or from superficial comparisons of similarities between components. A *transformation* relation is developed when students are aware that some structures in the Schema can be related in terms of changes in one another, or when students are able to group some Schema components into one or several groups

sharing a specific property or relation. A *conservation* relation involves the recognition of a property or properties that are shared by different components of the Schema. Students recognize these properties in different problems.

A Schema is said to be at the Intra- stage of development when its components are isolated or mainly related by correspondence relations. At the Inter- stage of development, groupings of different components and transformation relations start to appear. It is at the Trans- stage of development when conservation relations interrelate most components. A genetic decomposition (GD) for a Schema is a model given as a list of component structures, a description of relations between components, and a description of how the Schema may evolve. A GD is neither unique nor meant to describe the best way a student may learn a particular notion. It is only an initial hypothesis that must be tested with student interviews and rejected or revised based on the findings. This study is guided by the following research question: What can be learned about students' understanding of integrals in spherical coordinates based on a preliminary genetic decomposition?

METHODOLOGY

The study involved 21 university students enrolled in an introductory multivariable calculus course at a mid-tier Iranian university, meeting twice weekly for 90-minute sessions. The Stewart (2012) textbook was used as the reference for the course. An interview instrument with four questions about spherical coordinates was used (in the Results section, we show these questions when reporting the analysis of students' responses). The interview instrument was designed in terms of questions that may inform us about students' knowledge of spherical coordinates that could be used to design a first GD. Eleven students, out of the total 21, were selected by their instructor and agreed to participate in the interviews. The students were chosen based on their test average and course participation, so three were above average, five were average, and three were below average. The students participated in approximately one-hour-long semi-structured interviews one week after the semester ended.

To analyse the data, we first examined it through the lens of mathematical observations. This approach guided the development of codes grounded in those observations. Some of these codes were as follows: *geoSC* shows geometric understanding of spherical coordinates volume element; *SCe* appropriate use of spherical coordinate element in integration; ~ negation of a code; *~rho* difficulty with rho ρ; *~phi* difficulty with phi ϕ; *~theta* difficulty with theta θ. Since the interview instrument included very basic questions on integrals in spherical coordinates and our aim was to identify fundamental constructions, we proposed a preliminary GD during the second pass of data analysis, based on the Schema. This model, referred to as the TISC-Schema, is described in the next section. The relations within the GD, along with their types (correspondence, transformation, and conservation), were also employed as coding categories. After coding the relations and their types for each problem, a third data analysis was then conducted to classify students according to the Schema development stages (Intra, Inter, and Trans).

Genetic decomposition based on TISC-Schema

According to the analysis results a data grounded preliminary GD was designed. It consists of four components:

- Three-dimensional space (3D) as a Schema with points, planes, cylinders, graphs of two-variable functions, and equations as Objects.
- Spherical coordinates (SC) as a Schema, including conversion Processes from spherical to rectangular coordinates and vice versa in algebraic and graphical representations.
- Riemann sums (RS) as a Schema including Processes for the geometric or contextual meaning of the factors of a typical term of a Riemann sum in spherical coordinates, partitions, integrals as limits of Riemann sums, and differential elements as "small" or "infinitesimal" spherical wedges as Processes or Objects.
- Triple integrals (TI) as a Schema, including Processes for interpreting an integrand, setting up the limits of integration, and integrating in spherical coordinates.

The following relations may appear in students' work:

- 3D-SC: We consider this relation when students show they can interpret spherical coordinates in space.
- RS-SC: This relation is considered when students can relate a triple integral in spherical coordinates to the Riemann sum it comes from, and when they can interpret or represent the volume element $\rho^2 sin\phi d\rho d\phi d\theta$.
- TI-SC: We consider this relation when students can establish the limits of integration in spherical coordinates.

RESULTS

The analysis of interview responses focuses on the relations among the components of the TISC-Schema.

Construction of an Intra-TISC Schema

Seven students mainly showed isolated components and the construction of correspondence relations. We present S6's answers as a paradigmatic example of students who were classified within the Intra-TISC Schema level. In Question 1 (the geometrical interpretation of $\rho^2 sin\phi d\rho d\theta d\phi$):

> S6: It means we need to find the limits of integration with the indices ρ where ρ specifies the distance of the solid from the origin, also with ϕ where it's the angle between the positive part of the *z*-axis and each point on the this, and also with θ where it's as described in polar coordinates, I mean θ is $\tan^{-1}(y/x)$.

S6 interpreted the volume element of integration simply as "limits of integration with indices." Although he identified the roles of the variables, there was no reference to the volume element as a geometric wedge or part of a 3D region. He merely named individual variables with partial definitions, suggesting a correspondence relation of RS-3D. In Question 2a (drawing and describing the meaning of the equation $\rho=1$), S6

stated that $\rho=1$ represents a sphere of radius 1 centered at the origin, showing evidence of relating algebraic equation $\rho=1$ to a geometric object in 3D space which can be interpreted as a transformation relation 3D-SC regarding ρ. In Question 2b (drawing and describing the equation $\phi=\pi/4$):

S6: It means the angle between the line passing through the *z*-axis and the line passing through the origin is $\pi/4$, so we have two cones, one is in the above of the *xy*-plane, the angle between this cone and the positive *z*-axis is $\pi/4$, the other cone is in the below of the *xy*-plane, the angle between this cone and the negative *z*-axis is also $\pi/4$ [Figure 1.a].

S6 faced challenges in interpreting ϕ in relation to a cone and stated that the condition leads to two cones, one above and one below the *xy*-plane. While he attempted a geometric interpretation, his understanding of ϕ was flawed, as he claimed that the angle is also measured from the negative *z*-axis for the lower cone (Figure 1.a). S6's confusion about ϕ reveals a correspondence relation (habitual but flawed) for 3D-SC regarding ϕ, where the relation between ϕ and the geometric object is not fully grasped. In Question 2c (drawing and interpreting $\theta=\pi/4$):

S6: I draw a line in the *xy*-plane such that the angle between this line and the *x*-axis is $\pi/4$, the line should be like this, but we are in the spherical coordinates which is in 3D so this line will be changed to a plane like this [Figure 1.b].

S6 initially interpreted θ as the angle between a line and the *x*-axis, implying a 2D mindset. While in polar coordinates θ is indeed measured from the positive *x*-axis, it defines a direction (or ray) in the *xy*-plane, not a full line through the origin at an angle of $\pi/4$ to both directions. S6 described and sketched such a line in the *xy*-plane, then extended it to a full plane in 3D. However, the surface described by $\theta=\pi/4$ in spherical coordinates is not a full plane. It includes all points (x,y,z) such that their perpendicular projection onto the *xy*-plane lies along the ray at angle of $\pi/4$ from the *x*-axis, with any value of *z*. S6's explanation was consistent with a correspondence relation for 3D-SC regarding θ. However, by considering all parts (Questions 2a, 2b, and 2c), since S6 showed problems with the two variables ϕ and θ, this shows that he has not yet established the 3D-SC relation overall. In Question 3 (drawing the region of integration of $\int_0^{\pi/4}\int_0^{2\pi}\int_0^4(\rho^2)\rho^2\sin\phi\, d\rho d\theta d\phi$):

S6: The first integral shows the boundaries for ρ, it's from 0 to 4 so a sphere with radius 4, the second integral shows the limits of θ, it's from 0 to 2π, when we have a sphere it automatically includes the angles from 0 to 2π in the *xy*-plane, the last integral involves the changes of ϕ and it's from 0 to $\pi/4$ so it shows two cones in the above and the below of the *xy*-plane, the region of integration is two symmetrical regions respect to the *xy*-plane, the first region is above the *xy*-plane from the cone to the sphere, the second region is the same but below the *xy*-plane [Figure 1.c]

S6 identified ρ from 0 to 4 as representing a sphere and θ from 0 to 2π. His statement that a sphere "automatically includes all angles [θ] from 0 to 2π" shows the relation

3D-SC for θ in this question. However, in Question 2c, he previously showed challenges interpreting $\theta=\pi/4$. Regarding the angle ϕ, used to describe the region, he again interpreted $\phi=0$ to $\pi/4$ as forming two cones, as this way of thinking of ϕ was also shown in Question 2b. This response furthermore indicates challenges in describing the limits of integration for ϕ and in meaningfully connecting those limits to the region's geometry, thus failing to establish a TI-SC relation. These challenges in interpreting ϕ and its limits suggest that S6 may have relied on memorized fragments (e.g., "sphere with cones") without a meaningful understanding. Although some elements were recalled, the connections between coordinate bounds and the geometric region still require more student work. In Question 4 (establishing an integral for the volume of a unit sphere in spherical coordinates):

S6: I should consider the variables ρ, θ and ϕ, the boundaries for the inner integral will be for ρ and it shows the changes in radius of the sphere umm it's from 0 to 1, the angle θ is like the angle in the polar coordinates and since we have a complete circle so the angle θ changes from 0 to 2π and finally the angle ϕ will be for the outer integral, it shows the angle between the *z*-axis and the sphere and it's from 0 to π/2.

I: The volume of a sphere with radius r is $4/3\pi r^3$, so in this case the volume is $4/3\pi$. Could you solve your triple integral in spherical coordinates?

S6: The boundaries are constant numbers and we don't have any function in the integral so the answer will be 1 times 2π times π/2, it will be π [Figure 1.d], oh, it's not the same with $4/3\pi$, I don't know where my mistake is.

S6 did not write $\rho^2 sin\phi d\rho d\phi d\theta$, for the spherical coordinate volume element. He used ϕ from 0 to π/2, which corresponds to a hemisphere, not a full sphere, showing a lack of understanding of the variable ϕ. After the interviewer prompted that the volume should be $4/3\pi$, he performed a computation and then expressed confusion when the result did not match the known formula. It appears that S6 relied on some memorized procedures and struggled to properly describe the volume element, determine the limits of integration, and set up the integral appropriately. Altogether, this provides evidence that the TI-SC relation was not established.

Overall, most of S6's responses reflect isolated components or habitual correspondence relations. His explanations suggest a Schema in early development, at the Intra stage.

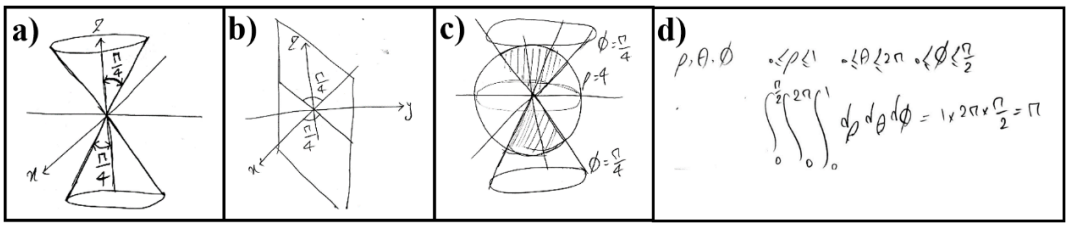

Figure 1: S6's work a) drawing $\phi=\pi/4$; b) drawing $\theta=\pi/4$; c) Question 3; d) Question 4

Construction of an Inter-TISC Schema

Four students gave evidence of the Schema at an Inter-TISC. We selected S4 who showed the construction of Inter-TISC as an example of the possible diversity of constructed relations. In the interpretation of the volume element in Question 1:

S4: It means to me that I should find the volume of a space in 3D with variables ρ, ϕ, and θ, to do this we need to consider the term $\rho^2 sin\phi d\rho d\phi d\theta$ in the integration.

I: Can you describe this term geometrically?

S4: I think this term is the volume of a small part of a solid, the summation of all such small parts can be reformed with a triple integral in spherical coordinates. We use spherical coordinates when the shape of the region in space includes a sphere.

S4 treated the differential element $\rho^2 sin\phi d\rho d\theta d\phi$ as a prescribed formula, a fixed expression to be used when integrating in spherical coordinates. This suggests that, in S4's construction, its use may stem from habit or repetition, likely due to frequent exposure to this element in spherical integration. Because she explicitly identified that this is the volume element in spherical coordinates and understood its necessity when integrating, we used the code *SCe* for her recognition of the element. While S4 mentioned "a small part of a solid" and correctly noted that spherical coordinates are useful for spherical regions, she did not elaborate on the differential volume element as a spherical wedge, nor did she refer to its geometry (shape, boundaries), thus lacking geometric interpretation (*~geoSC*). For this reason, we considered the overall RS-SC relation as a correspondence relation in S4's response to this problem.

Like S6 in Question 2a, S4 also drew, described and justified $\rho=1$ as a sphere with radius 1, showing evidence consistent with 3D-SC as a transformation relation. In Question 2b, describing $\phi=\pi/4$:

S4: It shows a cone where its angle with the positive segment of the *z*-axis is π/4, I mean the angle between the slant heights of the cone with the positive *z*-axis is π/4 [Figure 2.a].

I: What happens geometrically as ϕ changes from 0 to π?

S4: If the angle ϕ changes from 0 to π then it means the angle between the positive *z*-axis and the slant heights increases, so the cone goes down, I mean below the *xy*-plane.

S4 stated it is a cone with slant height forming an angle π/4 with the positive *z*-axis, showing a geometric description. She indeed related the algebraic condition ($\phi=\pi/4$) to a geometric figure in 3D space. She also elaborated on how changing ϕ from 0 to π alters the orientation of the cone, which gives evidence consistent with 3D-SC as a transformation relation.

In Question 2c (drawing and describing $\theta=\pi/4$):

S4: If in 2D I draw $\theta=\pi/4$ in the *xy*-plane then it will be a line like this passing through the first and third quadrants, but since we are in 3D and variable *z* can have any value so this line will be changed to a plane in 3D [Figure 2.b].

S4 began with a 2D interpretation. She drew $\theta=\pi/4$ as a line in the *xy*-plane. However, her description of this line "passing through the first and third quadrants" implied that she views $\theta=\pi/4$ as representing a full line (extending in both directions), rather than a ray originating from the origin and sweeping out an angle of π/4 from the positive *x*-axis. Indeed, she did not exclude points where $\theta=5\pi/4$. This interpretation reflects confusion about the directionality of θ. In spherical and cylindrical coordinates, $\theta=\pi/4$ defines a half-plane. S4's explanation regarding θ in Question 2c is consistent with a correspondence relation of 3D-SC. In general, although S4 provided justified explanations regarding ρ and ϕ, she showed some challenges with θ; therefore, overall, we consider 3D-SC as a correspondence relation in this problem. In Question 3, when we asked students to draw the region of integration of $\int_0^{\pi/4}\int_0^{2\pi}\int_0^4(\rho^2)\rho^2\sin\phi\, d\rho d\theta d\phi$:

S4: The first integral shows the boundaries for ρ and it's from 0 to 4, so we should consider the points on a sphere with radius 4 and its inside, the second integral shows the boundaries of θ, it's from 0 to 2π so we start from the point (4,0,0) and go around the sphere in the *xy*-plane to again the point (4,0,0) so it involves the boundaries for θ, the third integral shows the boundaries for ϕ and its lower limit is 0 and its upper limit is π/4, so we are inside the cone with angle π/4 from the positive z-axis, so finally the lower limit of the region is the cone and the upper part of it is the sphere [Figure 2.c].

S4 explained the meaning of the bounds on ρ, θ, and ϕ, and justified these bounds in spherical coordinates during her reconstruction of the 3D region, providing evidence of TI-SC as a transformation relation. She translated the integration limits to a geometric object bounded below by a cone and above by a sphere, describing the region precisely and visually. This provides evidence of establishing the relation 3D-SC as a transformation in this problem. In Question 4 (find the volume of the unit sphere):

S4: I can find one eighth of the volume of the sphere then I will multiply it by 8, I mean I'll find the volume of the part of the sphere where all *x*, *y* and *z* are positive. So I have the triple integral $\rho^2 sin\phi d\rho d\phi d\theta$, since it's a sphere with radius 1, the lower and upper boundaries for ρ are 0 and 1, the angle ϕ starts from positive *z*-axis and comes to the *xy*-plane, so its lower boundary is 0 and its upper limit is π/2, to find the limits for θ I need to see what we have in the *xy*-plane, a quarter of a circle with radius 1 in the *xy*-plane, so θ is from 0 to π/2 [Figure 2.d].

S4 chose to calculate one-eighth of the sphere and then multiply by 8, this indicates geometrical flexibility. S4 justified integration limits based on spatial orientation and structure of the first octant of the sphere, giving evidence of TI-SC as a transformation relation. She correctly included the entire spherical volume element $\rho^2 sin\phi d\rho d\theta d\phi$

(*SCe*). S4's ability to decompose the problem, justify limits, and connect them to geometry reflects a transformation relation of 3D-SC in this problem.

There were five situations (Questions 2a, 2b, 2c, 3, and 4) in which it was possible to analyse the presence of 3D-SC in the student's responses. In four of these situations, S4 provided evidence of a transformation relation, while in one situation, she showed a correspondence relation. Overall, we classified 3D-SC as a transformation relation in her Schema. In Question 1, it was possible to examine RS-SC, and in this case, S4 established a correspondence relation. The TI-SC relation was targeted in Questions 3 and 4, and for both, S4 demonstrated evidence of a transformation relation. Therefore, we considered TI-SC in her Schema as a transformation relation. Since S4 demonstrated transformation relations for both 3D-SC and TI-SC, and a correspondence relation for RS-SC, we classified her Schema as being in the Inter-stage of development.

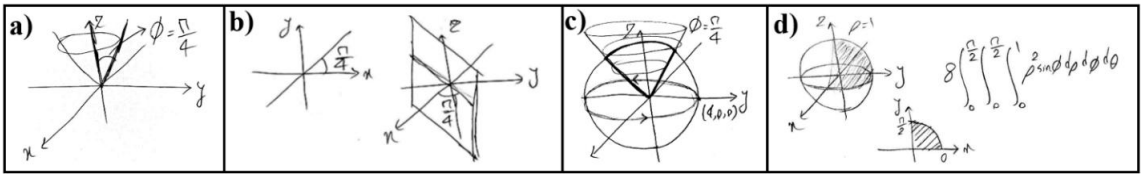

Figure 2: S4's work a) drawing $\phi=\pi/4$; b) drawing $\theta=\pi/4$; c) Question 3; d) Question 4

DISCUSSION AND CONCLUSION

The findings revealed significant challenges in visualizing spherical coordinates, interpreting angular variables (ϕ and θ), and understanding the volume element $\rho^2 sin\phi d\rho d\theta d\phi$. The research question was: What can be learned about students' understanding of integrals in spherical coordinates based on a preliminary genetic decomposition? Based on the interview data, we proposed a GD drawing on Schemas for categorizing and differentiating the sophistication of students' understanding. Seven students demonstrated Schemas at the Intra-TISC Schema, characterized by fragmented or superficial knowledge, often misapplying formulas or misinterpreting geometric relationships. Four students exhibited Inter-TISC Schema, showing the ability to justify and relate components in problem-specific ways, with partial coherence, for instance, correctly setting up integrals for symmetric regions but struggling with certain constructions such as θ and the volume element $\rho^2 sin\phi d\rho d\theta d\phi$. We had no students at the Trans stage. Broadly speaking, the findings support the claim that while students may memorize symbolic procedures, they often fail to reconstruct the underlying geometric and analytic reasoning necessary for a deep understanding of triple integrals in spherical coordinates. We designed a preliminary GD based on Schemas, which provided some findings that we have reported; however, further analysis is needed. As future research, one could use the APO portion of the theory to design a more detailed GD based on mental constructions and mechanisms, model students' understanding of this mathematical notion, and compare those results with the findings of this study. In conclusion, the study underscores the need for targeted instructional interventions to address challenges with spherical coordinates.

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Design of variational tasks: the case of continuous exponential covariation

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This paper adopts a variational and dynamic perspective to address continuous exponential covariation within an engineering Calculus course. Drawing on Harel's DNR model and a historical-epistemological analysis, it introduces a new approach that incorporates Variational Ways of Understanding and Variational Ways of Thinking as conceptual tools to guide the design and analysis of variational tasks. The pre-design stage is developed, in which configurations of thinking for reasoning about continuously changing quantities are identified. The model provides operational criteria for developing activities that promote covariational reasoning in exponential processes and lays the groundwork for subsequent stages of design and analysis.

Keywords: variational thinking, variable quantities, variational ways of understanding, variational ways of thinking, continuous exponential covariation.

DEFINITION AND RELEVANCE OF THE PROBLEM

The development of students' ability to understand and quantify phenomena of variation and change constitutes a fundamental goal in Mathematics Education. However, in school practice, this goal is often reduced to mechanized and static algorithmic procedures, which prevents students from constructing a deep understanding of mathematical concepts and connecting them to real-world situations. Particularly in the didactic treatment of exponential behavior phenomena, Khanh et al. (2020), who analyzed the teaching approach to exponential notions in textbooks, note that these tend to follow a pattern: first, the concept is defined; then, practical examples are presented; and finally, application problems are proposed. Within this pattern, a pedagogical transformation in the development of exponential notions becomes evident, in contrast to their historical-epistemological emergence and evolution. This transformation overlooks the temporal sequence in the development of Calculus and its objects of study (Khanh et al., 2020; Jiménez et al., 2022).

This issue becomes particularly relevant in engineering education. Calculus, understood as the mathematics of variation and change, constitutes an essential tool for analyzing and modeling real-world phenomena. However, its teaching at the university level reveals a mismatch between educational objectives and students' practical needs. On the one hand, curricula emphasize a formal and abstract approach characteristic of Mathematical Analysis. On the other hand, students require tools that enable them to interpret, model, and solve problems related to physical and technological processes within their discipline. This discrepancy creates a conflicting scenario that affects

motivation, retention, and academic performance, highlighting the need to rethink teaching strategies toward approaches more closely connected to professional practice.

From our experience, this curricular orientation hinders the development of skills needed to apply mathematical knowledge in relevant contexts. In this regard, Dray and Manogue (2010) propose that the teaching of Calculus should prioritize notions that capture its intuitive essence, making them comprehensible and coherent for students. Such an approach would facilitate the articulation between mathematical concepts and their use in solving engineering-related problems.

In contrast to the predominance of abstract approaches, it is pertinent to revisit the historical origins of the study of variation and change. During the Middle Ages, in the Oxford and Paris schools, scholars sought to understand phenomena of change—particularly non-uniform motion—through the notion of "degrees of intensity" of qualities such as heat, light, or velocity in relation to time or space. Later, Galileo Galilei introduced fundamental principles such as continuity—according to which a moving object passes through all intermediate degrees of velocity—and uniqueness—which establishes a correspondence between each degree of velocity and a specific instant (Crosby, 1998). These ideas laid the groundwork for the formalization of continuous change, later developed by Isaac Newton and Gottfried Wilhelm Leibniz through the calculus of infinitesimal variations (Kleiner, 2001). Thus, the study of change emerged from the need to understand real phenomena, rather than from the abstraction of relationships among numerical sets.

Based on this perspective, we argue that the teaching of Calculus should recover its variational and practical character, centered on the analysis of real phenomena and on the development of a type of mathematical thinking that enables students to understand and compute infinitesimal variations of quantities that are genuinely changing. In particular, this study aims to address the difficulties associated with the teaching and learning of continuous exponential covariation (Amador & Jiménez, 2025), a concept whose understanding is often hindered by static, formal, and algorithmic approaches that fail to reflect the inherent dynamics of exponential phenomena.

To this end, theoretical constructs are proposed based on Guershon Harel's (2008) conception of mathematical thinking, with the aim of establishing a theoretical and explanatory framework oriented toward the development of variational thinking, while recovering the dynamism required by this type of reasoning. In particular, *Variational Ways of Understanding* are defined as cognitive products associated with the construction of dynamic images of the quantitative properties of varying quantities in processes of change, and *Variational Ways of Thinking* as the cognitive characteristics that emerge from the repeated coordination of such variational ways of understanding. These constructs allow for the operationalization of variational thinking in terms of observable cognitive configurations in students' mathematical activity.

From this standpoint, the goal is to provide conceptual and methodological tools for designing learning tasks that foster both variational and covariational reasoning, moving away from the analysis of static mathematical objects. In this sense, the research question guiding this study is: What ways of understanding can be characterized as variational, and which correspond to non-variational ways in the context of continuous exponential covariation?

THE DNR MODEL: FOUNDATIONS AND CONTRIBUTIONS

Harel (2008, 2008a) argues that the main goal of mathematics education is to develop students' mathematical thinking, which consists of a set of mathematical ways of understanding and mathematical ways of thinking. He defines ways of understanding as the particular cognitive product of a mental act carried out by a person, and ways of thinking as the cognitive characteristics of a mental act, revealed through repeated actions of a person's ways of understanding.

This profound reflection by Harel on the characterization of mathematical knowledge and the fundamental purpose of mathematics teaching an idea he has actively promoted in recent years has led him to formulate a methodological framework to achieve such a goal. He has referred to this framework as "DNR-based instruction." The acronym derives from the three foundational principles on which the framework is built: the Principle of Duality (D), the Principle of Intellectual Need (N), and the Principle of Repeated Reasoning (R). Of particular interest here is the Principle of Duality.

The Principle of Duality states that students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they already possess (Harel, 2008, p. 19). This principle asserts that ways of understanding (concepts and skills) and ways of thinking (practices, dispositions, and beliefs) are interdependent there is a bidirectional relationship between them.

The theoretical constructs presented in Harel's view of mathematical thinking provide an operational path for promoting mathematical thinking. In particular, they have motivated us to define and characterize two new theoretical constructs that are necessarily related to variational mathematical thinking. These constructs are *Variational Ways of Understanding* (VWoU) and *Variational Ways of Thinking* (VWoT), a point to which we will return later.

THE VARIATIONAL THINKING APPROACH

The approach adopted in this study is distinguished by its emphasis on a dynamic way of reasoning about the fundamental ideas that gave rise to Calculus. Among these ideas are the notions of continuous quantity, variable quantity, and the handling of the infinitely small. From this perspective, promoting a type of mathematical thinking

linked to variation and change requires clearly specifying its object of study: variable quantities.

The object of study of variational thinking is variable quantities

In various natural and social phenomena, quantities arise that vary over time or space. By their nature, these variable quantities reflect the inherent dynamism of real-world processes and thus constitute the central object of variational thinking. Examples of such quantities include time, temperature, pressure, and velocity, among others, all of which can be measured or quantified using appropriate units and instruments; that is, variable quantities have units of measurement (Jiménez et al., 2022).

Variational thinking is concerned with understanding and mathematizing the behavior of these quantities by analyzing how they change, how they relate to one another, and what patterns emerge from their evolution. In this sense, Jiménez et al. (2022) point out that these quantities exhibit essential characteristics: the symbol representing them refers both to a quantifiable attribute (such as temperature or time) and to the set of values it may assume within a phenomenon. These values do not vary randomly; rather, they follow a coherent sequence over time or space, which may be discrete or continuous, the latter being of particular interest for variational thinking.

In contrast, mathematical work centered on numerical functions is characterized by abstract and static treatment. By nature, a real function of a real variable is conceived as a correspondence between two numerical sets, without necessarily requiring reference to real-world phenomena. As stated by Larson and Edwards (2016):

> Let X and Y be sets of real numbers. A real function f of a real variable x from X to Y is a correspondence rule that assigns to each number x in X exactly one number y in Y. The domain of f is the set X. The number y is the image of x under f and is denoted by $f(x)$, which is called the value of f at x. The range of f is defined as the subset of Y consisting of all images of numbers in X. (p. 19)

This definition highlights the purely abstract, numerical, atemporal, and static nature of functions. The numerical variables involved simply represent elements of sets, without necessarily being linked to measurements or the quantification of real-world phenomena. In contrast, the numerical values associated with variable quantities retain an intrinsic connection to the phenomenon that generates them and are inextricably tied to a temporal dimension. Unlike these, the values of numerical variables are selected arbitrarily and lack a meaningful temporal relationship, since in numerical sets the order in which elements are considered is of no particular importance.

The dynamic nature of variational thinking

From a critical analysis of the specialized literature on the subject, we have come to conceive that variational thinking requires a dynamic way of thinking (Vasco, 2003),

and that it is directly related to the study of variable quantities. We conceive variable quantities as perceptible and quantifiable properties or qualities of real-world phenomena and processes, which, by their intrinsic nature, are dynamic. Variable quantities truly vary. They are omnipresent in what are commonly referred to as phenomena or processes of variation and change. In variational thinking, the central role is played by variable quantities, which are mathematical objects of a nature essentially different from that of functions. Functions appear only as secondary actors emergent tools for the mathematization of variable quantities and of the processes in which they participate (Jiménez et al., 2022).

In the studies presented by Thompson and Carlson (2017) and by Jiménez et al. (2022), the authors conceive variational thinking as consisting of two components: variational reasoning and covariational reasoning, both related to the mathematical conceptualization of variable quantities. In the first case, the reasoning and mathematical work focus on a single variable quantity along the number line; in the second, the reasoning and work involve two variable quantities simultaneously in the Cartesian plane.

THEORETICAL PROPOSAL: AN OPERATIONAL FRAMEWORK FOR THE DEVELOPMENT OF VARIATIONAL THINKING

Based on the variational thinking approach and on Harel's (2008) vision of mathematical thinking, we propose that, in order to develop variational thinking in students, learning strategies must be designed that enable them to *generate Variational Ways of Understanding* and *Variational Ways of Thinking*. We define *Variational Ways of Understanding (VWoU)* as the particular cognitive product of the mental act of producing dynamic images related to the essential characteristics or properties of variable quantities present in ongoing phenomena of variation and change. We describe *Variational Ways of Thinking (VWoT)* as the cognitive characteristics of the student's mental act of producing dynamic images, revealed through the repeated actions derived from their Variational Ways of Understanding.

For example, if a student formulates an interpretation such as "the numerical values taken by the literal symbol representing a variable quantity are associated with a unit of measurement," they are demonstrating a Variational Way of Understanding, because it reflects a particular cognitive product of their mental act of interpreting an important characteristic of variable quantities. Moreover, if the student holds more than one VWoU, it is likely that they also possess the Variational Way of Thinking that "variable quantities progressively take numerical values that can be grouped into three categories: the current value, the previous values, and the subsequent values."

For the organization of didactic activities, the methodological-didactic structure is arranged into three stages: Pre-design, Design, and Analysis. This article develops only

the Pre-design stage, which is considered a research outcome in itself. The Design and Analysis stages are presented as a methodological projection for future studies.

Stage 1: Predesign

This stage aims to define the *Variational Ways of Understanding* (VWU) that are expected to be fostered in students, as well as to establish the conceptual and cognitive elements that guide the design of activities. In order to identify and characterize the VWU to be promoted, the following process must be carried out:

- First, the type of covariation to be addressed (linear, quadratic, exponential, etc.) must be characterized, along with the application contexts in which such covariation manifests. This makes it possible to establish the link between variable quantities and their phenomenological meaning; that is, VWU characterize the dynamic development of the process.
- Second, a historical-epistemological analysis must be conducted to examine the development of the mathematical concept over time and how it is structured within disciplinary knowledge.
- Finally, the VWU are formulated as specific configurations of thinking, expressed through variational language. These VWU are not presented as formal definitions or declarative knowledge, but rather as more or less precise statements about the quantitative properties of variable quantities.

Thus, VWU operate as criteria for designing tasks that are consistent with the dynamic and continuous nature of phenomena of variation.

Stage 2: Design

In this stage, the guidelines that direct the development of instructional activities aimed at promoting the previously formulated VWoU are specified. Two key components are defined: principles for designing mathematical tasks and principles for designing technological tasks.

Principles for the design of mathematical tasks

The design of mathematical tasks is guided by outlining the cognitively logical sequence involved in obtaining the cognitive products associated with students' mental acts, supporting the progressive construction and analysis of VWoU. This structuring begins with the identification of the desirable and expected mental acts in students and is translated into statements and actions that guide the creation of the tasks.

Principles for the design of technological tasks

Design principles for technology-based tasks are guidelines aimed at coherently integrating digital resources—such as GeoGebra, simulators, or video clips—that help establish a connection between the physical world and the abstract world of mathematics. Technology should facilitate cognitive experiences that not only present information dynamically but also enhance reasoning about the continuous variation of quantities (Swidan et al., 2019).

In this spirit, a preliminary GeoGebra applet has been designed to illustrate the gradual purification process of a substance (see GeoGebra simulation). In this task, students interact with a dynamic simulation to identify variable quantities, describe their behavior, and formulate conjectures about their covariation. The activity aims to activate prior knowledge and promote the construction of variational images, understood as dynamic mental representations that enable the interpretation of continuous covariation between quantities.

This approach fosters intellectual necessity and the construction of variational images through interaction with dynamic representations.

Stage 3: Analysis

This stage aims to assess the extent to which the activities promote *Variational Ways of Thinking* (VWT). It focuses on students' productions (arguments, explanations), from which their cognitive characteristics are inferred (Guershon Harel, 2008, 2008a). The objective is to identify evidence of understanding of quantitative properties and of covariation between quantities.

The theoretical proposal presented constitutes a novel approach to the design of variational tasks, differing from what has been reported in the literature, and inspired by the pragmatic-pedagogical perspective on mathematical thinking developed by Guershon Harel. Its distinctive character lies in the operationalization of *Variational Ways of Understanding* and *Variational Ways of Thinking*, employed as organizing constructs for the processes of pre-design, design, and analysis of tasks aimed at fostering students' variational thinking.

PRELIMINARY RESULTS

The Predesign Stage as a Foundation for Identifying Variational Ways of Understanding

In order to address the research question, the Pre-design stage was developed as part of the methodological-didactic implications for fostering variational thinking. This stage constitutes the central focus of the present study and enabled the construction of a conceptual and cognitive foundation for identifying, characterizing, and formulating the *Variational Ways of Understanding* (VWU) associated with continuous exponential covariation.

This stage is structured into three interconnected phases. First, the type of covariation—continuous exponential covariation—and its contexts of manifestation (physical, biological, and technological) were characterized. This made it possible to establish the link between variable quantities and their phenomenological meaning, recognizing their dynamic and continuous nature.

Second, a historical-epistemological analysis of the development of the concept of exponential variation was conducted, identifying relevant ways of reasoning for understanding proportional dependence, relative rate of change, and exponential dynamics as a continuous phenomenon. This analysis was key to distinguishing between variational and non-variational ways of understanding.

Finally, the VWU were formulated as configurations of thinking expressed through variational language—not as formal definitions, but as statements describing the quantitative properties of quantities in situations of change. These configurations guide the design of tasks by promoting reasoning consistent with the continuous nature of phenomena of variation.

Variational and Non-Variational Ways of Understanding Continuous Exponential Covariation

The results of the historical-epistemological analysis led to the identification of a set of *Variational Ways of Understanding* that can guide the design of variational tasks aimed at enabling students to reason about phenomena exhibiting exponential behavior. Some of these are presented in Table 1.

Variational Ways of Understanding are characterized by their focus on variable quantities, their quantitative dependence, and the continuous structure of change. In contrast, non-variational ways privilege relationships between numerical variables or functions, without explicit reference to the quantities involved in the phenomenon. Although the latter are mathematically valid, they do not foster variational or covariational reasoning, as they fail to capture the continuous dynamics and the quantitative relationships inherent to the quantities involved.

Table 1.

Non-Variational vs. Variational Ways of Understanding Continuous Exponential Covariation

Non-Variational	Variational
The exponential function is a continuous function whose relative rate of increase depends specifically on the increment taken from any value of x. Knowing the rate makes it possible to determine the base of the exponential function.	The average relative rate of change of a covarying magnitude with exponential behavior, for equal increments of the independent magnitude, is equal to a nonzero numerical constant.

The exponential function is a differentiable function that satisfies the differential equation $y' = ky$.	A covarying magnitude whose instantaneous rate of change is directly proportional to the current value of that magnitude, $\frac{dy}{dx} = ky$.

CONCLUSIONS

The characterization obtained provides a foundation for designing tasks aimed at promoting variational thinking in the context of continuous exponential covariation. In particular, the identified *Variational Ways of Understanding* (VWU) make explicit the essential quantitative properties of the phenomenon, the continuous structure of exponential change, and the dynamic interdependence among variable quantities.

In contrast, non-variational ways reveal forms of reasoning that are detached from the dynamism required by variational thinking (Jiménez et al., 2022). That is, they constitute mathematical ways of understanding but are centered on properties of numerical functions and abstract variables; therefore, they are not characterized as variational.

From a didactic perspective, the distinction between these ways of understanding allows for the derivation of operational criteria for task design. First, the importance of starting from real phenomena that enable the identification and analysis of variable quantities is emphasized, thereby fostering the construction of mathematical meanings grounded in contexts of variation. Second, the need to design tasks that promote knowledge construction based on intellectual necessity is highlighted, in coherence with the DNR model, avoiding the premature introduction of definitions and formalism detached from students' experience. Finally, the use of dynamic representations and technological tools, such as GeoGebra, is presented as a key means for visualizing and experiencing continuous covariation, overcoming traditional static approaches.

Overall, the results show that the Pre-design stage not only made it possible to address the research question but also provided a methodological framework for transforming conceptual distinctions into operational criteria for the design and analysis of variational tasks.

By shifting the focus from functions to covarying quantities, this study contributes to rethinking the epistemological foundations of Calculus instruction, offering a basis for the development of didactic proposals that are more consistent with the dynamic nature of phenomena of variation.

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Conflict identification from an Ontosemiotic Analysis: the case of improper integrals

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This work illustrates the potential of ontosemiotic analysis to anticipate possible semiotic conflicts that may arise during the teaching and learning process of improper integrals in teacher education. The methodology employed is based on content analysis applied to the ontosemiotic analysis of four institutionally valid solutions to an improper integral problem, which allows for the inference of possible semiotic conflicts that may emerge in hypothetical student practices when solving the problem. The results indicate that the analyses conducted make it possible to identify different types of semiotic conflicts, which do not necessarily refer to mathematical errors.

Keywords: Improper integral, ontosemiotic analysis, ontosemiotic approach, didactic of calculus.

INTRODUCTION

Over the past four decades the study of mathematics teaching in higher education institutions has attracted increasing interest within the specialist mathematics education community (Nardi, 2017). Among the most significant areas within this field, the teaching of calculus stands out with a considerable diversity of theoretical and methodological perspectives, as well as varied findings and contributions (Hanke & Martínez-Panell, 2024).

It is essential to recognize the inherent sophistication of the concepts and procedures that characterize working with calculus. Concepts such as the limit, the reconstruction and redefinition of the notion of equality in both algebra and analysis, the development of the notion of continuity of a function, and that of the integral represent characteristic aspects of the sophistication of this discipline (Artigue, 2003). The teaching of calculus, and specifically that of integral calculus, has been the subject of significant interest among researchers in Mathematics Education, as evidenced, for instance, by its ongoing relevance throughout the various INDRUM conferences to date (e.g. González-Martín et al., 2024; Hausberger et al., 2020).

More specifically, the improper integral is a central and challenging concept in the teaching and learning of advanced calculus, whose understanding is marked by conceptual difficulties, recurring errors, and epistemological obstacles (Mateus-Nieves & Font-Moll, 2021; González-Martín, 2005).

A crucial aspect for advancing the didactic-mathematical study of a particular concept, as well as for contributing to well-founded teaching practice, is the analysis

of problem solutions that engage such a concept. When real students engage in problem-solving, their approach often differs from what is expected or intended by instructors. Identifying the meanings attributed to the mathematical objects used in these solutions, as well as anticipating possible discrepancies with what is institutionally considered appropriate, not only enriches the didactic-mathematical knowledge of the mathematics education research community but also has the potential to enhance teaching practice.

Burgos et al. (2021), from the perspective of the Ontosemiotic Approach (Godino, 2024), study the ontosemiotic complexity of the notion of the Definite Integral, focusing on the practices involved in the initial encounters with this concept and, through the analysis of the elementary practices present in the definitions provided by textbooks. This analysis allows them to identify potential implicit semiotic conflicts in the treatment of this object. In a similar vein, de la Fuente et al. (2011) identify meanings for the object Improper Integral by analysing Spanish high school (16-18 years old students) textbooks, in which they recognize four epistemic configurations for the concept of integral and analyse potential semiotic conflicts a priori. The specialized literature reviewed shows a relative scarcity in the study of semiotic conflicts surrounding the improper integral. It is considered necessary to develop research that addresses, deepens, and diversifies didactic-mathematical knowledge about it (Burgos et al., 2021; de la Fuente et al., 2011; González-Martín, 2005). Aligned with these results and considerations, in this paper we address our main research question: which specific semiotic conflicts may arise in a mathematics classroom in which Improper Integrals are being studied?

Objective

The aim of this paper is to produce partial responses to our research questions by anticipating possible semiotic conflicts in the resolution of a mathematical problem, where hypothetical students must engage with meanings associated with the Improper Integral. This forms one of the potential contributions of applying ontosemiotic analysis to the resolution of a mathematical problem, within the context of teacher education and professional teaching practice. To this end, four institutionally appropriate solutions (available at https://goo.su/VHqo) to a problem posed to prospective mathematics teachers are taken as a starting point. An ontosemiotic analysis is applied to these solutions, which enables the anticipation of semiotic conflicts from an institutional perspective.

THEORICAL FRAMEWORK

The Ontosemiotic Approach (OSA) (Godino, 2024) is grounded epistemologically in the pragmatic nature of mathematical knowledge, which encompasses both a set of practices and a network of interrelated objects and processes which are institutionally contextualized. For the sake of brevity, within this framework, an *object* is any entity that participates in a mathematical practice. The idea of practice evokes any act done by someone to solve a mathematical problem, communicate its solution, validate it or

generalize it. To solve a specific problem, for example, the practician evokes a chain of such practices, which are individually considered as 'elementary practices' within the 'system of practices' associated and relative to that problem in the specific institution in which is involved. The idea of institution is taken in a deliberately loose sense, focusing on a community of people with shared interests, methodologies and ways-of-doing. These consideration, when the mathematical or educational institutions are taken into account, encompasses crucial aspects such as epistemic traditions, normative elements and ecological constraints.

The overall meaning of a mathematical object is then understood through the types of problems in which it is involved, and the different practices employed to solve them, together with a specific institutional setting. The construct of "ontosemiotic analysis" allows the researcher to take into account the six primary objects of the OSA: (a) the elementary practices, which consist of the different parts into which the resolution procedure can be divided; (b) the use and intentionality of these practices, referring to the purpose each part of the resolution serves or directs towards; (c) the objects referred to in the practices, which requires identifying the concepts, propositions and arguments activated during the resolution; and (d) the processes carried out to reach the solution. The correlated identification of these objects and processes is presented in a table, where the recognition of each elementary practice guides the manifestation of the other objects and processes involved in it.

Within this framework an ontosemiotic analyses was conducted (available at https://goo.su/A6Om) based on four different solutions to a problem concerning improper integrals. Furthermore, from the perspective of the OSA, the notion of "semiotic conflict" is interpreted as any disparity or discordance between the meanings attributed to an expression by two subjects (individuals or institutions). If the disparity occurs between institutional and personal meanings, this is referred to as an epistemic semiotic conflict, whereas if the disparity arises between practices that constitute the personal meaning of a single subject, this is referred to as a cognitive semiotic conflict. When the disparity occurs between the (discursive and operative) practices of two different subjects in communicative interaction (for example, student–student or student–teacher), this is referred to as an interactional semiotic conflict.

METHODOLOGY

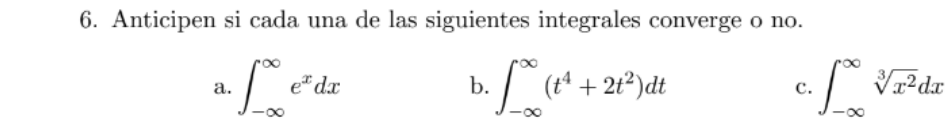
The present work consists of a content analysis (Krippendorff, 2019) applied to the ontosemiotic analyses referenced in Igolnikov and Rivas (2025). According to Krippendorff (2019, p. 20): "Content analysis is a research technique for making replicable and valid inferences from texts (or other meaningful matter) to the context of their use." Therefore, based on a problem regarding improper integrals -that will be shortly presented-, a content analysis is conducted on both the statement and the ontosemiotic analyses of four different solutions, with the aim of identifying possible semiotic conflicts. As will be seen in the following section, the content analysis of the

problem statement allows for the anticipation of potential interactional semiotic conflicts by identifying possible intentions or expected responses from the teacher in relation to the problem in question. Then, taking into consideration the ontosemiotic analyses of the various correct solutions, a content analysis of these is performed to identify possible discrepancies between those recognized in the ontosemiotic analysis and those that could arise in a classroom setting. To this end, the focus is placed on possible variations in some of the elementary practices involved in the solution, both from the perspective of the mathematical objects at play and the processes involved. This allows for the anticipation not only of cognitive semiotic conflicts but also epistemic and interactional ones, as shown in the following section.

In order to facilitate the inferences drawn from the content analysis applied to the ontosemiotic analyses of the four solutions considered, the tables of the ontosemiotic analyses of the initial elementary practices, in which the use of objects and processes involved takes place, are presented. For example, Table 1 presents the ontosemiotic analysis of the first elementary practice in solution R1. Likewise, the ontosemiotic analyses of the first elementary practices of the other solutions in question are presented. Due to space constraints, it has not been possible to include the analyses of the remaining elementary practices of the solutions analysed.

Results

To initiate the content analyses that will allow for the identification of potential ontosemiotic conflicts, the statement of the problem under study is presented. After a general initial analysis of the problem, attention is focused on b), for which possible conflicts will be identified.


6. Anticipen si cada una de las siguientes integrales converge o no.

a. $\int_{-\infty}^{\infty} e^x dx$ b. $\int_{-\infty}^{\infty} (t^4 + 2t^2) dt$ c. $\int_{-\infty}^{\infty} \sqrt[3]{x^2} dx$

Figure 1: Statement of the problem. The text reads: Anticipate if each of the following integrals converge or not.

It can be observed in Figure 1 that the three proposed improper integrals are defined over the entire real line. Thus, solutions that involve the calculation of limits will require considerably extended resolution procedures. This can be seen, for example, when comparing the solutions to the integral in item b presented in https://goo.su/VHqo. Likewise, this effect is reflected in the ontosemiotic analyses of solutions R3 and R4 for this same item. Instead of calculating limits, it can be recognized that the integrated functions possess the property of being unbounded, which could allow one to conclude their divergence without the need for exhaustive calculation. Related to these considerations is the fact that the problem statement asks to "anticipate" rather than "calculate". In this way, it is inferred that the focus is on solving the problem through a more qualitative analysis, avoiding an overly extensive calculation process rooted in algebraic manipulation. To achieve this, it is necessary

to carry out one of the first two elementary practices of R1 or R2 (see Table 1 and Table 2), as these are the ones that allow for this type of analysis.

Elementary practice to solve the task	**Use and intentionality of the practice**	**Objects involved in the practice**	**Processes**
1. The integrated function $f(x) = x^4 + 2x^2$ is a non-negative function, as it is expressed as a sum of non-negative terms.	Analyze the non-negativity of the function.	Concepts: non-negative function, summand. Propositions: the sum of non-negative expressions is a non-negative expression. Monomials of even degree are non-negative. Argument: the summed expressions are monomials of even degree.	Particularization: the particular function is analysed. Decomposition: the function is expressed as a sum of other two functions.

Table 1: Ontosemiotic analysis of the first Elementary practice of R1

In the case of not making use of the elementary practices presented in Tables 1 and 2 and choosing instead procedures that are predominantly algebraic in nature (such as those presented in Table 3 corresponding to solution types R3 and R4), a possible interactional ontosemiotic conflict is inferred, associated with the disparity of meanings attributed to the problem by the teacher and the student. It should be noted that this conflict may occur due to interpretations or difficulties related to carrying out practices R1 or R2, which leads to an approach different from that expected by the teacher.

It is necessary to point out that this disparity does not necessarily lead to an erroneous mathematical practice. That is to say, the manifestation of the referred interactional conflict does not imply that a mathematical error is incurred.

Elementary practice to solve the task	**Use and intentionality of the practice**	**Objects involved in the practice**	**Processes**
1. If the integrated function is non-negative, the question of whether the integral converges or diverges can be interpreted as asking whether the area of the region between the graph	Shift perspective, transform the question, guide the solution toward anticipation.	Concepts: Non-negative function, convergence, divergence, area under the curve. Propositions: If the function is	Signification: The object Improper Integral, and the question, is conceived within a geometric–cartesian framework.

of the function and the x-axis is finite or infinite. If the area under the curve is infinite, then the improper integral diverges.		non-negative, the result of the integral is the area under the curve. If the area under the curve is infinite, then the improper integral diverges.	

Table 2: Ontosemiotic analysis of the first Elementary practice of R2

When analysing the possible reasons why this conflict may arise, several emerge, including: (a) paying little attention to the statement and not considering/responding to the request "Anticipate"; instead, the focus is placed on what is presented by the symbolic representation of the integral (as posed in items a., b., c. of the statement) to be calculated; (b) interpreting the request "Anticipate" as an action that requires carrying out some calculation procedure beforehand; (c) difficulties associated with the development of the practices identified in resolutions R1 or R2, for example: using the involved properties or constructing the graphs of the integrated functions; (d) not considering it necessary to analyse the integrated function to recognize properties/propositions that could lead to solving the problem and only using the properties/propositions involved in resolutions R3 and R4, whose initial elementary practices are referred to in Table 3.

Elementary practice to solve the task	**Use and intentionality of the practice**	**Objects involved in the practice**	**Processes**
1. Since the integral is taken over the entire real line, it is necessary to decompose it as a sum of integrals, by means of…	Apply a decomposition of the improper integral into two improper integrals.	Concepts: improper integral, real line, domain of integration Proposition: the integral over the real line of a function is equivalent to the sum of the integrals obtained by splitting the real line into two half-lines. Procedure: decomposition of the integral as the sum of improper integrals.	Particularization: the expression for an integral over the real line is used, applied to the specific case at hand. Decomposition: the integral to be calculated is written as the sum of two integrals, which, when "combined," yield the integral in question.

Table 3: Ontosemiotic analysis of the first Elementary practice of R3 and R4

A content analysis applied to the ontosemiotic analysis carried out on resolution R1, according to its first elementary practice shown in Table 1, allows us to infer that this

practice is characterized by the recognition of two essential properties/propositions: (a) the non-negativity of each summand and (b) the non-negativity of the integrated function, which is expressed as the sum of two non-negative summands. At the same time, the appropriate use of these two propositions requires employing the processes of particularization and decomposition (see Table 1). If a hypothetical student does not recognize the usefulness of analysing the function in terms of the properties of the summands and, as a result, does not have or does not utilize the relationship between the exponent of the monomials (summands) and the positivity of each of them, the processes of particularization and decomposition/unitarization described in the ontosemiotic analysis (Table 1) would not take place in his/her practice. For this student, it can be inferred that an interactional semiotic conflict may manifest, if it is interpreted that the professor's choice of the function to be integrated is intended for the student to recognize the usefulness of addressing this object within the analytical-functional or geometric framework by reading the given expressions (Arcavi, 1994).

It can be inferred from the above that the possible causes for the emergence of this second interactional semiotic conflict, as evidenced in the ontosemiotic analysis, lie in the recognition and use of the objects (concepts, propositions) and processes (particularization, decomposition) involved in an analytical-functional resolution of the improper integral.

The case of resolution R2 is characterized by the evocation of the representation in the cartesian plane. If the hypothetical student responds that the integral converges based on “the mistaken belief that a definite integral is always an area, and therefore must yield a [finite] positive value” (González Martín, 2005, p. 424), an epistemic semiotic conflict may arise in relation to the institutional meaning assigned to the integral in question. This same consideration allows for the identification of another possible semiotic conflict, this time cognitive: procedures such as those in R1 and R3 may enable the student to obtain the correct result, although it may be inconsistent with the aforementioned mistaken conception.

In the case of resolutions R3 and R4, it may happen that a student, when attempting to carry out the calculation, does not have the proposition: “the integral over the real line of a function is equivalent to the sum of the integrals obtained by splitting the real line into two half-lines.” (Table 3). In this situation, one of the following analytic-algebraic propositions might come into play, as a result of extending the available practices:

P1: $\int_{-\infty}^{+\infty} (x^4 + 2x^2)dx = \lim_{t\to+\infty} \int_{-t}^{t} (x^4 + 2x^2)dx$

P2: $\int_{-\infty}^{+\infty} (x^4 + 2x^2)dx = \int_{a}^{b} (x^4 + 2x^2)dx$

P3: $\int_{-\infty}^{+\infty}(x^4+2x^2)dx=\lim_{x\to+\infty}\left(\frac{x^5}{5}+2\frac{x^3}{3}\right)-\lim_{x\to-\infty}\left(\frac{x^5}{5}+2\frac{x^3}{3}\right)$ (Martínez Abarca, 2008, p. 81)

These manifestations would give rise to disparities between institutional meanings (the use of the proposition referred to in the ontosemiotic analysis, Table 3) and the student's personal meanings (the use of propositions that differ from that proposition), so the conflict occurring in this case is epistemic in nature.

When investigating the possible reasons for the manifestation of this conflict, in addition to the possibility of intuitive use (Fischbein, 2002), at least four can be inferred, the last three depending on the first: (i) not possessing the objects (mainly the proposition associated with the previously mentioned additive decomposition) and processes (particularization and decomposition, also referred to in Table 3); (ii) making use of proposition P1, which coincides with to what is institutionally recognized as the Cauchy Principal Value, whose manifestation may be due to the use of an extension of the practice involved in the calculation of Type I improper integrals, which can be visualized as a substitution of $\int_a^{+\infty} f(x)dx$ by the compound expression $\int_a^t (x^4+2x^2)dx$, as well in the case of $-\infty$, which, by making both substitutions unitary, leads to the statement of proposition P1; (iii) making use of proposition P2, which coincides with the notion of a limit in two variables, relying on the typical roles played by the letters a and b in the definitions of $\int_a^{+\infty} f(x)dx$ y $\int_{-\infty}^{b} f(x)dx$ applied to $\int_{-\infty}^{+\infty}(x^4+2x^2)dx$ which, when unitarized and applied as in Type I integrals, leads to the statement of proposition P2; (iv) making use of proposition P3 identified from the findings reported by Martínez Abarca (2008, p. 18).

Regarding the manifestation of this conflict, it should be noted that the extension of an institutionally valid practice produces partially correct responses or, in any case, responses with a scope of validity not known in advance. In this way, the emergence of conflicts that do not result in mathematical errors becomes evident and, on the contrary, reveal innovative processes of mathematical generalization.

Discussion and conclusions

The findings of this study reveal the potential of ontosemiotic analysis for identifying semiotic conflicts (interactional, epistemic, and cognitive) that may arise in the context of teaching and learning the improper integral.

de la Fuente et al. (2011) identified a priori semiotic conflicts based on epistemic configurations regarding the improper integral. Burgos et al. (2021) also identify potential conflicts surrounding the definite integral in textbooks. In this work, where semiotic conflicts related to the improper integral are identified, a different and possibly complementary methodology to those authors has been adopted; through content analysis (Krippendorff, 2019) applied to the ontosemiotic analysis of institutionally appropriate practices (Igolnikov and Rivas, 2025), possible variations or disparities are identified between the meanings assigned by hypothetical students in the implementation of their personal practices. It has been observed that this methodology, complemented by a review of specialised literature, enables the production of didactic-mathematical knowledge necessary for its teaching, which may be useful both for the professional practice of teachers and for those in initial teacher training.

During the process of identifying the aforementioned conflicts, it has become evident that their manifestation may be due to disparities between the meanings of the objects and processes involved in problem-solving, which do not necessarily lead to mathematical errors. That is, the analyses carried out make it possible to establish a distinction between the notion of conflict and that of error, which are sometimes conflated. This fact should draw the attention of teaching staff, who, based on didactic-mathematical knowledge such as that presented in this document, can make informed decisions.

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Beyond Procedures: A Look at Students' Understanding of Applied Optimization Problems in Introductory Calculus

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An understanding of first-semester calculus is a prerequisite for success in STEM disciplines. However, many students find it challenging to achieve the deep conceptual understanding necessary for topics that require reasoning and modelling, especially when approaching applied optimization problems. These problems are particularly valuable as they allow students to recognize the real-world applicability of fundamental calculus concepts. A teaching experiment involving a series of tasks was developed and implemented with first-semester college calculus students, aiming to help students connect new learning of the optimization concept with their prior knowledge. This paper highlights findings of this pedagogical strategy, offering insights into effective methods for supporting student learning of this calculus topic.

Keywords: Calculus, Applied Optimization, Teaching Experiment, Prior Knowledge.

INTRODUCTION AND BACKGROUND

Success in Calculus I is seen as both a measure of mathematical aptitude and a predictor of future achievement in higher-level STEM coursework. Therefore, calculus is often viewed as a gateway to STEM majors (Baisley & Adams, 2019; Bressoud et al., 2015; Rasmussen & Ellis, 2013). Solving applied optimization problems is a key topic in Calculus I courses, which presents a significant challenge for many first-semester calculus students. These problems, often framed as real-world scenarios, require the integration of multiple mathematical notations and calculus techniques. Students frequently struggle with the complexities of understanding, navigating, and ultimately effectively solving these types of problems. According to Abadir and Maher (2025), coupling a fast-paced curriculum with traditional lecture-style instructions does not provide students with enough time to gain a conceptual understanding of optimization objectives and processes. This lack of depth is evident in the problem-solving phase; Abadir and Maher (2024) and Swanagan (2012) observed that students often go through an aimless phase where they verbalize strategies, but are unable to properly implement the approach and tend to rely on procedures that fail to produce a correct solution. These actions indicate that, while students understand the procedural demands, they lack the structural knowledge to efficiently navigate them. Furthermore, LaRue (2016) and White and Mitchelmore (1996) suggested that these challenges stem from students' inherent tendency to view variables as mere symbols rather than dynamic quantities. This conceptual gap is further demonstrated by Mkhatshwa (2016),

who discovered that students typically fail to distinguish between relative and absolute extrema in economic contexts and that students demonstrated a lack of quantitative reasoning by interpreting all extreme values as relative extrema problems, neglecting domain-based justifications. Taken together, these findings underscore a critical gap in traditional calculus instruction, highlighting an urgent need for a novel pedagogical approach that prioritizes conceptual depth and helps students bridge the divide between procedural recall and meaningful problem-solving.

To address that critical need, a teaching experiment was conducted during the fall 2024 semester to facilitate Calculus I students' learning of applied optimization problems by promoting the construction of a knowledge representation structure aiming to help students connect new knowledge to their existing mathematical knowledge. Four pairs of students participated in a series of five extra-curricular sessions, from which one was selected as a representative sample for detailed analysis as presented in this paper to address the following research question: How did two Calculus I students integrate and leverage prior knowledge to make sense of and successfully solve an applied optimization problem?

THEORETICAL FRAMEWORK

This study builds on Davis's (1984) cognitive science approach to mathematics learning, which emphasizes building strong knowledge structures by connecting new ideas to students' existing mental representations. For calculus-applied optimization problems, these pre-existing mathematical ideas include knowledge about variables, functions, derivatives, and extreme values. Davis and Vinner (1986) argued that students learn calculus best when they understand the meaning behind the symbols. They also noted that important ideas should be taught in a way in which students can actively develop simple, yet powerful, ways of thinking about the concepts. The goal is to help students build complex knowledge that they can easily access and use in diverse situations. However, historically, mathematics instruction has emphasized the use of fixed universal procedures that centralize the mathematics instead of placing individual students and their growth in mathematical knowledge at the heart of the learning process (Louie et al., 2021). Traditional lecturing in higher-level mathematics often results in what Skemp called "instrumental understanding" (1976), yielding the ability to apply procedures without true conceptual understanding. Hagman (2017) asserted that the breadth of topics covered in calculus courses and the pressure of common final exams create significant time constraints for instructors, often forcing a shift toward procedural coverage at the expense of conceptual depth. This superficial understanding tends to be fragile and hinders knowledge transfer to novel problems, ultimately limiting mathematical proficiency. By drawing on Davis's (1984) cognitive science perspectives, this study aimed to investigate how a thoughtfully designed intervention can improve calculus students' understanding of applied optimization, considering not only how students use language to express their ideas, but also other types of representations, and the role of reflection in their learning.

METHODOLOGY

This study drew upon design research methodology (Cobb et al., 2003; Creswell & Creswell, 2017; Gravemeijer & Prediger, 2019; Sandoval, 2014) to create an intervention for learning to solve applied optimization problems. Students worked in pairs during five task-based sessions. This decision was grounded in Vygotsky's (1978) theory of the social construction of knowledge, recognizing that learning occurs first socially and then individually. An extensive body of research supports the benefits of paired problem-solving activities for developing mathematical reasoning (Maher, 2005; Weber et al., 2010; Schoenfeld, 2022). Students, working in pairs, provide opportunities to test conjectures, justify approaches, and build upon each other's ideas. Engaging in peer discourse and exploring together encourages ownership of learning. Students working with a partner can offer a more manageable social dynamic (Slavin, 1996), facilitating the establishment of learning norms and maximizing time spent on cognitive processes (Webb, 1989).

A sample of eight Calculus I students was recruited during the 2024 fall semester at a large research university on the East Coast of the United States. These participants, assigned pseudonyms for data analysis, were paired to engage in five task-based research sessions. Each session utilized carefully designed tasks to address previously identified conceptual gaps as discussed in earlier sections of this paper, fostering a learning trajectory that connected new learning of the applied optimization concept to prior knowledge.

The first session focused on two fundamental tasks: translating a verbal representation into symbolic function notation and then negotiating its domain in relation to a specific quantity. The second task was to obtain a function's derivative using several derivative techniques. In the second session, students explored how to determine the extreme values of functions, distinguishing between absolute and relative extreme values. Students' initial encounter with an optimization problem occurred during the third session, allowing them to build on the knowledge they gained in the prior two sessions and motivating them to construct a general method without receiving formal instructions about the topic. In the fourth session, the students solved an optimization problem independently before engaging in collaborative error analysis; in the final session, they had the chance to apply their knowledge of optimization to create a physical object of a box with the maximum volume.

The first author was the primary researcher and facilitated the task-based sessions, taking a low-impact approach to allow students to navigate the tasks without interference. She purposefully avoided interrupting the participants' active work, allowing their natural problem-solving processes and collaborative dynamics to continue uninterrupted. However, at the end of each exercise, she led the participants in a structured debriefing. During these debriefs, she used targeted, probing questions to learn about their underlying thought processes and to promote deeper thinking. This reflective approach enabled students to provide explicit arguments for their

mathematical choices, thereby solidifying their conceptual knowledge and closing the gap between procedural performance and meaningful understanding.

Following established qualitative methods that include transcription, deductive and inductive coding, and thematic analysis, a data analysis framework based on Davis (1984) was adopted (Abadir, 2026). The analytic framework breaks down the cognitive process into observable behaviours per the coding scheme shown in Table 1, below. The primary goal was to illustrate how students leveraged their prior knowledge to make conceptual sense of novel concepts in applied optimization and successfully solve the problem task.

Code	Category	Description	Example
C-SM	Contextualize and Sense Making	Identifying given and missing quantities plus any posed conditions in the problem; Analyzing how quantities relate; Recognizing underlying mathematical concepts; Clarifying the goal and expected outcome	"So, there's a wall…, the barn. Do we know how long it is?"
UoR	Use of Representation	Using words, gestures, sketches, formulas, equations or graphs to represent the problem or any of its features	"So let's just draw a wall for the barn."
R-CPK	Retrieve and Connect to Prior Knowledge	Recalling a specific concept, idea, formula, or procedure along with how that piece of prior knowledge relates to a new idea or problem	"What is the equation for perimeter again, like, what: x plus x plus y plus y, so 2x plus 2y, right? "
APK	Applying Prior Knowledge	Applying an established rule or schema in solving the problem	"I think, if we can isolate one variable, like if we do 100 minus x is equal, or,..like 100 minus 2y is equal to x, and then substitute that back in and then derive

			from there. So, in one variable.”
CR	Conceptualizing Results	Understanding which mathematical principles are necessary at various stages in the optimization solving process as well as the rationale for their use	“A prime, but we're setting that equals zero because we want something for the maximum, right? Because that's the horizontal tangent line, right. When it's a maximum, it has a horizontal tangent line.”
JS	Justifying Solution	Being able to use calculus techniques to justify the optimal solution	“second derivative of the first derivative just negative four, which is concave down, which means there will be a maximum.”

Table 1: Coding framework used in the data analysis

FINDINGS

Data from one student pair in the third session are presented here as a representative sample of the patterns observed across all four participating pairs. Dan and Anna engaged in solving a classic "Farmer’s Fence" optimization problem, bridging their foundational knowledge to a real-world application. This session explored their ability to apply concrete concepts to a practical scenario.

> A farmer plans to make a rectangular pen for cattle. The wall of a barn will serve as one of the sides of the pen. He has 100 feet of fencing that he will use to create the other three sides. What are the dimensions of a pen of maximum area?

Throughout the session, Dan and Anna actively drew upon prior knowledge, employing various representations to solve the "farmer" problem. They sketched a visual representation, formulated an objective area function, and created a constraint equation based on the 100 feet of fencing. Subsequently, they successfully integrated

the constraint equation to express the objective function in terms of a single variable, enabling them to differentiate and identify critical numbers. Furthermore, they connected their analytical solution to the graphical representation, noting that the calculated value corresponds to where the function will have a horizontal tangent line. The following exchange and students' handwritten work provide an example of Dan's and Anna's processes through which they made sense of and solved the problem:

Dan: So, what key takeaways do you notice about? [asking Anna]

Anna: So, there's a wall…, the barn. Do we know how long it is?

Dan: No! It doesn't matter. I recommend just drawing a line. So, let's just draw a wall for the barn, and they told you he has 100 feet of fencing. since this has to be 100 feet, that's the perimeter… perimeter, right?

Anna: Perimeter, yes, excluding the side of the barn. Do we set the sides x and y?

Dan: What is the equation for perimeter again, like, what: x plus x plus y plus y, so 2x plus 2y, right?

Anna: Right! But we have one.

Dan: So, we have two y's, but only one x. So, P equals 2y plus one x, right? And we know P is 100, right? So, 100 equals 2y plus x, right? And we're told, what were the dimensions for the pen of the maximum area?

Anna: So, x times y.

Dan: The maximum is a horizontal tangent line [draws a concave down parabolic curve and indicates a horizontal line at its maximum] or maximum area over here, right? That's what they want to solve.

Figure 1: Dan's graphical illustration

Anna: We have two variables.

Dan: I think, if we can isolate one variable, like if we do 100 minus x is equal, or,..like 100 minus 2y is equal to x, and then substitute that back in and then derive from there. So, in one variable.

Anna: Okay, yeah, x equals 100 - 2y. Oh, yes, 100 - 2y times y. 100 y - 2y squared, and then you differentiate that.

Dan: Derive in terms of y, it's fine. A prime, but we're setting that equals zero because we want something for the maximum, right? Because that's the horizontal tangent line, right. When it's a maximum, it has a horizontal tangent line. So, x equals 50 and y equals 25.

Anna: Okay, we solved it. Why did we differentiate the area, not the perimeter?

Dan: In the task, it's asking you for the maximum area, not the maximum perimeter.

Anna: So, the perimeter function, we just used...

Dan: Yeah, we used it only for the variable.

Anna: Yeah, double --second derivative of the first derivative just negative four, which is concave down, which means there will be a maximum.

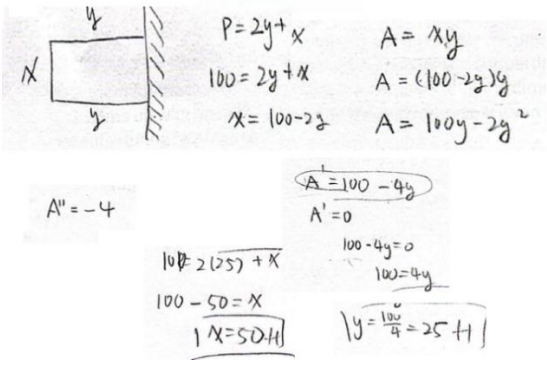

Figure 2: Anna's handwritten work

DISCUSSION AND IMPLICATIONS

The mathematical behaviours of Anna and Dan illustrate the promising potential of utilizing a pedagogical approach designed to prioritize conceptual depth in learning to solve applied optimization problems, bridging the gap between procedural recall and meaningful problem solving. While this study used a small sample of eight students, and these findings specifically reflect Anna's and Dan's engagement in the 'Farmer's Fence' task, the nature of their conversations—particularly their ability to deliberate over strategies and provide strong mathematical justifications—is revealing. By providing students with the collaborative space to defend their strategic choices, this intervention demonstrates a viable pathway for fostering the conceptual depth necessary to master applied optimization.

As Anna and Dan interpreted the problem, they recognized that their goal was to maximize the area of a rectangular pen with a fixed perimeter and one side defined by an existing barn wall and created a diagram as a visual representation of the situation. Anna's work (Figure 2) shows a diagram of the penned area adjacent to the barn, reflecting concrete understanding of how to utilize the available fence. Whereas the graphical connection that Dan made as they interpreted the problem was likely

stimulated through his experience during session two, in which students worked on tasks prompting them to reason about how critical numbers are possible locations for the existence of extreme values in where the function has a horizontal tangent line or not differentiable. Each student in the pair initially was attuned to different aspects of the problem in a complementary way. Through talking aloud while assigning variables and establishing the relationship between the pen's dimensions and the given perimeter, Anna and Dan demonstrated foresight regarding the procedural techniques required for differentiation. Drawing on their previous calculus learning in their regular classes and tasks they completed in earlier intervention sessions, they recognized the need to express the standard area formula ($\boldsymbol{A = xy}$) in terms of a single variable, y, so that they could differentiate the area function as a step in finding the maximum. To find critical numbers, they equated the derivative equation to zero and solved for the dimensions that could yield to the maximum area. The students' dialogue revealed a sophisticated synthesis of symbolic and graphical representations. By characterizing the area function as a downward parabola (Figure 1), Anna and Dan bridged their algebraic work with a visual conceptualization of the problem. They successfully connected the property of a horizontal tangent line at the parabola's vertex to the requirement that the derivative must equal zero at the absolute maximum. This connection demonstrates an ability to draw upon prior calculus knowledge to validate their procedural steps through a multi-representational lens. Additionally, Anna's and Dan's ability to justify why they differentiated the area function rather than the perimeter equation demonstrated a clear awareness of the problem's objective. This distinction suggests a successful transfer of knowledge from the intervention's earlier tasks, where they discussed the importance of identifying problem's goal and mathematical processes needed to achieve it. Their reasoning reflects a move beyond mechanical application of rules toward a conceptual understanding that the function being optimized must be the one subjected to the derivative test. To finalize their solution, Anna and Dan applied the second derivative test, using the resulting negative value and the concept of concavity to verify that their dimensions yielded a maximum area instead of a minimum. This final step demonstrated an internalized expectation for the role of justification. Their commitment to reasoning about the nature of the extreme value rather than merely accepting a procedural result reflects the impact of the researcher's debriefings from previous sessions, which emphasized the validation of solutions within the specific context of the task.

These findings suggest that when students engage in well-designed, meaningful mathematical tasks—and are given structured opportunities to reflect on prior learning—they are better able to connect pre-existing knowledge to new ideas. It is noteworthy that the researcher-led debriefings in earlier tasks helped the students internalize a standard for rigor, making such prompts unnecessary in later sessions. Anna and Dan began leading these discussions themselves, holding themselves accountable by proactively questioning their strategies and articulating justifications for their mathematical actions. Such reflective engagement supports the development of deeper conceptual understanding, particularly for complex topics like applied

optimization, and it can become a normative behaviour for doing mathematics at the Calculus I level. Anna and Dan's successful solution and justification of the maximum-area configuration in the Farmer's Fence Problem illustrate the effectiveness of this pedagogical approach. Their work demonstrates how students can not only identify the mathematical ideas needed at each stage of an optimization problem but also articulate why each idea is applicable. Moreover, their ability to connect the algebraic solution to its graphical representation, specifically the horizontal tangent at the extremum, underscores the value of integrating multiple representations to reinforce core calculus concepts. The challenge to mathematics educators lies in creating the space for engaging students in the tasks used in this teaching experiment, which likely will necessitate providing professional development for the teaching and learning assistants who typically facilitate the recitation sections to support their implementation efforts.

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TWG1B: The teaching and learning of proof and algebra

Learning to think and teach functorially: categorical covariation and effective noticing in Advanced Mathematics Education

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This study examines how university algebraic topology courses can simultaneously develop advanced mathematical thinking and representational competence for teaching. Through an Inquiry-Based Learning intervention, we investigate the development of categorical representational competence, the ability to recognise and manipulate systematic relationships between algebraic and geometric representations. Using a multi-layered observation design, we analyse how students develop functorial thinking and categorical covariational reasoning and how observer groups transform these insights into pedagogical knowledge through embodied, relational noticing. Results demonstrate that explicit attention to functorial connections supports both mathematical maturity and the development of pedagogically-oriented awareness.

Keywords: Advanced mathematical thinking, categorical representation, functorial thinking, teacher noticing, algebraic topology.

INTRODUCTION

Recent research demonstrates that university mathematics courses can strengthen connections between advanced and school mathematics (Wasserman et al., 2023). In this perspective, advanced mathematics education serves dual purposes: developing mathematical maturity (Miranda, 2024) while shaping pedagogically-oriented mathematical thinking Wasserman et al., 2023). Developing representational fluency across mathematical domains, for instance the ability to move systematically between geometric and algebraic languages, is a central challenge in advanced mathematics education. This study addresses advanced transitions within and across university courses and paths (Gueudet et al., 2016), focusing on algebraic topology where students coordinate algebraic and geometric representations through functorial relationships: thinking moves on transformations of categorical registers into others (Duval, 2017; Tall, 2013). The fundamental group, introduced by Poincaré (1895), provides an ideal context: it establishes a functorial connection between topology and algebra, assigning group structures to spaces in ways that these algebraic objects are topologically invariant. it. The fundamental group classifies spaces by analysing loops, capturing 1-dimensional homotopy information by forming a group of equivalence classes of closed paths, where two loops are equivalent if they can be continuously deformed into each other. The aim is to support undergraduates' transition from bachelor's to master's level, developing the representational fluency that characterises mathematical thinking and that, in courses including prospective teachers, can inform future pedagogical practice.

Research Question: *How can university algebraic topology instruction support advanced mathematical thinking through categorical representational competence,*

and how can this competence inform the pedagogical awareness of students who are also prospective teachers?

Specifically, we examine how students develop functorial thinking and categorical co-variational reasoning when working with fundamental groups and covering spaces, how observation experiences transform into pedagogically-oriented noticing, and what instructional principles support both mathematical maturity and teacher noticing.

THEORETICAL FRAMEWORK

Our framework integrates two primary theoretical strands: categorical representational competence (grounded in Duval's semiotic framework and Tall's three worlds) and teacher noticing in its enactivist reconceptualisation.

Categorical representation and covariation.

According to Duval (2017), the analysis of mathematical knowledge is in its essence the analysis of the cognitive synergy between different kinds of semiotic representation registers, because the mathematical way of thinking and working is based on transformations of semiotic representations into others. Based on this assumption, he proposes the use of semiotics to identify and develop the specific cognitive processes required for the acquisition of mathematical knowledge.

> The transformations of semiotic representations are at the heart of the mathematical way of working, and to teach it is crucial to develop students' cognitive skills to put different representations in correspondence (Duval, 2017).

Specifically, *treatment* is transforming a representation within the same register, while *conversion* is transforming a representation from one register to another.

Building on Duval's (2017) theory of semiotic representations, we define *categorical representation* as a meta-representational framework encompassing systematic, structure-preserving relationships between different mathematical languages connecting domains. Unlike Duval's *conversion*, categorical representation focuses on *functorial connections*: how objects and morphisms in one category systematically correspond to those in another while preserving essential structures.

We introduce *categorical covariation* as the ability to recognise, coordinate, and manipulate systematic variation objects and relationships between representational categories while maintaining structural coherence (extending Thompson & Carlson, 2017). Unlike traditional covariation, and recent development (Bagossi, 2024), which concerns quantities varying simultaneously, categorical covariation concerns how mathematical *structures* (objects in the category of topological spaces) transform while preserving essential relationships across representational categories (Miranda, 2026).

In algebraic topology, this manifests as understanding how topological structures' properties covary with algebraic ones in functorially preserved ways; for instance, how the geometric periodicity of the circle covaries with the algebraic cyclicity of the integers via $\pi_1(S^1) \cong \mathbb{Z}$, besides how topological spaces covary with their fundamental groups. It is important to note, however, that the fundamental group functor π_1 is

covariant but neither full, nor faithful, nor exact in the categorical sense: not every group homomorphism arises from a continuous map, and the functor does not preserve all structure. This has significant consequences: it means that while the functor translates topological problems into algebraic terms, the translation is partial and directional. Recognising this asymmetry is itself part of developing *functorial thinking*: students are guided to understand not only that continuous maps induce homomorphisms, but also what is *not* preserved, and why. This epistemological nuance distinguishes genuine functorial thinking from mere analogy.

These concepts connect to Tall's (2013) "three worlds of mathematics", that is, embodied-conceptual, proceptual-symbolic, and formal-axiomatic, by emphasising how students can be lead to build explicit bridges between them through structure-preserving relationships, moving fluidly from embodied experience (rope-wrapping) through symbolic manipulation (group computation) to formal axiomatic reasoning (covering space theory). Categorical covariation names the cognitive capacity that makes this movement systematic rather than ad hoc.

Noticing: from AID to an Enactivist Perspective.

Teacher noticing, *attending to, interpreting, and deciding* (AID) in response to significant classroom features, is well established as central to effective teaching (Jacobs et al., 2010; Mason, 2002, 2025). However, as Coles and Helliwell (2025) note, the AID framework was developed within a cognitive-psychological paradigm that treats noticing as an individual, internal process. Their enactivist reconceptualisation proposes instead that noticing is co-emergent and relational, arising through embodied interaction between teacher, students, and environment.

Rather than replacing the AID framework, we use it as a descriptive structure (what did the observer attend to, interpret, and decide?) while interpreting the *quality* of noticing through an enactivist lens: we ask not only *what* was noticed but also *how* it became noticeable, through embodied participation, relational coupling, and co-emergence rather than detached observation.

We adopt the term 'effective noticing' to describe noticing that is embodied, relational, and responsive to the co-emergent nature of mathematical activity in line with Coles and Helliwell (2025). In this view, body and gesture *constitute* rather than merely illustrate mathematical meaning-making. This perspective is particularly relevant in algebraic topology, where students' rope-wrapping, path-tracing, and bodily enactment of covering projections are integral to mathematical thinking, not supplementary.

METHODOLOGY

Twenty third-year Bachelor's mathematics students at an Italian university participated during their 2024–2025 Algebraic Topology course. The cohort included students pursuing both pure mathematics degrees and potentially the mathematics teacher education track, as among them a high percentage of students (80%) chose this degree programme because they would like to teach mathematics in high school in the future;

this composition meant that developing representational fluency was relevant both as mathematical maturity and as nascent pedagogical awareness. The students have already completed the two-year program, during which they acquired the foundations of calculus, algebra, and geometry. They had completed General Topology covering topological spaces, continuous functions, and topological properties.

The Inquiry-Based Learning design (Artigue & Blomhøj, 2013) centred on the question: "Given two topological spaces, how can we tell if they are homeomorphic?" Participants organised into: University Teacher (UT) who designed and facilitated sessions; Solver Groups (G1, G2, G3) who engaged in problem-solving, designed peer lessons, and presented them; and Observer Groups (O1, O2, O3) who observed respective solver groups with explicit focus on mathematical thinking and pedagogical moves, then designed and implemented lessons for second-year students (S, about 50).

The instructional sequence:

(1) Launch: open-ended problem posed;

(2) Exploration: small-group work with manipulatives;

(3) Interim sharing: groups shared emerging ideas;

(4) Formalisation: teacher-facilitated synthesis;

(5) Designing and leading: solver groups designed and implemented lessons to peers;

(6) Interim sharing: pedagogical discussion;

(7) Designing and implementing: observer groups designed and implemented lessons for second-year students;

(8) Final reflection on learning and teaching experiences.

DATA COLLECTION AND ANALYSIS

Data included video recordings, photographs of student work, written solutions, lesson plans, observer field notes, and post-course questionnaires. Mathematical Learning Analysis examined discourse, work, and representations for evidence of functorial thinking and categorical covariational reasoning. We identified episodes showing connections between topological and algebraic structures, coded representational activities (embodied, diagrammatic, symbolic, formal), and analysed transitions for structure-preservation awareness.

Teacher Noticing Analysis used the AID framework (Jacobs et al., 2010) as a descriptive scaffold, tracking three observational flows: O→UT (observers noticing the university teacher), O→G (observers noticing peer groups), and O→S (observers noticing second-year students). For each flow, we coded AID dimensions (attending, interpreting, and deciding) and additionally examined enactivist dimensions: whether the observer's noticing arose through embodied co-participation (mirroring gestures and feeling mathematical tensions) rather than detached observation. We traced progressions from *pragmatic noticing* (embodiment engages students) through

epistemological noticing (embodiment creates understanding) to *effective noticing* (embodiment constitutes mathematical meaning). Given the small sample size (n=20, with data drawn primarily from groups O2 and O3), findings are interpreted as illustrative rather than generalisable, and conclusions are framed accordingly.

RESULTS

Selected episodes illustrating the development of functorial thinking, categorical co-variational reasoning, and noticing across the instructional sequence, are presented.

Episode 1: Embodied Discovery of Invariants.

When posed "How can we tell if a sphere and a torus are homeomorphic?", students explored with physical manipulatives (rope and surfaces). G1 observed:

G1: Both compact and connected, everything suggests they're the same! But…

G3 discovered through rope manipulation:

G3: Imagine a closed path on the sphere. You can continuously deform it, shrinking to a point from any position! But on the torus, sometimes it remains a circle; you can't shrink it.

This embodied exploration provided cognitive roots for the fundamental group; that is, a first-contact intuition for the concept of contractibility. Students physically distinguished loop behaviour on the two surfaces, discovering that on spheres all loops contract while on tori some do not. Importantly, this physical activity provided the experiential grounding from which the formal concept of homotopy equivalence would later be constructed. It was a beginning, not yet a mathematical understanding. The conceptualisation of homotopy equivalence as a precise equivalence relation required subsequent formalisation, as shown in Episode 2.

Episode 2: Formalising Through Covariation.

After embodied exploration, UT facilitated formalising the fundamental group $\pi_1(x, x_0)$ as equivalence classes of loops based at x_0 with representatives path composition class as group operation. Students worked to connect formalisation with intuitions.

G2: On the sphere, all loops are equivalent, they can deform to each other, so there is only one equivalence class, the identity. $\pi_1(S^2) = \{e\}$.

G1: For the torus, loops wrapping horizontally can't deform to those wrapping vertically. Multiple equivalence classes. In fact, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ with each pair (m,n) representing a loop wrapping m times horizontally and n vertically. Can they be homeomorphic? Sphere and torus? They are not!

UT: Compare 'being homeomorphic' with 'having the same fundamental group'.

G3: If spaces are homeomorphic, they have the same fundamental group. But different fundamental groups guarantee non-homeomorphic… So, it's a tool for proving non-homeomorphism!!

This episode illustrates categorical covariational reasoning: students articulated how geometric structure (holes) systematically determines algebraic structure (the group)

and understood the *directionality* of this relationship: homeomorphism implies group isomorphism, but not conversely. Students created tables mapping topological spaces to their fundamental groups, a representational activity that itself enacts categorical covariational reasoning: the table is a diagrammatic representation of a functorial correspondence. They also grasped a key limitation of the functor: π_1 is not faithful, so the contrapositive is the useful direction for distinguishing spaces, not a biconditional.

Episode 3: Computing via Covering Spaces.

With fundamental groups defined, the question became computation. One of the strategies to adopt is based on the Covering Spaces Theory. UT began introducing covering spaces starting from a specific manipulative example: wrapping a wire ($\mathbb{R}$) around a cylinder; and, discussing the mathematical properties of the map projecting every point on the wire in the base circle of the cylinder (S^1), the identified collaboratively the exponential map $p: \mathbb{R} \to S^1$, $p(t) = e^{2\pi it}$. G2 analysed:

G2: Geometrically, every point of S^1 has a small neighbourhood whose preimage under p is a disjoint union of intervals in $\mathbb{R}$. Each interval projects homeomorphically onto the neighbourhood, so $\mathbb{R}$ 'unrolls' the circle, showing its cyclic nature in linear form. Algebraically, this reveals why $\pi_1(S^1) \cong \mathbb{Z}$: the integers count wrappings around the circle.

After this construction, groups were involved in reflecting about other maps with the properties, such as sending the 2-sphere onto the real projective plane, up to the shared formalisation of the concept of covering map: a continuous surjection $p: \tilde{X} \to X$ where every point in X has evenly covered neighbourhoods. Students then explored whether various maps were coverings, solidifying understanding through identifying local triviality with discrete fibres.

Episode 4: Peer Teaching and Noticing Pedagogical Functoriality.

Solver groups presented lessons to classmates while observers documented noticing. G2 presented covering maps emphasising visual representations and videos of deformations. Observer O2 noticed:

O2 (A): Look at their drawings! They drew pictures and, in those pictures, found ways to explain a covering of the circle.

O2 (I): A picture can suggest insights that support arguments. Visual representation made local triviality clear: you could literally see disjoint intervals mapping to neighbourhoods. Visualisation was crucial for understanding.

O2 (D): In our future lesson, we'll emphasise visual representations, especially for concepts involving local vs. global properties. We'll use videos to show continuous deformations that static images can't capture. Also, some ropes.

O2's noticing illustrates *epistemological noticing*: recognising not merely that visualisation engaged students, but that it supports the specific cognitive move of seeing how local structure determines global properties, precisely the local-to-global

reasoning that characterises functorial thinking. Their pedagogical decision was grounded in this understanding.

G3 focused on fundamental groups using physical demonstrations with rope. Observer O3 noticed:

O3 (A): When they wrapped rope on the sphere, there were no obstacles; it reduced to a point without tearing. But on the torus, you couldn't avoid the hole!

O3 (I): The embodiment, using objects and gestures, created intuition for abstract concepts…and made visible what's invisible in formal definitions. Students could feel the difference between contractible and non-contractible loops.

O3 (D): We'll use deformable manipulatives to explain equivalence and fundamental group computation. Physical interaction grounds understanding before formalisation.

O3's interpretation (embodiment "made visible what's invisible") still frames embodiment as illustrative, making something pre-existing accessible, rather than constitutive. This is characteristic of *epistemological noticing*: recognising that embodiment has cognitive value without yet grasping that meaning itself co-emerges through embodied action.

Episode 5: Teaching Second-Years: Effective Noticing Enacted.

Observer groups designed and implemented introductory lessons for second-year students. O3 provided physical surfaces and rope, asking students to enact the covering projection bodily.

O3 (A): Student engagement was highest during manipulative exploration with spontaneous hypothesis testing. When they used bodies to enact the covering projection, arms outstretched showing how $\mathbb{R}$ unrolls onto S^1, their gestures revealed understanding. One student traced a loop with one hand on the torus while her other hand lifted and traced the corresponding path in the covering space above.

O3 (I): When students enacted relationships with bodies, they weren't illustrating understanding they already had; understanding emerged through enactment. Physical 'wrapping' became a cognitive root for understanding the functorial nature of the fundamental group. When that student used both hands to coordinate paths in base space and covering space simultaneously, she was enacting the functorial correspondence. Not representing it but doing it.

O3 (D): Embodied exploration should come first before definitions. But I also need to recognise body and gesture aren't just preparation for formal understanding; they remain part of mathematical thinking even after formalisation.

This episode illustrates *effective noticing*: O3 recognised that mathematical meaning *co-emerged* through embodied enactment rather than existing prior. The observation

that the two-handed gesture "was enacting the functorial correspondence, not representing it but doing it" reflects enactivist understanding that body and gesture *constitute* mathematical thinking. Notably, O3 reported mirroring students' gestures by tracing imaginary loops alongside them, suggesting that their own noticing was itself co-emergent: what became noticeable arose through O3's embodied participation, not from detached observation.

Episode 6: Reflective Awareness.

Post-course individual questionnaires revealed metacognitive awareness. As an example, student A on categorical covariation:

> Student A: We created a table associating each topological space with its fundamental group and explained why (covariational reasoning and functorial thinking). Categorising and linking helps you think and solve problems. We felt like mathematicians and able to teach mathematics.

Student B on representational fluency and teaching:

> Student B: I didn't believe university mathematics could be used to understand how to teach. We realised this when we noticed the teacher's method of using objects to explain concepts, and when my group designed a lesson. Surprisingly, teaching a complicated concept to a class helped us understand it better. Going back and forth from one environment to another, from one category to another, was powerful!

These responses reflect awareness that working across representational registers, the core of categorical covariational reasoning, strengthens both mathematical understanding and pedagogical awareness. Student B's language ("going back and forth from one category to another") is particularly notable: it spontaneously captures the bidirectional, structure-preserving movement that defines categorical covariation, suggesting that this form of thinking had become intuitive through the course design.

DISCUSSION AND CONCLUSION

This study demonstrates that algebraic topology courses can simultaneously develop advanced mathematical thinking and pedagogically-oriented noticing when designed to foreground functorial connections and create opportunities for embodied, relational observation and enactment. Students developed categorical representational competence, the capacity to coordinate multiple representational systems in structure-preserving, functorial ways, that extends Duval's framework beyond facility with individual registers.

The progression from embodied exploration (rope-wrapping) through formal functorial understanding (covering spaces) to metacognitive awareness (questionnaires) extended Tall's (2013) three worlds framework by emphasising the *relational* nature of the transitions between worlds. Embodied experiences created cognitive roots, but the crucial developmental step was recognising relationships as systematic and structure-preserving, and, importantly, as *directional and partial*: the functor π_1 translates

topology into algebra, but not fully or faithfully. Understanding what is preserved and what is not is itself constitutive of functorial thinking.

Functorial thinking, as defined here, represents a meta-mathematical competence: the capacity to move between representational domains while tracking what is and is not preserved. Our findings suggest this is accessible to undergraduates when grounded in concrete embodied experience and developed through sequenced exploration, challenging the assumption that categorical thinking requires prior mathematical sophistication beyond the domain content.

Observer groups' development illustrates a progression from *pragmatic noticing* (embodiment engages students) through *epistemological noticing* (embodiment creates understanding) to *effective noticing* (embodiment constitutes mathematical meaning). This progression was itself co-emergent: what became noticeable arose through the observers' own embodied participation in the teaching-learning situation, not from detached analysis.

The multi-directional noticing analysis (O→UT, O→G, O→S flows) reveals that noticing developed not through isolated cognitive reflection but through active, embodied participation across multiple observational contexts. Each flow shaped the others: noticing the university teacher informed noticing of peer groups, which in turn shaped noticing of second-year students. This co-emergent, relationally-constituted character of noticing development extends beyond what video-based noticing studies can capture and suggests that embodied live participation is a productive condition for developing effective noticing.

It is noteworthy to note that the context of this study is intra-university: students taught second-year peers, not secondary school students. We therefore do not claim a direct contribution to Klein's second discontinuity, which concerns the gap between university mathematics and secondary school teaching (Grenier-Boley & Robert, 2024). However, we suggest that developing categorical representational competence, the fluency to move between mathematical domains in structure-preserving ways, constitutes a form of mathematical maturity that, for prospective teachers in the cohort, may have longer-term relevance: teachers who have developed functorial thinking may be better equipped to help school students build connections across representations, in algebra, geometry, or beyond. This connection warrants further investigation with studies that directly track outcomes in school teaching contexts.

The distinction between "learning mathematics" and "learning to teach mathematics" may be less sharp than assumed when courses are designed around representational fluency and embodied, relational noticing, as both involve moving systematically between domains and attending to structure-preserving relationships. Further research should investigate the development of categorical representational competence across different mathematical domains and student populations and examine more directly how it informs practice in school teaching contexts.

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Teaching the concept of root of a polynomial at the beginning of higher education in France: a study of practices

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The introduction of the concepts of polynomial and polynomial function are the subject of various challenges at the beginning of higher education. In this research, we are interested in a related concept, namely the concept of a root of a polynomial, and we study the discourse of two teachers when they introduce this notion. To do so, our reference is the 'relief' of this notion, resulting from the combination of epistemological, curricular and cognitive studies. By means of this reference, we envisage 'proximity opportunities', which are possible attempts to bring together students' prior knowledge and the notion of root. Our analyses allow us to identify attempted proximities in their discourses.

Keywords: algebra, polynomial roots, teachers' practices, activity theory, qualitative analysis.

THE TEACHING OF POLYNOMIALS

Polynomials and polynomial functions are central objects in mathematics and in the teaching of mathematics, in secondary and tertiary education. Whereas it may be considered as a simple object, as it is only made with product and sums, previous research (e.g. Best & Bikner-Ahsbahs, 2017; Pleština 2023; Veuillez--Mainard, 2024) show students face several types of difficulties concerning polynomials, the nature of these objects and related notions. The first author's PhD focuses on the teaching and learning of polynomials in France at the secondary-tertiary transition, a period that has been identified as presenting numerous didactical challenges (Gueudet, 2008).

Polynomials and polynomial functions play an important role in most areas of mathematics (Pleština & Milin-Sipus, 2023; Veuillez--Mainard, 2024). In algebra, they are considered through polynomial equations and constitute important objects of structural algebra, in particular of ring theory. Polynomial functions are elementary objects in analysis, relevant to approximate other functions. The variety of points of view on polynomials leads to their wide representation in curricula through high-school and university. They are first encountered in secondary education through polynomial equations of small degree in algebra, and through monomial and polynomial functions that are reference functions in analysis. In tertiary education, the notion of a formal polynomial is introduced and studied, while polynomial functions begin to be used to approximate new functions (Lagrange interpolation, Taylor expansions etc.). In this contribution, we focus on the concept of a root of a polynomial. It is central in the

French curriculum: the first considerations of polynomials appear through polynomial equations, whose resolution is equivalent to finding the roots of a polynomial. The concept of a root in itself is defined in 11th grade (for quadratic functions), and developed in tertiary education. Roots are related to polynomial factorisation, which is itself useful for simplifying and solving other tasks involved in the study of polynomials (study of functions, arithmetic study, etc.). Moreover, knowing the roots of a given polynomial allows to (almost) entirely know it.

While little research focuses on the concept of polynomials as such (Pleština, 2023), the concept of a root of a polynomial appears in research (see below) be it implicitly or explicitly, as it can be expressed both in terms of divisibility – and this concept is thus linked to that of polynomial factorisation or of irreducible polynomials – or in terms of zero of the underlying polynomial function.

In a previous research, we studied students' conceptions of polynomials (Veuillez--Mainard, 2024). In this contribution, we focus on the teaching and learning of the concept of root of a polynomial at the beginning of higher education in France through the teacher's discourse during lectures. We will first present our theoretical framework and methodology, which will enable us to state our research question. We will then specify the relief of a notion, which is the reference we will use to study the teaching of this concept. We will use this reference to study the discourse of two teachers, one at university and the other in 'classe préparatoire', before stating some results and then some perspectives.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

We inscribe our research within the theoretical framework of Activity Theory, adapted to the didactics of mathematics (Bridoux et al., 2016). Thus, students' learning is modelled through the notion of activity, defined as everything a student does (or not) while performing a mathematical task (Rogalski, 2013). Within this framework, it is hypothesized that students' activities are favoured by autonomous work on mathematical tasks (Piaget) and via the collective discussions within students' Zone of Proximal Development organised by the teacher (Vygotsky) and the corresponding institutionalisations. Moreover, students' learning with respect to a notion is associated to the conceptualization of this notion in the sense of Vergnaud (2009).

However, during lectures, students' activities are often difficult to observe and we thus hypothesize that the study of the teacher's discourse is a legitimate intermediary in order to study their activities and consequently their learning. By mean of her/his discourse, the teacher might indeed try to 'stay close' to the students or to highlight different types of connections. The connections we study in this research are called 'discursive proximities'. As in Bridoux and Grenier-Boley (2024, p. 792),

> "three types of proximities are distinguished. *Bottom-up proximities* lie between what students have already done and the introduction of a new object or property. In this case, the teacher's discourse therefore aims to move from the contextualised to the decontextualised by generalising the particular case. *Top-down proximities* are situated

between what has been explained and examples or exercises. The teacher can then explain how the particular case fits into the general case, moving from the decontextualised to the contextualised. Finally, *horizontal proximities* do not lead to any change between contextualised and decontextualised. They consist of reformulations, explanations of the links between concepts, comments on the structure of the course, etc."

To study the teacher's discourse during lectures, we elaborate a reference which is called '*relief on the concepts to be taught'* (Bridoux et al., 2016). A relief study of the teaching of a notion in this sense combines epistemological, curricular and didactic analyses at the specific moment of teaching. The didactic analysis includes the study of the conceptualization of the notion and of their known learning difficulties.

In this study, the determination of the relief of the notion of root of a polynomial allows us to identify *proximity opportunities* which will be compared to the *attempted proximities*, those that are attempted by the teachers during the course. In this regard, our research questions are the following: What is the relief of the notion of (root of) a polynomial at the beginning of higher education in France? What are the proximity opportunities and the proximities attempted by the two teachers via their discourse?

ELEMENTS OF RELIEF ABOUT THE ROOT OF A POLYNOMIAL

Due to the limited size of this contribution, we will only provide a few elements about the relief of this notion during the first year of French higher education, starting with the curricular analyses, followed by a few elements of the epistemological and didactic analyses on the concept of the root of a polynomial or – if relevant – on polynomials.

Curricular analyses

The French curriculum concerning polynomials is presented in Veuillez--Mainard (2024). In France, students pursuing higher education in mathematics mainly attend two types of institutions: universities on the one hand, and 'classes préparatoires' (CPGE for short) leading to and engineering studies on the other. In the latter, a national official curriculum is defined, whereas in university the syllabus is defined in each university. However, on the subject of polynomials, our previous research shows that there is little difference between the curricula of French universities. We summarize in Table 1 the curriculum of the University of Montpellier concerning polynomials during the first year. In secondary education, quadratic polynomial functions are defined and their variations are studied. In grade 12, integer power functions are used as reference functions, polynomial functions being linear combinations of those. The notion of root of a quadratic polynomial function is introduced in grade 11, and of a polynomial function in general in grade 12, in 'experts mathematics' option. In higher education, as polynomials become formal objects, the concept of specialisation is introduced before the one of root. It is then linked to polynomial factorisation and in CPGE the arithmetic of the ring $\mathbb{K}[X]$ is studied in detail, this topic not being covered at the University of Montpellier. Moreover, real quadratic equations are solved in $\mathbb{R}$ in grade 11, in $\mathbb{C}$ in grade 12 in 'experts mathematics', and complex quadratic equations are solved in tertiary education. The curricular studies conducted in our PhD thesis show

that polynomial factorisation based on the determination of a root is rarely carried out in high school. At the University of Montpellier, polynomials are taught at the very beginning of the first year, whereas in CPGE it is studied at the end of the first trimester.

High school	Tertiary education, 1st year of Bachelor
- Solving in $\mathbb{R}$ (grade 11) and $\mathbb{C}$ (grade 12, 'experts' mathematics option) of quadratic equations - Quadratic and integer power functions as reference functions in analysis - Definition of a polynomial function of any degree (grade 12, 'experts' mathematics option)	- Definition of polynomials and of the algebra $\mathbb{K}[X]$ - Root of a polynomial - Arithmetic of the ring $\mathbb{K}[X]$ (CPGE) - Fundamental Theorem of Algebra (CPGE) - Polynomial functions as reference functions or reference approximation functions (Lagrange polynomials, series expansions…) in Calculus

Table 1: french curriculum regarding polynomials

The concept of root of a polynomial is therefore an important issue at the secondary-tertiary transition. In high school, the root of a second-degree polynomial f (and of degree n in 'experts mathematics') is defined as a real number x such that $f(x) = 0$. The characterisation in terms of divisibility is presented, but not elaborated upon. For instance, in the grade 11 textbook we studied, 140 tasks involve solving quadratic equations, while only 17 focus on factorisation by means of a root. In higher education, several characterisations of a root of a formal polynomial P coexist, given in several representation registers (Duval, 2006) involving polynomial functions or polynomials: $P(x) = 0$, zero of the associated polynomial function, $(X - \alpha)$ divides P, $P = (X - \alpha) \times Q$. The Fundamental Theorem of Algebra is stated and used in tertiary education, whose 'elementary' cases are those studied in high-school. The notion of root of a polynomial is thus explicitly present in secondary education, but in a limited way, and defined and linked to the factorisation of polynomials in higher education.

Epistemological, didactic and cognitive analyses

From an epistemological point of view, polynomial arithmetic developed by Hilbert has contributed to the emergence of structuralism developed by Noether in the 1920s (Hausberger, 2016), in particular to the definition and systematic study of $\mathbb{K}[X]$ or to its presentation via a universal property related to the evaluation morphism.

Polynomials are an important object of study at the beginning of higher education in France, mainly in analysis and algebra. In what is called 'classical analysis', the focus is "the study of *functions*, with an emphasis on so-called elementary functions, of which the most important are polynomials" (Winsløw, 2016, p. 164). The regularity and simplicity of polynomial functions make them a major focus of attention and a tool in analysis via functional approximation, whether this term is understood in the sense of interpolation, series expansion or density, as in the Stone-Weierstrass theorem. In algebra, polynomials with coefficients in a field become formal objects of study, first structurally, as $\mathbb{K}$-algebras, and related to the notion of polynomial functions through the evaluation morphism. Polynomials and their roots are essential tools in higher algebra, for example in the context of endomorphism or matrix polynomials or annihilator polynomials in linear algebra, or minimal polynomials of a number in Galois theory.

Later, polynomials and their roots are also crucial in algebraic number theory via the study of algebraic and transcendental numbers or of Galois groups.

Polynomials - Pleština (2023) highlights that algebra and analysis induce two distinct praxeological organizations of polynomials in secondary school, whereas the relation of first-year university students with respect to polynomials is associated to an almost empty logos block. The definition of polynomial itself is sometimes ambiguous or even conflictual in several regards (Bolondi et al., 2020), a result that has been confirmed by our own research (Veuillez--Mainard, 2024). According to Weiss (2016), the Factor Theorem (α is a root iff $(X-\alpha)|P$) is crucial in the connection between the algebraic aspect of polynomials and the analytic aspects of functions.

Equations and arithmetic - In secondary education, one main object of study is the solving of quadratic equations in $\mathbb{R}$ and $\mathbb{C}$ for which "an important *raison d'être* […] is the factorization of polynomials" (Chaachoua et al., 2024, p. 125). In this regard, some research points out the importance for students to make connections between different forms of these equations, which are also related to different techniques such as the discriminant technique, the null product technique and the square root technique. In higher education, solving quadratic equations becomes a 'tool' for other types of tasks. The curriculum focuses on the arithmetic of $\mathbb{K}[X]$, up to the notion of multiple roots at university, and up to the study of the Euclidean ring $\mathbb{K}[X]$ in CPGE.

Polynomial functions - The research Best and Bikner-Ahsbahs (2017) shows that many concepts are intertwined in the learning of polynomial functions – in particular the semiotic registers associated to the notion of function (algebraic, graphic, table, verbal) – that may result in a fragmented understanding of polynomial functions. The research of (Constantin, 2021) has shown that the concept of substitution in an algebraic expression, connected to that of root, could be considered as *transparent knowledge*.

Proximity opportunities - Higher education teachers could base the definition of the root of a polynomial on that of the root of a polynomial function seen in secondary school (Grade 11 or 12) or on the solutions to the underlying equations, which offers opportunities for bottom-up proximities. Depending on how they have defined the concepts of polynomial and polynomial function, they could also draw on top-down proximities with these concepts for the notion of root, or horizontal proximities in terms of changes in semiotic registers or with the evaluation morphism, including the use of substitution. Moreover, the Fundamental Theorem of Algebra and the Factor Theorem could be linked with the elementary cases of factorisation that are taught in high school with bottom-up or top-down proximities. Similarly, the Intermediate Value Theorem applied in high school on third-degree polynomial functions provides a particular case of the Fundamental Theorem of Algebra. The Euclidean division of $\mathbb{K}[X]$ offers a way of characterising the notion of the root of a polynomial, which can be linked to or motivated by the Factor Theorem, which may also lead to proximity opportunities. Finally, the arithmetic properties of $\mathbb{K}[X]$ and $\mathbb{Z}$ can be connected, and even generalized to the arithmetic of rings with a bottom-up proximity.

ANALYSES OF TWO LECTURES

General elements about the lectures

Our experiment takes place during the first year of higher education, both in university and CPGE. In CPGE, whereas the curricula are defined nationally, some elements may differ depending on the teachers. For example, the very definition of a formal polynomial is different in the three CPGE classes we observed. Although we studied various teachers in our Phd Thesis (secondary school, university and CPGE), we have limited our focus to two teachers – one at the university of Montpellier and the other at CPGE – in order to conduct a detailed analysis of their discourse on the subject of roots. We briefly outline the differences in approach between each of these two teachers.

Teacher 1, CPGE - Polynomials are introduced as null sequences from a certain rank, the indeterminate X being defined as the sequence $(0,1,0,0\ldots)$. Operations on $\mathbb{K}[X]$ are defined based on those for sequences, and therefore the writing $P = \sum_{k=0}^{n} a_k X^k$ is consistent with the sum and the multiplication of $\mathbb{K}[X]$. This then allows to define the polynomial function $x \to \tilde{P}(x)$ and a root of a polynomial as a zero of the associated polynomial function. Once the injectivity of the morphism $P \to \tilde{P}$ is established in the case where $\mathbb{K}$ is infinite, a polynomial and its polynomial function are identified.

Teacher 2, university - Real polynomial functions are first defined, then a polynomial is "informally" defined as an object considered through *its form, its writing*, via an expression of the form $P = \sum_{k=0}^{n} a_k X^k$. The formal definition with finite sequences is then given, but the theoretical link with the expression $P = \sum_{k=0}^{n} a_k X^k$is not explicitly made. The elementary operations on polynomials are defined on the coefficients of the polynomial. The polynomial function associated to a polynomial is not defined, these objects are hence not identified. The evaluation of a polynomial in a scalar x is then defined as $P(x) = \sum_{k=0}^{n} a_k x^k$, and a root is a scalar such as $P(x) = 0$.

RESULTS

Teacher 1, CPGE

We first want to emphasise that in CPGE, polynomials are taught at the end of the first semester. Thus other courses prior to this one may have been related to the concept of root. The definition of root of a polynomial is introduced during the sixth course, after a section on polynomial arithmetic. The definition is given, without any particular introduction, the only comment made by the teacher is that 'it's not a big surprise, […] there is just the notation $\tilde{P}$ that you didn't know (our translation)'. We conclude that there is no attempted proximity for the definition of a root, in particular no relation with prior high-school knowledge. Then, the teacher repeatedly compares the arithmetic of $\mathbb{K}[X]$ with the arithmetic of $\mathbb{Z}$ (both of which are Euclidean rings), in definitions, properties and proofs. For example, in the proof of the characterisation of a root by divisibility, he points out that to show that an element divides another one, one may show that the remainder of the Euclidean division is null, and that this method

has already been used earlier in the course, as well as in the course on the arithmetic of $\mathbb{Z}$. Later, he compares the notion of multiplicity of a root with the p-adic valuation:

> In the same way, we saw that d is the p-adic valuation of n if and only if p^d divides n, that is, there exists q such that n equals $p^d \times q$, and q is not divisible by p. Well here, it's the same thing, except that it's formulated in terms of evaluation. […] All of these are properties of the p-adic valuation.

Lastly, the teacher emphases the importance of what he calls the *Rigidity Theorem of polynomials*: the number of roots of a polynomial counted with their multiplicities is majorized by the degree. In an interview, he outlined that the section on roots is organised with respect to the Rigidity Theorem and its numerous applications, notably the Fundamental Theorem of Algebra and the identification of a formal polynomial with its polynomial function (when $\mathbb{K}$ is an infinite field). This leads him to make meta comments, some about the notion of algebraic closed fields. He states for instance:

> It is this […] equivalence between root and divisibility that will have many consequences on the structure of polynomials, in particular the Rigidity Theorem.

Again, no connection is made with the properties of quadratic equations seen in high school, neither when presenting the Factor Theorem, nor when presenting the Rigidity Theorem and the Fundamental Theorem of Algebra. We therefore interpret that there is a missed proximity opportunity. However, two top-down proximities are attempted subsequently: an example of application of the characterisation of a root by divisibility is given (to show that $X-1$ divides another polynomial), and an example of computation of multiplicity. The teacher also attempts a horizontal proximity that was not anticipated in our analyses between the Rigidity Theorem and the Intermediate Value Theorem: the IVT proves the existence of a zero for any continuous function that has a positive and a negative value, and the Fundamental Theorem of Algebra proves the existence of a complex root for any polynomial.

In conclusion, few proximities are attempted in this lecture to establish connections with the notion of root. In particular, only one reference is made to secondary school knowledge (IVT) and therefore no bottom-up proximity is attempted. The arithmetic connections highlighted between $\mathbb{Z}$ and $\mathbb{K}[X]$, in relation to their ring structure, reflects the notion of a root in terms of divisibility. This shows an attempt of horizontal proximity between this notion and its expression in a register linked to divisibility. When the characterisation of a root by divisibility is given, two top-down proximities are attempted. In the rest of the course, roots are studied mainly as objects and are rarely used as a tool for other concept related to polynomials, or in exercises.

Teacher 2, university

The definition of root (and multiple root) of a formal polynomial is introduced and defined in the second lecture, and then developed in the fourth and final lecture. In the latter, the characterisation with divisibility is stated and proved.

After defining the evaluation of a formal polynomial as $P(x) = \sum_{k=0}^{n} a_k x^k$, the teacher introduces the notion of a root. For this purpose, as at the very beginning of the course, he compares the uses of polynomials in analysis and algebra. He clarifies that the focus of the course is not function analysis (monotonicity, graph…). He then says: 'we will focus on… It's something you've known for a long time, namely whether it can take the value 0'. He then defines a root of a polynomial and give an example: the polynomial $X - a$ has a single root a. We interpret this example as a top-down proximity attempt which, however, only gives a limited outlook on this concept. Moreover, he does not explicitly state that $P(x) = 0$ on the example, which we interpret as missed opportunities of horizontal proximities between different registers.

After this definition, the teacher introduces the Taylor polynomial formula: $P(X) = \sum_{n=0}^{+\infty} \frac{P^{(n)}(a)}{n!}(X-a)^n$. He justifies its usefulness by introducing another point of view on polynomials, namely finding the roots of a given polynomial. Indeed, knowing a polynomial as a linear combination of the powers of $(X - a)$ allows to quickly see whether a is a root of the polynomial. However, to know this form of the polynomial, one may have to evaluate first $P(a)$, which allows to know whether a is a root.

The notion a root reappears in the fourth lecture as an object, with the characterisation by divisibility and the definition of multiplicity. The characterisation is presented as an application of the Euclidian division, that is the division of a polynomial by $X - a$. An example of this theorem is given by explaining the factorisation of a defined polynomial into $(X - a) \times Q$, which demonstrates an attempt of several horizontal proximities. It should be noted here that the teacher finds the factorisation by doing a Euclidean division, whereas in high-school the technique used is the expansion of $(X - a) \times Q$ followed by the identification of the coefficients (for polynomials of second- and third-degree polynomials). However, the teachers do not comment on this technique, thus missing opportunities of proximity.

Finally, during the third session of exercises on polynomials, the teacher reminds the concept of multiple roots. He introduces the definition with a bottom-up proximity, based on the case of quadratic polynomials. This allows him to make a connection with the high school curriculum, recalling the method of the discriminant to solve second-degree equations. At the request of a student, he also describes the case of third-degree polynomials. Later, and for the first time in the course, he explains the links between polynomial equations and the roots of a polynomial, which we consider to be an attempt of horizontal proximity in the natural language register:

> All the vocabulary you already know about the solutions of an equation, we're going to transpose it into terms of roots because it means the same thing. It's just a different word.

During this teacher's lectures, we identify more attempted proximities than in the previous one. We observed a significant attempt of bottom-up proximity with secondary equations in high-school, and horizontal proximities between equivalent formulations of the notion of a root,. As for Teacher 1, the concept of root is mainly used as an object in the lesson and is not directly exemplified.

CONCLUSION AND OUTLOOKS

Our relief study showed several proximity opportunities. Bottom-up proximities might be attempted to link the notion of root of a polynomial to high-school knowledge. Top-down proximities might allow the teacher to show how the definition is manipulated like an object or a tool. The articulation of different semiotic registers concerning this notion by means of horizontal proximities in the teacher's discourse might help students to give sense to the new notion.

Our analyses show that teachers do not attempt the same proximities, which may be due to the way the concepts of polynomial and polynomial function are defined by each of them. If proximities are nevertheless attempted, they remain rare and unwritten for Teacher 1, more varied and sometimes written for Teacher 2. Since these connections are mostly oral, we hypothesize that their impact on students' learning is limited.

Our research shows how the tools presented make it possible to apprehend the teacher's discourse and to formulate hypotheses about the way in which students may receive this discourse. The aim now is to continue this study with the concepts of polynomials and polynomial functions, and to compare them with students' conceptions (Veuillez--Mainard, 2024). This will be the subject of future work.

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On phylogenetics of matrices in finite mathematics textbooks

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Keywords: finite mathematics, textbooks, quantitative methods, matrices.

In the U.S., Finite Mathematics (FM) is a non-STEM-major course, typically covering matrices, finance, linear systems, and probability, and is a common alternative course to Business Calculus. FM's eclectic mix of topics makes textbook selection nontrivial. Textbook analysis studies benefit institutions by supporting the implementation of these courses (Mesa & Griffiths, 2012). Prior studies (e.g., Lockwood et al., 2017) qualitatively analyzed a combinatorics concept in FM texts. However, our study investigates how the matrix narrative is developed across textbooks through the lens of quantitative biology. *Phylogenetics* (PG) examines organisms' evolutionary history, often visualized as trees (Owen, 2008). While PG tree analysis has been applied in math and quantitative biology research (Owen, 2008), to our knowledge, it has not been applied to math textbook analysis. This study expands math textbook analysis to FM and introduces PG methods as a quantitative approach. One of the authors teaches FM and is interested in the alignment of matrices to the business curriculum, specifically, which topics are most emphasized. We begin to explore this by answering the research question: How similar are the structures of FM textbooks' presentation of matrices?

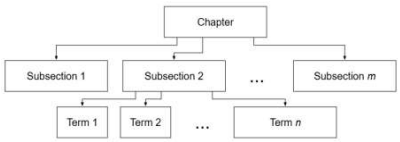

Figure 1: PG tree structure of textbooks.

METHODOLOGY

In this study, we analyzed three popular FM textbooks: (L) Lial et al. (2021), (T) Tan (2018), and (SB) Sekhon and Bloom (2016). We constructed (PG) trees to model the structure of each textbook's chapter on matrices (Ch 2 in all three texts). We included section titles, subsection headers, and emphasized terms as nodes since these items indicate the author's intended hierarchy of information. Figure 1 shows the general structure of the trees we constructed. The full diagram for each text is available at https://tinyurl.com/PhylogeneticMatrices. We analyzed subtrees corresponding to four topics common to the texts: Gauss-Jordan Method (L, T, SB: 2.2), Matrix Arithmetic (L: 2.3/2.4, T: 2.4/2.5, SB: 2.1), Inverses (L: 2.5, T: 2.6, SB: 2.4), and Input-Output Models (L, SB: 2.6, T: 2.7). In PG, the *Robinson-Foulds distance (RFD)* between two trees is the size of the symmetric difference between their edge sets, or the union of the edge sets minus their intersection (Owen, 2008). In this context, RFD quantifies the

similarity between two texts' structures. To account for size variation, we computed pairwise *relative* RFDs, dividing the RFD by the trees' total number of edges.

RESULTS, DISCUSSION, & IMPACTS

The pairwise RRFDs for all four topics are visualized in Figure 2 as triangles drawn to scale compared to an equilateral triangle of side length 1 (the max RRFD). Our results suggest that FM texts differ substantially on matrix topics since the side lengths of the blue triangles are all ≥0.5 (50% dissimilar). Also, the triangles' interior angles capture the consistency of the differences between texts; e.g., for inverses, Tan's angle is smaller than the others, signifying that its approach diverges from the other two texts. To help reach mathematicians with this research, quantifying these differences may help inform institutions' text selection for FM courses. For instance, they could choose the cheaper option of two similar texts to benefit students. Since texts can influence how a course is presented (Mesa & Griffiths, 2012), text selection is impactful on math courses.

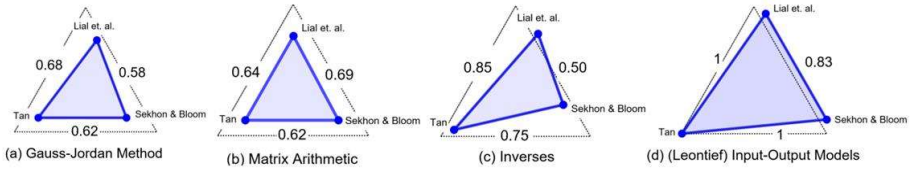

Figure 2: Pairwise relative RF distances for four topic themes.

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Supporting students' conceptual understanding in linear algebra through peer discussion and dual-mode exercises: an APOS-based approach

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Keywords: linear algebra, APOS theory, ACE, STACK, conceptual understanding.

INTRODUCTION

Many first-year university students experience persistent difficulties in linear algebra, particularly in transitioning from procedural computation to conceptual understanding. Concepts such as linear independence, kernel and image are often treated symbolically without sufficient conceptual grounding (Tall & Vinner, 1981). APOS theory (Dubinsky & McDonald, 2001) describes learning as a progression from Action to Process and Object. However, instructional approaches that explicitly support these transitions in classroom practice remain underdeveloped. This study explores an instructional approach integrating peer discussion with dual-mode exercises (online and paper-based) and examines how this design may support students' conceptual understanding in linear algebra.

THEORETICAL FRAMEWORK AND DESIGN

APOS theory is used as the theoretical framework and informs this study in three ways. First, it guides task design: online exercises target Action-level engagement, while worksheets and peer discussion promote Process-level reflection. Second, it structures classroom activity through the ACE cycle (Activities, Class discussion, and Exercises). Third, it provides a lens for interpreting students' responses, particularly in identifying transitions from procedural action to conceptual explanation.

The study was conducted in a first-year linear algebra course for engineering students, with over 80 students enrolled. Each session consisted of: (1) Online exercises using STACK (Sangwin, 2013) with immediate feedback; (2) Paper-based worksheets requiring written explanations; (3) Peer discussion in small groups. This dual-mode design was intended to connect computational practice with conceptual reflection. In introducing vector spaces, formal definitions were postponed, and students examined the conditions for computing linear combinations. They computed not only with numerical vectors but also with polynomials and other sets, and discussed in groups when such operations are possible, which were later formalised as axioms of vector spaces. This design supports the transition from Action to Process in APOS terms.

METHODOLOGY

This exploratory study adopts a design-based approach to evaluate the effectiveness of the APOS-ACE cycle in a linear algebra context. Data were collected from a first-year engineering course ($n > 80$) through mid-semester questionnaires and classroom observations. The questionnaires utilised 5-point Likert-scale items to capture students'

perceived understanding and interest after each session, achieving a response rate of 80–90%. The analysis focuses on the pedagogical alignment between task design and student response. Specifically, we identified class sessions with significant fluctuations in reported understanding to examine how the transition from computational Actions (via STACK) to Process-level reflection (via peer discussion and worksheets) was facilitated or hindered by the level of abstraction of the topic. While this preliminary phase relies on self-reported data, the findings are interpreted through APOS theory to refine the instructional approach.

FINDINGS AND DISCUSSION

Questionnaire responses across the first 17 class sessions showed variation in students' reported understanding and interest, depending on the topic of instruction.

In the initial sessions on abstract vector spaces and linear combinations (Sessions 1–2), students reported high interest with some variation in understanding. In sessions on spanning sets and subspaces (Sessions 3–4), both measures were relatively high, and this trend continued for bases and dimension (Sessions 5–6). A marked decline in both understanding and interest occurred in sessions on linear maps, kernel, and image (Sessions 7–10), indicating increased conceptual difficulty. In subsequent sessions on representation matrices (Sessions 11–14), both measures improved, although some variation remained. Finally, in sessions on inner products (Sessions 15–17), understanding recovered to relatively high levels, while interest remained moderate.

The results indicate that understanding declined in sessions on linear maps, kernel, and image. This decline is also reflected in lower average scores in online exercises during this period. Classroom recordings further suggest that peer discussion was not sufficiently directed towards deeper exploration. Together, these observations indicate limitations in the instructional approach in these sessions and suggest a need to redesign tasks and discussion prompts to better support Process-level understanding.

NOTES

1. This study is supported JSPS KAKENHI Grant Number JP24H001681.

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Referential Linking and Clarifying Descriptive Equivalence

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Equivalence is ubiquitous in the mathematics curriculum, underlying important ideas and processes. Scholars have examined undergraduate students' understandings of mathematical equivalence in specific domains. Recent work (Cook et al, 2022) has emphasized the importance of attending to students' understandings of equivalence across multiple mathematical domains, with an emphasis on attending to the mental operations students employ when interpreting equivalence between mathematical objects. We explore three episodes in which students refer to mathematical objects in new contexts to justify the equivalence of objects in a current contextual system. We call this phenomenon referential linking, and we present models of referential linking built from conceptual analysis on the students' data episodes.

Keywords: Equivalence, Mathematical Thinking and Learning, Theory.

INTRODUCTION AND GROUNDING LITERATURE

Equivalence, the idea that different objects should be considered the "same" in some way, is a fundamental aspect of mathematical problem solving. Researchers have examined students' understanding and leveraging of equivalence in specific domains such as arithmetic and algebra (e.g. McNeil & Alibali, 2005), abstract algebra (e.g. Cook, 2018) and combinatorics (e.g. Lockwood & Reed, 2020).

More recently, researchers have examined ways that students understand equivalence across mathematical domains (e.g. Cook et al, 2022). We are similarly motivated to examine ways to engender students' generative understandings and ways of thinking (Thompson et al., 2014) about equivalence across mathematical domains.

A focus on cross-domain ways of thinking

Many high-level mathematical ways of thinking (Thompson et al., 2014), practices, or meanings transcend topical domains and have the potential to be tools for students at all levels of mathematical study to leverage productively. Charles (2005) defines a big idea as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10). Such cross-cutting foundational "big ideas" of mathematics can enable students to treat mathematics as a coherent network of ideas (e.g. Charles, 2005; Thompson, 2008). Such a view of mathematics can be productive for students, who might view mathematics as providing them domain-specific instances of "the same phenomenon,

regardless of differences in specific contexts" (Zandieh et al., 2017, p. 24) rather than giving myriad techniques to employ in very specific situations.

In this vein, we present episodes taken from clinical interviews in which students interpreted equivalence between multiple mathematical objects. We focus on episodes where students employed novel (to us) mental actions when establishing and justifying the equivalence. The resulting model contributes to the literature on the various ways students might attribute equivalence between mathematical objects.

THEORY

Thus far studies have largely focused on underlying notions of "sameness" in students' interpretations of equivalence. Drawing on Cook et al. (2022), when one observes students making a claim of equivalence between two mathematical objects "… using descriptors such as *same, common, similar …*" (p. 3) we find it useful to infer from the students' language and mathematical activity possible cognitive features of the students' equivalence claims. This is the analytical process of *conceptual analysis* (Thompson, 2008) that we articulate in the Data Analysis section below.

We are interested in exploring instances where students justify or describe a claim of equivalence by appealing to a context that might be seen as distinct from the context of the equivalence claim. For example, students have been observed attributing equivalence to fractions by noting that two fractions represent the same shaded area within a region (Kara et al., 2018). In such a justification for the equivalence of 1/2 and 3/6, it is the area-context equivalence of the partitions that theoretically grounds students' interpretations of equivalence for the fractions. We are interested in examining the subtleties of students' reasoning when their interpretations of equivalence are specifically rooted in appeals to contexts distinct from (in the mind of the student) the context of the original mathematical objects.

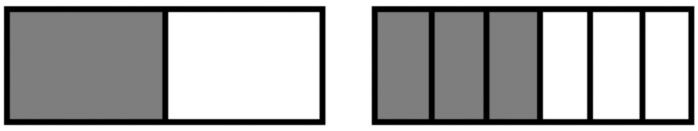

Figure 1: An area-based appeal to equivalence of 1/2 and 3/6.

Radical Constructivism and Students' Meanings

We are primarily concerned with how students understand and think about the mathematical objects of interest when making claims of equivalence, and also how ideas and mental images are being brought to bear when they appeal to outside contexts (such as area) for justifying such equivalence. As such, we ground our work in a radical constructivist (Glasersfeld, 1995) epistemology and attend specifically to students' mathematical meanings (Thompson et al., 2014) as they make interpretations of equivalence.

Radical constructivism relies on three interconnected and interdependent mental processes of assimilation, accommodation, and abstraction (Glasersfled, 1995). An assimilation "comes about when a cognizing organism fits an experience into a conceptual structure it already has" (Glasersfeld 1995, p. 62). This assimilatory mechanism underlies the idea of students' *meanings* and *understandings*.

When we speak of a student's understanding, we refer to a student assimilating a situation to their current meanings . The student's meanings for an idea is "the space of implications that the current understanding mobilizes – [mental] actions or schemes that the current understanding implies" (Thompson et. al., 2014, p. 13). All meanings are enacted in-the-moment, but some in-the-moment meanings are stable. Stable meanings have repeatedly proved useful for students whereas some in-the-moment meanings arise when assimilations are to temporary cognitive structures that might be fleeting or arise from momentary accommodations made during the process of thinking and learning. For space we omit elaborations of the constructs *accommodation* and *abstraction*.

In the episodes we present, the students develop in-the-moment meanings for the equivalence between the presented mathematical objects and we infer that the students drew from operations (mental actions) and imagery borrowed from more stable meanings to establish a justification for their claim of equivalence.

Contexts and Referents

When we speak of *context* we do so in line with radical constructivist epistemology. We infer that the students in our episodes are drawing from borrowed mental operations and imagery. That, students borrow from their meanings related to ideas that differ from their meanings for the mathematical objects subject to the equivalence interpretations. The students will appeal to mathematical objects from a more established (stable) meaning (partitioning areas) and borrow operations and mental images to facilitate accommodations for the new equivalence-related in-the-moment meaning that they are developing. That is, we speak of context in a fluid and personal way to note that in the moment of equivalence attribution a student has likely established an imagined contextual system (a cognitive structure) in which they imagine the objects they are examining exist.

The student's problem context can be different from person to person or from problem to problem. This is analogous in principle to Duval's (2006) idea of registers and shifting registers, but we are more concerned about the process of students' progressive adoption of new meanings and do not find beneficial for our purposes many of the underlying theoretical assumptions of Duval's uses of semiotics.

We will also speak of *referents*. For instance, in the example in Figure 1, the left area is a referent for 1/2 and the right area is a referent for 3/6. By referent, we mean the mental objects in a contextual system that the student imagines to correspond to an object in another contextual system. We will make this more explicit in the results when we establish the idea of *referential linking*, but we want to frame our terminology

in a radical constructivist perspective, noting that we are dealing with students' mental constructions and seek to emphasize students' mathematics (Steffe & Thompson, 2000).

We ask the following research question: *What models of student thinking describe and explain students' justifications of equivalence by appeal to and manipulation of referents in a new contextual system?*

METHODS

We draw from interviews spanning multiple studies targeting student understanding of equivalence within and across mathematical domains. While the primary domains explored in the interviews are abstract algebra and discrete mathematics, students were also asked at times to solve problems that required interpretations of equivalence related to algebraic equations, calculus, and geometry and set theory. The interviews were task-based clinical interviews (Hunting, 1997), where students solved various mathematics problems and the researchers frequently probed for understanding and asked follow up questions or prompts in line with the intuitive and responsive interactions described by Steffe and Thompson (2000). In retrospective analysis the researchers built second-order models of the students' thinking (Steffe & Thompson, 2000; Thompson, 2008; Clement, 2000).

We present multiple episodes in which participants made claims of equivalence between mathematical objects. We specifically feature interpretations of equivalence made by three postsecondary students, Katie, Brian, and Howard (pseudonyms); all students enrolled in universities in the United States. Katie was a multivariable calculus student, Brian was a mathematics major who had previously taken a course in discrete mathematics, and Howard was a mathematics graduate student.

Data Analysis: conceptual analysis

The explicit method of analysis is conceptual analysis (Thompson, 2008), which is a method of model building that seeks to form theoretical models that articulate "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 43). Our aim is to describe a model for how one might understand interpreted mathematical equivalence in the moment of making a claim of equivalence. Conceptual analysis at its core is student-driven. That is, the model building inherent to conceptual analysis should foreground the experiences and reasoning of students in the context of solving mathematics problems, when possible.

Here "models" are built in retrospective analysis (Steffe & Thompson, 2000) and are framed by Clement as "provid[ing] only a viable plausible explanation for a set of observations" (Clement, 2000, p. 555). These models of a student's meanings posit concrete answers to the question "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p. 78). Such models are useful by both researchers and teachers in that they can be useful

to the reader as grounds for viable explanations of future observed student discourse and activity (Steffe & Thompson, 2000).

The video records were spliced together so that we could view the students and their written work together in retrospective analysis (Steffe & Thompson, 2000). Transcripts were enhanced with images, references, and comments. We then reviewed the videos and transcripts of the interviews to build the models of the students' meanings.

KATIE'S CLOCK ARITHMETIC

In our first presented episode, Katie (also featured in Authors, Date) was attempting to explain why $3 *_{12} 11 = 9$ ($*_{12}$ represents multiplication in Z_{12}) by repeatedly appealing to 33 and 9 occupying the same position on a 12-hour clock.

> Katie: I mean, on the 12 hour system, they'd be the same number. … I'm trying to think of how I want to draw it. I'll do a circle I guess. … It'd be going around the clock two times. … How does that relate to 9 and 33? Like, if you started at one, and you went 33, you counted up to 33 … you would end up with 9.

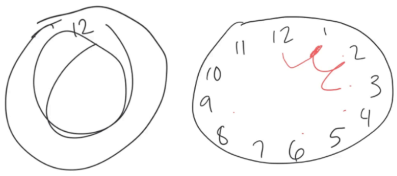

Figure 2: Katie's descriptive justification that $3 *_{12} 11 = 9$.

Katie is justifying equivalence between 33 and 9 (mod 12) because both numbers refer to the hands on a clock at the position that Katie understands as 9:00 for the purpose of telling time. We will further examine this case below when we elaborate the model of *referential linking*, but for now we note Katie's underlying justification for the equivalence interpretation.

Katie indeed interprets equivalence according to Cook et al.'s (2022) framework, both interpreting equivalence between the integers 33 and 9 as well as the time referents. That is, the prompt requires Katie to make an equivalence interpretation between 33 and 9, which she justifies via equivalence between the time referents for 33 and 9. Katie has a stable meaning for time passing and the figurative imagery associated with circular clocks and leverages mental operations from her understanding of clocks and time passing to establish an in-the-moment meaning for equivalence of 33 and 9.

BRIAN AND HOWARD'S INTEGRALS

In their respective interviews, Howard and Brian were asked to explain a series of integral equations, including the equation $\int_0^1 2x\,dx = \int_0^1 2 - 2x\,dx$. After first attempting to describe the equality in terms of odd functions or in terms of function manipulation (i.e. an initial thought that $2 - 2x$ was a shift of the function $2x$), Howard drew the graphs in Figure 2 and explained the equation in terms of triangular areas.

Howard: These are just triangles now that I draw the picture … these are just triangles with the same – we just turn the triangle around and it doesn't change the area. … So the equal sign is just telling us that the area of a triangle doesn't change when you turn it around.

Brian reasoned similarly, drawing the graphs in Figure 3b saying:

Brian: 2x is going to look something like that, and 2-2x is going to look something like that. So this one [2x] is this area and this one is this area [2-2x], which is going to get me .. you know if I – up to some transformations I think those are the same … reflection and a rotation [reflect the upper left triangle], stick it right there and then [rotate].

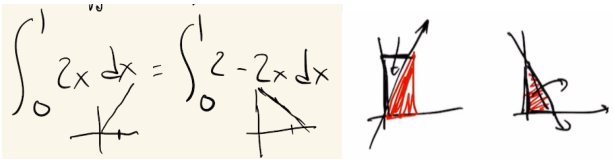

Figure 3a and 3b: Howard's (left) triangular areas and Brian's (right) triangular areas.

Both Brian and Howard note that the integrals (or more specifically the integrands) describe the same triangle after manipulation, such as a rotation. These are equivalence interpretations of the triangular referents rooted in geometric transformations. They then borrow the equivalence status of the transformed triangles to imply equivalence for these two integrals. Thus, the students engage in two interpretations of equivalence (according to Cook et al.'s framework) but the justification of the integral equivalence relies on the result of the triangular area equivalence.

Note that both Brian and Howard demonstrated stable meanings for area interpretations of integral measures, however the in-the-moment meanings being developed in these interviews is an additional interpretive layer of equivalence of the integrals being expressed by the equal sign. These were not the only means of interpreting equivalence that Brian and Howard demonstrated in the interviews, but these were uniquely reliant on geometric transformations as underlying the equivalence, as opposed to numerically-driven or transformation-driven (Howard's original attempt to apply integral transformation rules on the integrand functions).

Referential linking

We now present our model of the students' meanings and elaborate the explanatory power of the model for these empirical cases. A commonality of these interpretations of equivalence is that students borrow operations and mental images from stable and established meanings. In each case, there is a clear link between present and imagined mathematical objects that enable this productive borrowing. Two links are established in each case, between referents in both contextual systems. We call this phenomenon *referential linking*, which we define in detail below.

We infer that, to Katie, 33 and 9 were in the same contextual system, and to Howard and Brian, the integrals in the equation $\int_0^1 2x\,dx = \int_0^1 2 - 2x\,dx$ were in the same

contextual system. To attribute equivalence between these objects, all students linked the present objects to paired objects in a new mathematical (or imagined) contextual system. It is the context jump that characterizes the referential linking. Figure 3 gives a diagram of the relationships between mental objects at play during the students' interpretations of equivalence.

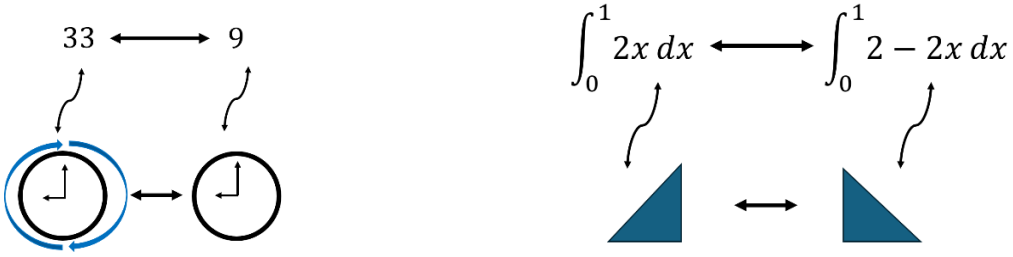

Figure 4(a, b, c): Referential linking for integers and integrals.

In each case, the attribution of equivalence depends on the interpretation of the mathematical objects being considered as representable or interpretable in a personal context (hand-orientation on a clock or bounded geometric areas) introduced by the student. That is, the integers and integrals were interpreted as holding referential meaning in the appealed-to context. In some cases, as with integrals and area, the context might be readily available through standard mathematical associations, though we note that both Brian and Howard introduced geometric transformations on the triangles that extend beyond the typically static representations of areas measured by integrals.

We emphasize this point by noting that Katie and Howard, when appealing to the geometric or temporal contexts above, established a link between the numeric or symbolic expressions being observed (33 and 9, the two integrals) and specific instances of the mental objects in the referential system (times and triangles). Howard and Brian used different imagined triangles and mentioned geometric actions that preserve area. Katie mentioned rotations of the hour-hand along the clock, and examples abound in the literature where equivalent fractions represent shaded areas that share the same area partition (Figure 3) but are sub-divided in different ways.

In each example, there was an expectation that transformations could occur in the referential space (flipping triangles, rotating clock hands) that would correlate to instances of symbolic or numeric expressions and that the referential link between the two (vertical in Figure 4) instances would be maintained along transformation of objects in the referential system.

We thus might say that the students established an in-context equivalence between the original objects (integers and integrals) by first making a referential link between the two objects with paired objects in the new contextual system in which they have developed stable meanings (clock orientations and triangular areas). Leveraging these stable meanings, they then utilized transformational equivalence to justify the equality of the objects in introduced context. They then appealed to the referential link to justify a final attribution of equivalence between the original objects. The link between objects

in the original context is tentative at first, then is justified by moving referentially, making an equivalence in the new system, and then moving back to the original context. This process is succinctly visualized in Figure 5 for Katie's argument.

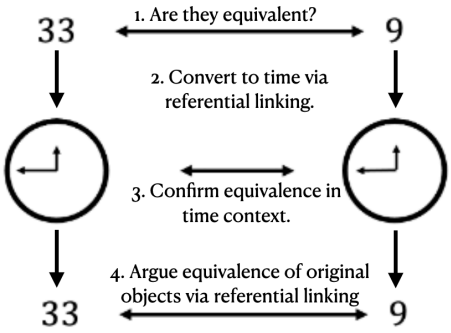

Figure 5: Justifying equivalence via referential linking for modular arithmetic.

DISCUSSION AND CONCLUSION

We sought to address the research question: *What model of student thinking describes and explains students' justifications of equivalence by appeal to and manipulation of referents in a new contextual system?*

We found that when asked to compare two present mathematical objects, the students in our study linked these objects to referents in a different contextual system where they anticipated being able to make the comparison between the referents. They then use the comparison made in the referential context to make conclusions in the original space; we called this activity referential linking.

We find referential linking to be potentially beneficial for students, particularly in instances of developing in-the-moment understandings of equivalence in a particular new context. We think that students might be empowered to make and justify claims of equivalence if they have broader access to other meanings, mathematical or experiential to which they can link the more abstract mathematics. That is, while Katie could likely carefully systematize the underlying operations of translating between times, her referential context was grounded more in experience and the intuitive social phenomenon of telling time. This ability to ground interpretations of equivalence in more intuitive meaning might open up new possibilities for students as they develop initial informal understandings or ways of thinking about equivalence.

We highlight that referential linking could be a tool researchers adopting a radical constructivist perspective could leverage while exploring students' moving between registers (Duval, 2006). Arguing that a graph, equation, and context somehow represent equivlanet relationships could entail referential linking. We call for future research exploring students' potential productive uses of referential linking to justify equivalence given equivalences' role as a big ideas in mathematics (Charles, 2005)

Finally, we note that referential linking as a construct is likely to be applicable in research beyond that of establishing the equivalence of two objects. For example, referential linking could be important in supporting students graphing activity as they

imagine how quantities varying in a situation can be linked to representations of these changing quantities on axes. Future research can explore ways students may leverage referential linking to explicitly articulate cognitive aspects of observed student activity in various domains of research.

ACKNOWLEDGEMENTS

This research is based upon work supported by the NSF – 2055590.

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Which understanding of the scalar product do preservice teachers intend for pupils at upper secondary level?

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The scalar product is a significant topic at upper secondary level in many countries that is required in several STEM-programs immediately from the start. Therefore, it is important to qualify prospective teachers to equip pupils with a solid understanding of the concept – especially since many beginning undergraduates have problems to assign the concept a meaning. We conducted a study to explore the understanding preservice teachers wish their pupils to acquire at upper secondary level. The data show that although almost all our participants intended to cover many properties of the scalar product that can contribute to an understanding of the concept, they intended to do so mainly for enabling pupils to perform certain calculations, rather than to use the properties to help them give the concept a meaning. Consequences are then discussed.

Keywords: linear algebra, concept image, Grundvorstellungen, mixed methods.

INTRODUCTION AND EMBEDDING OF THE RESEARCH

The scalar product or dot product is an important topic at upper secondary level in many countries. One reason is that it is needed in many STEM-programs directly at the beginning, for example in basic mechanics courses. But it also plays a significant role in the mathematics major program, as it is the basis for angle-related notions in analytic geometry. Therefore, students enrolling in STEM-disciplines should at best be already equipped with a solid understanding of the scalar product when entering university.

However, there is some evidence that many beginning undergraduates struggle with the concept – even if it has been covered at school. First data was provided by Knight (1995). In a test administered at a US university ($n = 286$), only 12% of the beginning undergraduates who had learned about the scalar product at school were able to evaluate a scalar product of two vectors given geometrically. Barniol and Zavala (2014) obtained more precise results by administering a 20-item test on understanding vectors – the TUV – to 423 students in a physics course at a Mexican university. For example, only 33% selected from five options the correct interpretation for the value of the scalar product of two vectors given geometrically using orthogonal projections – an approach that is essential for being able to interpret values of the scalar product in physical contexts. In a further study with isomorphic items involving such physical contexts (work and electric flux), Zavala and Barniol (2013) found that many students had indeed problems to interpret the scalar product in these contexts. For Germany – the location of the study presented here – alike results were obtained. For instance, in a study on beginning undergraduates' understanding of the scalar product by Feudel et al. (2025), only 9.5% of the 74 first-semester participants referred to orthogonal projections when asked for their associations related to the scalar product.

One reason for these results is that calculation is stressed more than this interpretation in the teaching of the concept, as indicated in a textbook analysis by Clark (2019). For instance, nine of the ten analyzed mathematics textbooks defined the scalar product arithmetically as a sum of products, while the relationship to the geometric definition $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\theta)$ that links to angles and orthogonal projections was covered only later – and in three books even not at all. But even if the latter formula is covered, the relationship to orthogonal projections might stay hidden, as shown in a case study by Foster & de Villiers (2016). She observed how a secondary teacher introduced the scalar product. Although the teacher started with the geometric definition via the cosine formula, the connection to orthogonal projections was not made. Furthermore, the teacher did not link this definition to the practical calculation procedure via vector components that relies on the distributive law. The latter, however, is essential for understanding the scalar product as an algebraic structure. Hence, there are some indications in the literature that several associations that are relevant for acquiring an understanding of the mathematical concept of the scalar product – especially its algebraic properties as well as its interpretation in geometric and physical contexts – might not be conveyed at upper secondary level even if the concept is covered there.

To explore this assumption further, we examined the following research question:

Which understanding of the scalar product do preservice mathematics teachers intend for their pupils at upper secondary level and why?

An answer to this question helps, on the one hand, to find explanations on why beginning undergraduates often have a limited understanding of the scalar product as a mathematical concept. On the other hand, the results can point out more generally what preservice teachers would like pupils at upper secondary level "to take away" in connection with mathematical concepts (which is important for teacher education).

THEORETICAL FRAMEWORK

As background framework, we used the construct of *concept image* by Tall and Vinner (1981) that we utilized to conceptualize understanding theoretically: A person's concept image is the total cognitive structure associated with the concept, including all the mental pictures, associated properties and processes. Selden and Selden (2008) highlight several important components of a concept image (p. 103): examples, non-examples, facts, properties, relationships, and visualizations. In addition, it typically includes a personal reconstruction of the definition (Tall & Vinner, 1981).

The German mathematics education community furthermore highlights special kinds of associations that contribute to an understanding of a concept: *Grundvorstellungen,* translatable as *basic ideas.* Greefrath et al. (2016) define a *Grundvorstellung* (GV) of a concept as "a conceptual interpretation that gives it meaning" (p. 101), and emphasize that GVs should be part of a valid concept image. GVs are usually put forward by mathematics education researchers on the basis of a mathematical analysis of the concept with a special emphasis on establishing connections between approaches to

define the concept and relevant applications in outer- and inner-mathematical contexts – grounded in German traditions of subject-based didactics ("Stoffdidaktik"), but also take observations of students' learning processes into account (Salle & Clüver, 2021). The literature highlights three features of GVs (Vom Hofe & Blum, 2016): they should link the concept to familiar knowledge and experiences, generate a mental representation of the concept, and link the concept to real-life applications. In addition, there is a distinction between primary and secondary GVs. Primary GVs are GVs that *evolve* through handling real objects (e.g., dividing apples between childen), while secondary GVs evolve through mental actions with mathematical objects and means of representing these (e.g., observing mentally the behavior of secants lines through the graph of the function when the two intersecting points approach each other).

Based on a mathematical analysis of the definitions of the scalar product, its structural properties, and its applications, Frohn (2020) has put forward four (secondary) GVs pupils should acquire at upper secondary level:

1. *Generalized product:* The scalar product is considered as an operation between two vectors that fulfils certain algebraic properties, such as the distributive law.
2. *Orthogonality indicator*: The scalar product is a number that indicates whether two vectors are orthogonal or not.
3. *Angle indicator:* The scalar product yields a measure for the angle between two vectors. Its sign especially tells whether this angle is acute, right, or obtuse.
4. *Orthogonal projection:* The scalar product $\vec{a} \cdot \vec{b}$ represents the (signed) product of the magnitudes of $\vec{b}$ and the orthogonal projection of $\vec{a}$ onto $\vec{b}$.

The first GV is essential for grasping the algebraic nature of the scalar product, the second and third give the concept a meaning in geometry, and the GV *orthogonal projection* is significant for its contextual interpretations in physics. All of these link relevant applications with approaches to define the concept. We especially used these four GVs as a reference with which we compared the associations that preservice mathematics teachers intend for their pupils, since fostering the development of GVs is formulated as a major guideline in the school curriculum in Germany.

METHODS OF THE STUDY

Participants and data collection

The study took place in a master course "Didactics of Mathematics for upper secondary level" at a large German university. Its participants had passed all of the mathematics modules in the teacher program, especially the linear algebra modules containing the scalar product. All were familiar with the constructs *concept image* and *Grundvorstellungen,* which had been introduced at the beginning of the course, but not yet specified for the scalar product. Instead, as a prototypical example, the instructor had discussed with the participants GVs of the derivative (Greefrath et al., 2016) and further associations related to the derivative that should become part in pupils' *concept images* at upper secondary school. In this process, he also practiced with the

participants a possibility to visualize one's concept image following the idea of a concept map by Novak (2010), namely drawing different components of the concept image into different boxes, such as examples, non-examples, visualizations, or relationships to other concepts.

In the middle of the course, the students received two compulsory homework tasks:

1) *State a concept definition that you would use to introduce the scalar product at upper secondary level. Explain why you consider this definition appropriate.*
2) *Visualize a concept image for the scalar product that your pupils should have after the treatment of the topic in your teaching. Number the components and justify for each component why it should be part of your pupils' concept images.*

The students were allowed to use all material they wanted. 44 students handed in a solution, and 42 agreed that their data may be used for this study. They were familiar with this kind of task at that time, in particular with the above-described possibility to visualize a concept image. A sample visualization is shown in Figure 1.

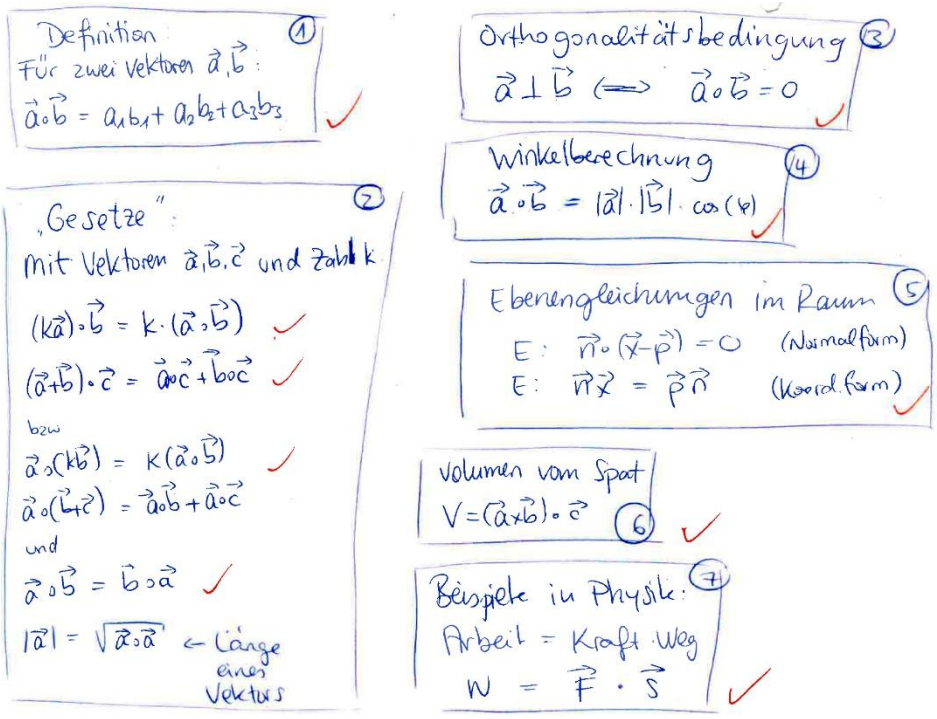


Figure 1: A participant's visualization of the concept images their pupils should acquire

Data Analysis

Concept definitions: First, the first author categorized the definitions our participants would choose to introduce the concept to pupils. As categories, he used the two typical approaches to define the scalar product at school: the arithmetic definition as a sum of products and the geometric definition via $\vec{a} \cdot \vec{b} := |\vec{a}| \cdot |\vec{b}| \cdot \cos(\gamma)$. The reasons for our participants' choices were categorized on the basis of the data. If more than one reason was mentioned, the person received multiple codes. Afterwards, a student from another university recoded all responses, the codes were then compared, and disagreements were resolved in a discussion.

Concept images: First, the lead author categorized the visualizations of the concept images our participants intended for their pupils according to the different components contained. The basis was a category system developed in a previous study on beginning undergraduates' associations related to the scalar product (Feudel et al., 2025). It consisted of superordinate categories based on our theoretical framework: the four *Grundvorstellungen* by Frohn (2020) and further types of associations that are typically part of a *concept image* but not directly relatable to one of these four GVs: the arithmetic formula, relationships to other concepts, and applications. Each superordinate category comprised subcategories based on what the students actually wrote, e.g., the criterion $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$ as a subcategory of the GV *orthogonality indicator*. In the current study, these subcategories were extended by statements that were mentioned by our master students but not in the previous study, e.g., the Cauchy-Schwartz inequality. In addition, one new superordinate category evolved: the link between the different definitions of the concept. Our participants' reasons for including different aspects into the intended concept images were categorized inductively.

Again, a student recoded the data afterwards using the system developed. The results were then compared and disagreements were resolved in a discussion.

RESULTS OF THE STUDY

Results regarding the chosen concept definitions

The concept definitions with which our participants would introduce the scalar product in their own teaching are displayed in Figure 2.

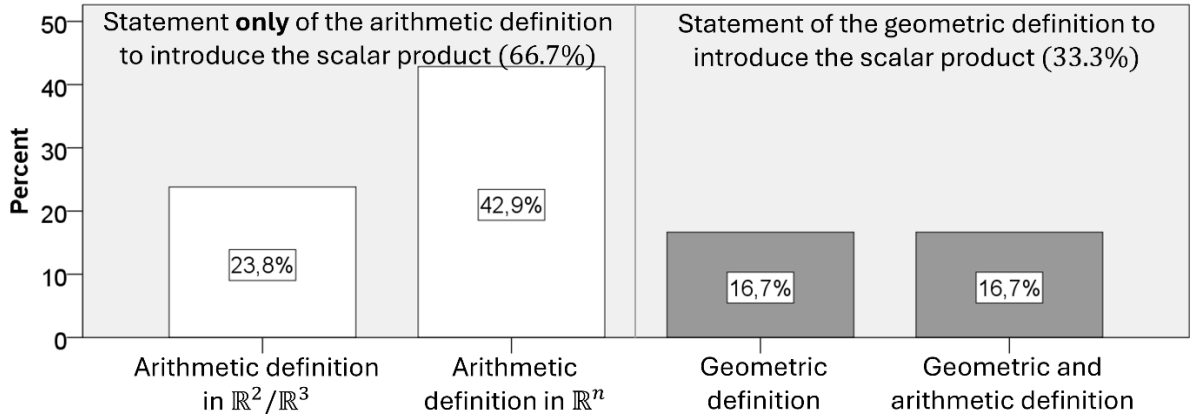


Figure 2: Stated definitions for introducing the scalar product to pupils ($n = 42$)

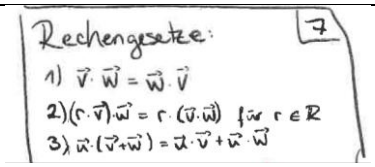
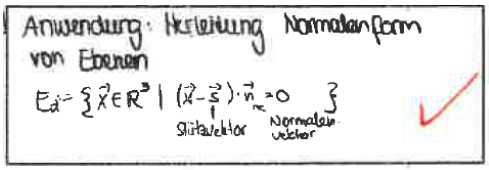
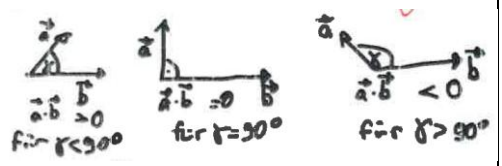
The clear majority stated (only) the arithmetic definition as their approach to introduce the concept, although the geometric definition relates much closer to the GVs *orthogonality indicator, angle indicator,* and *orthogonal projection*, giving the concept a meaning in geometric and extra-mathematical contexts. The main reasons for choosing the arithmetic definition were that this definition is well or better accessible (23.8%) and that the result is straightforward to obtain (14.3%). However, only four students justified this choice by making explicit that the arithmetic definition facilitates an exploration or the justification of the algebraic properties of the scalar product, which could promote the development of the GV *generalized product*. At least, five

further students also highlighted the importance of emphasizing that the result is a number, which would be a first step towards the GV *generalized product*.

Only 33.3% stated the geometric definition as intended approach to introduce the scalar product. The majority of these argued that the connection to orthogonality and the angle becomes immediately clear when using the geometric definition already at the start, which could promote the GVs *orthogonality indicator* and *angle indicator*.

Results regarding the components in the intended concept images

Table 1 shows the associations our participants liked pupils to have after a teaching unit on the scalar product. The numerations show the subcategories found in the data.

Category	Description and subcategories	Examples
Arithmetic formula	The participant mentioned the arithmetic formula of the concept 1) as a definition or 2) in an example with numbers.	"$\vec{x} \cdot \vec{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n$"
Generalized product	The participant mentioned 1) at least one of the algebraic laws of the scalar product, or 2) that it yields a number, or 3) compared it to other products.	
Orthogonality indicator	The participant mentioned 1) the orthogonality criterion involving the scalar product, 2) that the scalar product is used to check for orthogonality, or 3) referred to normal vectors.	"$< v, w >= 0 \iff v \perp w$" 
Angle indicator	The participant mentioned 1) the geometric definition of the scalar product, 2) the angle formular involving the scalar product, 3) that the scalar product is used to determine angles, 4) a relationship between the type of the angle and the sign of the scalar product, or 5) drew a picture of two vectors and the angle in between or referred to this angle verbally.	"For $\vec{a}, \vec{b} \in \mathbb{R}^n$ holds: $\vec{a} \cdot \vec{b} = \lvert\vec{a}\rvert \cdot \lvert\vec{b}\rvert \cdot \cos(\sphericalangle(\vec{a}, \vec{b}))$" „$\varphi = arccos\left(\frac{\vec{a} \cdot \vec{b}}{\lvert\vec{a}\rvert \cdot \lvert\vec{b}\rvert}\right)$" „Application: Determination of the angles" 

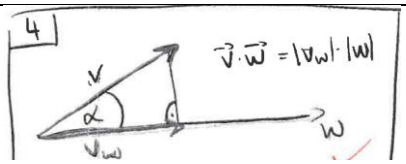
Orthogonal projection	The participant referred to an orthogonal projection verbally, in a corresponding formula, or drew a suitable picture.	
Link between definitions	The participant referred to the cosine theorem as a link between the concept's definitions.	"Derivation of the calculation formula via a triangle and the cosine theorem"
Link to other concepts	The participant mentioned a link to one of the following other concepts at the structural level: 1) the absolute value of a vector, 2) the cross product.	"$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$ with $\|\vec{a}\|$ being the length of the vector" "Cauchy-Schwartz inequality: $\| < u, v > \| \leq \|u\| \cdot \|v\|$"
Other inner-mathematical application	The participant mentioned that the scalar product is used for 1) calculating areas or volumes, 2) calculating lengths/distances, 3) proving geometric statements, or mentioned an unspecific phrase indicating another inner-mathematical application.	"Volume of a parallelepiped: $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$" "Determination of distances between points, lines, and planes" "Proofs in geometry (e.g. Thales)"
Extra-mathematical application	The participant referred to a concrete extra-mathematical application of the scalar product.	"Example in physics: Work = Force · Distance, $W = \vec{F} \cdot \vec{s}$"

Table 1: Categories and subcategories in our participants' intended concept images for pupils ($n = 42$)

Figure 3 shows the categories related to the associations our participants would like their pupils to acquire after a teaching unit on the scalar product at upper secondary.

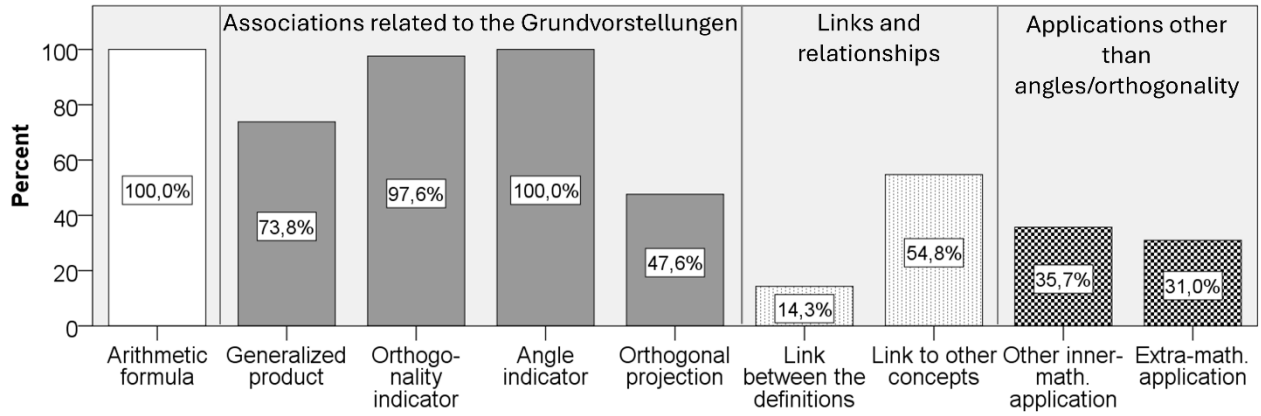


Figure 3: Categories for which our participants visualized associations that should be part in their pupils' concept images after a teaching unit on the scalar product ($n = 42$)

Assuming that the subcategories in Table 1 represent corresponding contents our participants intend to convey as a teacher later, this indicates that all of them aimed to cover the arithmetic formula and the relationship between the scalar product and angles or orthogonality. 92.9% would teach the criterion $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$ that is the central part of the GV *orthogonality indicator*, and even 97.6% would cover the cosine formula $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\gamma)$ (maybe rearranged for the angle), which is an essential component of the GV *angle indicator*. However, only 28.6% wished their pupils to make a link between the sign of the scalar product and the type of angle.

Figure 3 furthermore shows that the majority liked to foster associations related to the GV *generalized product*, namely the algebraic laws the scalar product fulfills (73.8%). However, less than half of our participants intended associations relating to orthogonal projections – and hence to the GV *orthogonal projection*. Interestingly, these were not the same as the 26.2% who liked to cover the extra-mathematical application "work" of the scalar product, although this GV is the mathematical core of this application.

Links to other concepts like the absolute value were only stated by about half of our participants. And finally, only 16.7% mentioned proofs of statements from elementary geometry, which can link the scalar product to pupils' previous knowledge from lower secondary level and illustrate its usefulness within geometry.

Now we discuss the most important reasons why our participants wished to promote certain associations. For the arithmetic formula, the majority argued that this formula is the basis to calculate the scalar product or the basis of the concept overall (52.4% of the cohort). Regarding the algebraic laws, the most often stated reasons for their treatment were that these facilitate calculations (23.8%) or provide an opportunity to handle the definition/practice proofs (21.4%). Only 14.3% expressed the aim of covering these to make the properties of the scalar product as an operation explicit, which would be essential for promoting the GV *generalized product*. Concerning the orthogonality criterion, the most often stated reason was that this is an important application of the concept (52.4%). The same applies to the cosine formula involving the angle (47.6%). Although these two aspects can indeed contribute to assigning a meaning to the scalar product in geometric contexts – based on the GVs *orthogonality indicator* and *angle indicator* – fewer students emphasized this in their reasons for covering the corresponding formulas. For instance, only 28.6% stated to cover the cosine formula to foster an understanding and/or provide a visualization of the concept. Regarding associations related to the GV *orthogonal projection*, on the contrary, the most often stated reason was geometric interpretation/visualization (31.0%).

SUMMARY AND DISCUSSION

Overall, our data indicate that the preservice teachers in our study intended to convey lots of properties of the scalar product that can help to build rich concept images. These associations did not only cover the arithmetic formula, but also geometric properties that can help to build the GVs *orthogonality indicator* and *angle indicator* and give the concept a meaning in geometric contexts. However, the subcategories our participants

mentioned indicate that these might be mainly reduced to the corresponding formulas. This also matches to the main reasons our participants stated for why they wanted to teach these aspects: calculation and practical application, rather than using these to give the concept a meaning, as formulated in the GVs by Frohn (2020). In particular, only 28.6% wished their pupils to make a link between the sign of the scalar product and the type of angle, and no one expressed a verbal idea that the scalar product gives an indication of the size of the angle between two vectors (if the vectors are normed), which is the central idea of the GV *angle indicator*.

For the GV *generalized product*, the results are similar. The majority wanted to convey the algebraic laws that are an essential part of this GV. However, only a few (14.3%) referred in their reasoning to the GV's central idea that the scalar product is a new kind of operation that fulfils certain laws while others are violated. While covering these laws to practice the concept's definition and to practice proof is certainly valuable, it does not yet foster the GV *generalized product* as formulated by Frohn (2020).

Finally, regarding the GV *orthogonal projection,* our data show that only a minority intended to promote corresponding associations. Assuming that our participants' intentions reflect current teaching practices, since preservice teachers tend to teach mathematics in the way they were taught (Prescott & Cavanagh, 2006), this yields one explanation for why (beginning) undergraduates rarely associate orthogonal projections or have trouble using these to interpret scalar products (see introduction).

Overall, our results suggest that preservice teachers for upper secondary level intend to convey to their pupils many properties of the scalar product that *can contribute* to make meaning of the concept in geometric and extra-mathematical contexts. However, the data also indicate that many (not all!) might tend to convey these properties with the aim that pupils can use these effectively in calculations and practical applications, rather than to make use of these to promote an understanding of the concept itself. Therefore, it might be important to discuss in a didactics course how specific properties of the scalar can help to make meaning of the concept in different contexts, e.g. through the means of GVs. This discussion could be accompanied by a suggestion of explicit activities that foster this meaning-making, as for instance proposed by Donevska-Todorova (2015). An interesting starting point for future research is the extent to which our findings from this study are concept-specific or generalizable, in particular if preservice teachers' orientations towards calculations and applications rather than on meaning-making is specific for the scalar product or a more general phenomenon.

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Investigating De-ritualisation in Students' Proof Evaluation Discourse on Mathematical Induction

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Students often approach proof by mathematical induction (MI) as a procedural routine rather than as logical reasoning, which from a commognitive perspective reflects an overemphasis on ritualised participation. This study examines the suitability of a commognitive analytical framework for investigating emerging de-ritualisation in students' discourse through a proof evaluation task involving MI. Data from first-year undergraduate students' group discussions were analysed using five characteristics of de-ritualisation. The analysis reveals emerging objectification, substantiability, bondedness, flexibility and agentivity as students evaluated and ranked arguments based on generality and logical coherence, alongside persistent ritualised routines. These findings indicate that proof evaluation and ranking tasks afford meaningful opportunities for de-ritualisation to emerge, and that the proposed analytical approach provides a productive lens for examining both students' discourse and task design in the context of MI.

Keywords: mathematical induction, commognition, de-ritualisation, proof evaluation task

INTRODUCTION

Proof by MI is a fundamental method in undergraduate mathematics, yet it is considered a difficult method of proof for students (Arnesen & Skartsæterhagen, 2025). Several studies report that many students often approach MI as a sequence of memorised steps rather than as a logical method of proof (Knuth, 2002; Stylianides et al., 2007). In particular, students tend to reproduce the procedural structure of proof by MI, verifying a base case and carrying out an inductive step, without fully understanding why these components are necessary or how they establish generality (Stylianides et al., 2007). Within the Commognitive Theory (Sfard, 2018), this tendency reflects an overemphasis on *ritualised* learning.

Addressing this challenge requires pedagogical approaches that indicate the importance of mathematical analysis of MI while simultaneously creating opportunities for students to move beyond procedural reproduction of proof by MI towards engagement with reasoning and justification, in other words providing opportunities for *de-ritualisation*. As part of a larger study on teaching and learning of MI, we designed a series of activities to provide opportunities for such opportunities and to investigate students' discursive practices evolution (Lestyanto & Pfeiffer, 2025).

One of these activities is a proof evaluation task, which encourage students to reflect, discuss, and evaluate several given proof attempts. The task also includes a request to rank the arguments in order of preference. This task invites students to attend to the

logical structure, generality, and communicative clarity of the arguments, and thereby provides opportunities to reveal both ritualised and de-ritualisation forms of discourse.

This study contributes by operationalising de-ritualisation characteristics as an analytical lens for investigating students' discourse in MI, and by illustrating how these characteristics can be identified in the context of a proof evaluation task. Accordingly, this study addresses the following research question: To what extend is the proposed commognitive analytical tool suitable for investigating emerging de-ritualisation in students' discourse, and how effectively does the design of a proof evaluation task involving MI afford opportunities for identifying such de-ritualisation?

STUDIES ON MATHEMATICAL INDUCTION

MI is a proof method to prove that a statement $P(n)$ holds for all integers in $D = \{n \in \mathbb{Z} | n \geq n_0\}$. It requires establishing two interdependent components: the base case, which establishes $P(n)$ for an initial value $n = n_0$, and the inductive step, which proves the implication $P(k) \rightarrow P(k+1)$ for an arbitrary k in D. Research has documented students' difficulties in understanding both the meaning and the logical necessity of this method. Early studies (e.g., Knuth, 2002) show that students often rely on memorised templates when constructing MI proofs, reproducing familiar procedures without engaging with the underlying reasoning. From a commognitive perspective, such behaviour can be interpreted as the enactment of *ritualised* routines, where participation is guided by established procedures rather than by the mathematical meaning of the task. Similarly, Stylianides et al. (2007) report that even pre-service teachers frequently struggle with the essence of base case of proof by induction and the meaning associated with inductive step in proving the implication $P(k) \rightarrow P(k+1)$ for an arbitrary k in the domain discourse of $P(n)$. These difficulties indicate a gap between *ritualised* and *exploratory* mathematical discourse.

While various instructional approaches have been proposed to address these challenges, such as *quasi-induction* (Harel, 2001), *explanatory* proofs (Stylianides et al., 2016), and the use of visual representations (Relaford-Doyle & Núñez, 2021), much of this work focuses primarily on cognitive aspects of learning. Less attention has been given to research that considers social aspects of learning MI (Arnesen & Skartsæterhagen, 2025). Telloni and Malara (2021) suggest a comprehensive 'teaching path' and represent an early effort in this direction.

On the other hand, recent research has emphasised the importance of engaging students not only in constructing proofs but also in evaluating and understanding mathematical arguments. Proof evaluation tasks, in which students analyse and judge the validity of given arguments, have been shown to support the development of proof comprehension and critical reasoning (Moralı & Filiz, 2023; Pfeiffer & Quinlan, 2015). Such tasks shift the focus from producing proofs to examining their logical structure, encouraging students to attend to issues such as generality, justification, and coherence. In the context of mathematical induction, these activities may help students move beyond procedural reproduction by prompting them to reflect on the role of the base case, the

inductive step, and the establishment of general validity. Building on this perspective, our study extends the social aspect of teaching and learning on MI by investigating how students' discursive practices unfold during a proof evaluation task involving MI, with particular attention to how students rank, compare, and justify different arguments. Investigating these processes requires a theoretical framework that accounts for both mathematical reasoning and discourse, which motivates the use of commognitive theory in this study.

THEORETICAL BACKGROUND

Commognition is a theory that views learning as the development of discourses, meaning that it is viewed from sociocultural perspectives, emphasizing communication and cognition as intertwined processes. In this theory, discourse is characterised by unique language or *keywords*, *visual mediators*, *routines* and *endorsed narratives*, which together define how mathematical objects and processes are communicated and understood (Sfard, 2018). Commognition views mathematics as a form of communication shaped over time by communities of people (Sfard, 2018).

Within this framework, particular attention is given to *routines*, defined as the pair of *task* and *procedure* (Lavie et al., 2019), where the task is what the participant wants to achieve, and the procedure is the way she achieves it. These routines may be *ritualised* or *explorative. Ritualised* routines are performances which include mimicking or rigidly following previously known procedure and *explorative* ones include exercising agentivity, making independent choices, and flexibly adapting procedures (Nachlieli & Tabach, 2022).

Within the commognitive theory, **de-ritualisation** refers to the process through which ritualised routines are transformed into explorative ones (Lavie et al., 2019). The six key characteristics of this transformation include *flexibility,* where multiple methods emerge for performing a task; *bondedness*, where steps in a routine are interconnected, with outputs of any step carried out in the procedure are used as input to the next step; *applicability*, as routines become applicable in different contexts; *performer's agentivity*, where learners independently make decisions and initiate tasks without external guidance; *objectification*, where learners increase the level of abstraction of a mathematical object; and *substantiability,* where learners justify their actions based on outcomes rather than merely describing processes (Lavie et al., 2019). Facilitating this transformation poses one of the key challenges for mathematics teaching (Lavie et al., 2019). Accordingly, there is a call for more research on de-ritualisation processes and task design to support these. Several studies operationalised de-ritualisation characteristics, e.g. Nachlieli & Tabach (2022) when considering classroom learning of arithmetic topics, and Karavi et al. (2022) identified such characteristics in a lecturer's discourse of proof teaching. Our study contributes by developing a proof evaluation task involving MI to promote de-ritualisation and developing an analytical framework to analyse students' de-ritualisation evolution during group discussions.

In this study, de-ritualisation is understood as a transformation in students' engagement with proof by MI. Indicators of de-ritualisation in students' MI discourse may involve moving from listing steps to explaining logical mechanism, articulating why assuming $P(k)$ is legitimate for hypothesis, discussing the method itself, or recognising when proof by induction applies, or comparing it with other proofs.

THE PROOF EVALUATION TASK

Proof evaluation tasks, in which students analyse and compare given arguments, have been used to support students' understanding of mathematical proof (Moralı & Filiz, 2023; Pfeiffer & Quinlan, 2015). Building on this approach, our proof evaluation task, see Appendix, was designed to engage students in analysing and comparing different arguments for the same mathematical statement in the context of MI. Rather than constructing proofs themselves, students were asked the following questions about those arguments: 1) 'Rank the arguments in order of your preference, include some comments to explain your ranking', 2) 'Include your comments about these arguments', and 3) 'Find any errors in the arguments and advise to resolve the errors'. This task was intended to emphasise proof as an object of reflection and discussion, and to foster de-ritualisation processes in students' discourse on MI.

The task involves six arguments that differ in their mathematical validity, structure, and level of clarity. Some arguments follow the standard structure of MI, some present valid but different forms of proof methods, while others contained specific limitations or flaws. These variations are deliberately designed to highlight key aspects of MI, particularly the role of the base case, the logical function of the inductive step in establishing the implication $P(k) \rightarrow P(k+1)$, and the requirement that a proof must establish the truth of a statement for all values in the domain. By comparing arguments with different features, students are expected to engage with questions such as: What makes an argument valid? What role do the base case and inductive step play in ensuring validity of an argument?

Framework for Analysing De-ritualisation

In this study we operationalise five of the six de-ritualisation characteristics[1] and use these as an analytical tool to investigate how they appear in student discourse: namely flexibility, bondedness, agentivity, objectification, and substantiability. These characteristics are interpreted below in relation to both the mathematical content of induction, and the demands of evaluating proofs and ranking task.

In the proof evaluation task, **flexibility** is evidenced when students compare different arguments and recognise that mathematical validity is not restricted to a single procedural form. Ritualised routines occur when students accept only arguments that match the standard template of MI, whereas explorative routines involve evaluating alternative reasoning and comparing their strengths.

[1] We did not examine applicability since our activities do not encourage students to use mathematical induction in different contexts.

Bondedness is observed when students attend to the logical coherence between different components of a proof, particularly in MI, the relationship between the base case, inductive hypothesis, and inductive step. Ritualised routines focus on the presence of steps, while explorative routines involve examining whether these steps are meaningfully connected and justified.

Agentivity refers to students' willingness to take initiative in proposing, challenging, and revising evaluations of arguments. In the proof evaluation task, this is reflected in students making independent judgements about the ranking. Ritualised routines involve ranking of arguments based on familiarity, whereas explorative routines involve active questioning and justification.

Objectification occurs when students treat proofs as objects of reflection and discussion rather than as procedures to reproduce. In the context of proof evaluation, ritualised routines focus on recognising procedural forms, while explorative routines involve comparing and critiquing proofs as mathematical objects.

Substantiability is evident when students justify their evaluations based on logical reasoning or communicative considerations. For example, comments like "this proof is correct" or "this proof is wrong" without explanation can be indications of ritualised routines, whereas explorative routines involve explaining why a proof is valid/invalid, citing logical steps, clarity, or communicative quality.

METHODOLOGY

As part of a larger study on teaching and learning of MI, we conducted a series of activities involving MI consisting of three sessions, each 100 minutes long. The proof evaluation task is one of the tasks in Session III. The participants were 14 first-year volunteers of undergraduate students enrolled in a 4-year mathematics-education program at a public university in Indonesia which prepares students to become secondary mathematics teachers. In their first year, students take a compulsory course that introduces logical thinking and mathematical proofs. Data collection was conducted outside the schedule for this course.

The participants were divided into four groups and engaged in collaborative group discussions. Data collected for this study include video recordings of group discussions and students' written responses to the proof evaluation task. For the purposes of this paper, we focus on selected excerpts from two groups (called Mawar and Tulip), which provide insights into students' discursive engagement during the proof evaluation task. We refer to individual participants within the Mawar group as M1, M2, M3, and M4 whilst within the Tulip group as T1, T2, T3 and T4.

The video recordings were transcribed in the original language (Indonesian), and selected excerpts were translated into English for reporting. The analysis focuses on students' discursive practices during group discussions, particularly how they evaluate, compare, justify, and rank different mathematical arguments. Students' discourse was analysed using the de-ritualisation framework described in previous section.

FINDINGS

In this section we present how the chosen five characteristics of de-ritualisation appear in students' discourse when they are engaged in the proof evaluation task.

A prominent feature in students' discourse was the emergence of *objectification*, as students began to treat proofs as objects of evaluation rather than as procedures to reproduce. This was evident when students recognised that the task required them to judge arguments rather than to determine whether statements were true. For example, T2 noted that "*The task doesn't ask us to decide if it's true or false, but about its validity*". Such remarks indicate a shift toward viewing proofs as objects to be analysed, compared, and discussed.

Students' discourse also indicated the emergence of *substantiability*, as students increasingly justified their evaluations based on logical and communicative criteria. Initially, both groups evaluated arguments based on their structural completeness and ranked arguments according to whether they followed the familiar structure of mathematical induction. For example, T2 argued "*Because it* (Bowo's argument) *has steps and fulfil inductive step requirement*", or M2 proposed *"Ira is first because the steps are complete, from the base case to the inductive step."* However, as the discussion progressed, students began to refine their criteria. For instance, when comparing two similar arguments (Bowo's and Ira's), some noticed that one argument explicitly stated the general conclusion, while the other did not. M1 realised "*Oh, does it mean that it should be Bowo's argument as the first rank? I didn't see that difference before. Because he states that* $n > 0$", and M2 added "*While Ira doesn't show where the* n *starts*". This shift indicates movement from judging based on procedural completeness to evaluating whether an argument establishes generality, a key aspect of *substantiability*. Further evidence of *substantiability* emerged when students rejected arguments based solely on examples. When discussing Ajeng's example-based argument, T2 remarked "*She just picked some numbers… this cannot prove the statement"*.

While students showed progress in evaluating arguments, their discourse shifted between ritualised and de-ritualisation performance. In some cases, students attended to the connections between components of a proof, indicating emerging *bondedness*. For example, when discussing logical errors in Ira's argument, students commented on its inductive step:

> T4: The base case is correct, but in the $S(k)$, she … (pause)
>
> T2: Overall, the inductive step in Ira's argument is correct, but she does not specify the *k*.

Some students began to reflect on the underlying logic of proof by induction; however, they quickly reverted to their prioritised ritualised routines. A notable instance is students' evaluation of Ira's argument that contains significant logical errors in the inductive step and lacks clarity regarding the base case. Despite these issues, some students initially ranked the argument highly and considered it as a valid argument

because it follows the familiar structure of MI. Students frequently focused on the presence of procedural elements rather than on their logical function. In terms of *bondedness*, the base case and inductive step were recognised as required components, but none of the students critically examined their role in establishing the initial value and the implication $P(k) \rightarrow P(k+1)$. This shows that ritualised routines remained a significant part of students' discourse, where validity of arguments is associated with form rather than with logical coherence.

Students' discourse also revealed occurrence of *flexibility*, by considering alternative forms of reasoning beyond the template of MI. For instance, when encountering Razak's valid argument that does not use MI, some students recognised its validity while also commenting on differences in clarity ("*It is true, but the way is too difficult to be understood"*, M2).

Students' discourse also includes instances of *agentivity* through active participation when proposing, challenging, and revising their rankings. In several instances, students questioned initial rankings and suggested changes based on outcomes of their discussion:

M1: Should this be first? I didn't notice that difference before.

Such exchanges indicate that students take initiative in negotiating their evaluations rather than simply accepting given arguments, reflecting a more explorative orientation.

However, *flexibility* and *agentivity* in students' discourse were not always sustained across discussions. In some cases, students focussed on familiar criteria, such as preferring arguments that appeared simpler or structurally complete, even when these lacked sufficient justification. This reflects how students' discourse shifts between established procedural orientations and emerging reflective considerations, highlighting the dynamic interplay between ritualised and explorative participation.

Discussion and Concluding Remarks

Common mathematical induction exercises are to prove given statements using MI. Consequently, many students view MI as a process (Arnesen & Skartsæterhagen, 2025). Our findings indicate that the suggested proof evaluation task provides opportunities for students to move beyond procedural reproduction toward engagement with reasoning and justification on MI.

Our choice of five characteristics of de-ritualisation for investigating students' discourse proved useful. A key finding is the emergence of *objectification* and *substantiability*, as students began to treat proofs as objects of evaluation and justify their judgements using logical and communicative criteria. Through comparing arguments, students attended not only to the presence of procedural steps but also to issues of generality and clarity. Previous research has shown that students often rely on empirical verification and may treat checking examples as sufficient to establish generality (Stylianides et al., 2007). In our findings, students' discourse illustrates

moments in which this orientation is reconsidered, as some students recognised that arguments based on checking specific cases do not establish the truth of a statement for all values. This suggests that engaging in proof evaluation and ranking activities can create opportunities for students to reflect on the limitations of empirical verification and to attend more closely to the requirement of generality in mathematical induction.

Students' evaluation of Ira's argument indicates a dynamic interplay between ritualised and explorative routines in their discourse, rather than a linear transition between the two. Despite containing logical errors in the inductive step and lacking a clearly justified base case, the argument was initially judged as valid because it followed the familiar structure of MI. This suggests that students' participation was still partly guided by procedural recognition, consistent with findings that learners often rely on template-based approaches to MI (Knuth, 2002). This pattern underscores the inevitability of ritualised routines in mathematical learning and their role as stepping stones for new discourses (Lavie et al., 2019), and in line with Nachlieli and Tabach's (2022) report of partial substantiation in students' solutions.

At the same time, moments of *flexibility* and *agentivity* were observed when students compared, ranked, and revised their evaluations of different arguments. This shift supports the view that learning mathematics is a gradual process of de-ritualisation (Lavie et al., 2019).

Taken together, these findings suggest that proof evaluation and ranking tasks have the potential to foster de-ritualisation processes. At the same time, the persistence of ritualised tendencies indicates that such activities require careful design to more explicitly foreground the logical structure of MI. In particular, students' acceptance of example-based arguments (as in Ajeng's case) and their overlooking of errors (as in Ira's argument) point to the need for tasks that explicitly address the distinction between empirical verification and proof, as well as the role of each component of MI in establishing generality.

In conclusion, this study shows that students' discursive practices in a proof evaluation task reveal both ritualised and emerging de-ritualised performances in their discourse on MI. By integrating a commognitive perspective with deliberate task design, the study demonstrates that de-ritualisation can be productively operationalised as an analytical lens, and that proof evaluation and ranking tasks provide meaningful opportunities for identifying such discursive shifts. These findings indicate that the proposed analytical approach is suitable for examining both students' discourse and the affordances of proof evaluation and ranking tasks in supporting de-ritualisation.

Acknowledgment

Indonesia Endowment Fund for Education or LPDP for funding this study.

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Appendix

Proof Evaluation Task

Santi, Ira, Alex, Bowo, Ajeng, and Razak are thinking about proving the following statement.
Statement: The sum of the first n positive odd integers is n^2.

Santi's Argument

The sum of the first n positive odd integers can be written as $\sum_{k=1}^{n}(2k-1)$, and we get $\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = 2\frac{n(n+1)}{2} - n = n^2$.

Ira's Argument

Let S_k be the statement $1+3+5+\cdots+(2k-1)=k^2$.

- Base Case: $1+3=4=2^2$ is true.
- Assume that statement S_k is true for all k.
- Inductive Step: $1+3+5+7+...+(2k-1)+(2k+1)=(k+1)^2=k^2+(2k+1)=S_k+2k+1$

So the statement is true.

Alex's Argument

Let $n=9$.
Then we get for the LHS:

$$1+3+5+7+9=25,$$

and for the RHS:

$$9^2=81$$

As $25 \neq 81$, the statement is wrong.

Bowo's Argument

Let $P(n)$ be the statement $\sum_{i=1}^{n}(2i-1)=n^2$.
Then $P(1)$ is true as $2(1)-1=1=1^2$ is true.
Let $k>0$ and assume $P(k)$ is true.
Then $P(k+1)$ is true as

$$\sum_{i=1}^{k+1}(2i-1)=\sum_{i=1}^{k}(2i-1)+[2(k+1)-1]=k^2+2k+1=(k+1)^2$$

Consequently, $P(n)$ is true for all $n>0$

Ajeng's Argument

The first seven odd positive integers are $1,3,5,7,9,11,13$. Adding these numbers up gives 49, which is equal to 7^2. So the statement is true.

Razak's Argument

The sum of the first n positive odd integers is the sum of the first $2n$ positive integers minus the sum of the first n positive even integers. The sum of the first n positive integers is $\frac{n(n+1)}{2}$, and the sum of the first n even integers is 2 times the sum of the first n integers. Putting this together gives $\frac{2n(2n+1)}{2} - 2\frac{n(n-1)}{2} = n(2n+1)-n(n+1)=n^2$.

Design research on a card-based learning environment for exploring congruence classes and modular arithmetic

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Keywords: modular structures, conceptual understanding, design research

RESEARCH TOPIC

The concepts of congruence relations and modular arithmetic are highly relevant mathematical concepts in the introductory phase of study, since firstly, they provide the basis for simple but non-trivial examples that go beyond school mathematics for many algebraic concepts in university mathematics (e.g. (finite) groups, quotient structures), secondly, they are the starting point for relevant more advanced areas of mathematics (e.g. finite geometry) and thirdly, they form a central basis for highly relevant areas of mathematical application (e.g. cryptography). For many students, developing an understanding of this area at the beginning of their studies often involves significant conceptual difficulties (e.g. Asiala et al., 1997; Migallón et al., 2025; Schüler-Meyer, 2019, Siebert & Williams, 2003).

The poster presents innovative learning material that aims to build conceptual understanding of congruence classes and modular arithmetic. Specifically, it involves the design and exploratory research of card-based manipulatives for use in bridging courses and first-year courses, as well as in outreach activities for school students. The basic idea is to design cards with a natural number on the front and symbols and color codes on the back representing the remainders of this number when divided by 2, 3, 4, 5, 6 or 7. The cards function as versatile mathematical manipulatives in a gamifiable setting: they provide a tangible visualization of modular structures and serve as a basis for concept-related visualizations and explanatory actions ranging from competitive sorting tasks to investigating the well-definedness of modular operations.

THEORETICAL FRAMEWORK

The conception and research of the material and its use align with topic-specific design research (e.g., Gravemeijer & Prediger, 2019). In this iterative framework, learning environments are designed and implemented based on the current state of research, and then studied empirically with a focus on the design's objectives. This yields both research insights and approaches for redesigning the learning environment, as well as a new cycle of design research. Grounded on existing literature and a theoretical analysis (Hoffmann, in preparation), this poster presents both the theory-based design and the initial results of empirical studies on local theory formation.

DESIGN AND RESEARCH AIMS

The project pursues two interconnected aims: the theory-based design of the cards and associated activities, and the empirical investigation of learning processes that arise when students work with the material, particularly regarding congruence classes and

the well-definedness of arithmetic operations. An important research goal here is to assess its suitability for fostering conceptual understanding. The poster will present key design principles, card prototypes, selected student artefacts, and short tasks sequences to illustrate the design rationale and typical learning pathways.

EXPECTED OUTCOMES AND CONTRIBUTION TO THE FIELD

By achieving the design and research aims, a basis for empirically sound further development of the material and the underlying design principles is created in line with the cyclical design research approach. In addition, empirical insights into individual learning paths are to be expected, which will make a relevant scientific contribution beyond the specific project.

ACKNOWLEDGEMENTS

Funded by the German Research Foundation (Project ID 491392403 – TRR 358). I thank Tobias Weich, Marceau Speczyk, Lea-Marie Rosanka, and Jana Hamade for their contributions to the material development.

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Exploring the features of linear algebra Peer Instruction questions

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Abstract: Peer Instruction (PI) is widely used in undergraduate mathematics, yet its effectiveness depends heavily on the design of its questions. This study analyses a bank of 132 linear algebra PI tasks, building on a previously identified set of 20 expert-perceived features. We use factor analysis to investigate the structure of these features. The results yield a concise set of six underlying factors. These factors offer a practical framework to support the design of effective PI questions.

Keywords: linear algebra; formative assessment, task design, factor analysis.

INTRODUCTION

Classroom assessment is an important component of active learning approaches at undergraduate level. We consider one such approach: Peer Instruction (PI) (Mazur, 1997). This method consists of the lecturer posing a multiple-choice question, which students answer individually using a classroom response system (often called "clickers"). The lecturer reviews the voting results and, if there is disagreement, invites the students to discuss before voting again.

The design of questions is crucial to productive discussions in PI, with Mazur (1997) noting that "the success of the method largely depends on [their] quality and relevance" (p. 26). Prior work has shown that task features can influence student engagement and reasoning (Henningsen & Stein, 1997), but little is known about the features of questions that encourage productive discussions in lectures.

Here we consider a large bank of linear algebra questions which was developed, revised and used for PI for more than a decade (Docherty, 2023). Questions in this bank are known to have generated productive discussions, so it makes sense to study their features. In a prior study (Docherty et al., 2024), we used a structured process to elicit a set of 20 constructs that expert instructors perceived in this collection. We categorised these constructs into two broad types, based on whether they focused on *Format* (e.g., "options look very similar") or on the *Mathematical Thinking* involved in the task (e.g., "multiple ways to approach"). Our present study builds on this prior work by exploring these constructs in more detail, across the collection of 132 questions.

We address the research question: *How do the features of questions vary in a collection of questions used for Peer Instruction in linear algebra?* To address this question, we classified each of the 132 questions in relation to each of the 20 constructs. We analyse the distribution of scores for each construct, and carry out a factor analysis to explore how these features are related. Our main contribution is to present a short list of (six) factors which can be used both to evaluate existing question banks and to inform task designers' work to develop useful Peer Instruction questions.

LITERATURE REVIEW

Design of Peer Instruction questions

There is a growing body of literature concerned with the design of PI questions. In a recent systematic review, Akveld, Docherty et al. (2025) synthesised existing guidance on the construction of multiple-choice questions (MCQs) for university classroom voting. Since PI was a prominent approach, several findings from the review are pertinent to our study.

The review highlighted the importance of considering the intended purpose of each question and designing the question accordingly. Prior literature describes a range of purposes, including introducing new topics, reviewing previously taught concepts, providing opportunities for in-class practice, and stimulating discussion. Several studies focused on stimulating discussion; they suggested that questions that demand higher-order thinking are more likely to spark discussion, although empirical evidence is limited. For example, Cline et al. (2020) tried to identify features that led to large numbers of wrong answers, since those questions might best elicit discussion, and found that "investigations of general cases or problems without numbers" was significantly correlated with voting outcomes.

The review also underscored the importance of other design features, such as the use of confidence data and well-chosen incorrect options. Stewart et al. (2013) found that asking students to self-rate their confidence in their responses can provide diagnostic information to support future question design. Furthermore, including distractors that reflect common misconceptions was identified as an important design feature. Akveld, Heimer and Wachter (2025) also report evidence that variation in the nature of incorrect options can influence student outcomes in formative quiz contexts.

Elsewhere, Gustafsson and Ryve (2022) sought to develop design principles and task types for classroom voting in secondary schools. They identified three design principles ("Conceptual Understanding", "Procedural Understanding" and "Misconceptions and Errors") and six task types (including "Multiple Defendable Answers" and "Evaluating Statements"). The findings are in broad agreement with those of Akveld, Docherty et al. (2025) in that "there seems to be a consensus among several researchers that CRS tasks in MC format should address a specific learning goal […] have logical distractors that elicit conceptions and misunderstandings […] and promote a wide range of responses" (p. 3046).

Analysing the features of questions

We see two broad approaches to analysing task features in the literature. The first approach is based on applying general frameworks to analyse features of mathematical tasks. For example, the MATH Taxonomy proposed by Smith et al (1996) offers a hierarchical model for analysing the complexity of mathematical tasks, according to their cognitive demand. The second approach is to seek the views of experts. Indeed, in a systematic review of research on task design, a key theme was the need to involve

teachers as task (re)designers (Johnson and Ohtani, 2025). As an example of this approach, Gustafsson (2024) investigated the whole-class discussions that took place following votes on MCQ, involving teachers in the design of the tasks. The present study adopts this second approach, drawing on our previous work (Docherty et al., 2024), in which we elicited the views of experts about the features of PI questions using Kelly's Repertory Grid technique.

THEORETICAL FRAMEWORK

Our approach to analysing PI question features is grounded in Personal Construct Theory (Kelly, 1955) and employs the Repertory Grid Technique which was developed as a tool within this framework. Kelly posited that we interpret the world through personal constructs, that is, dimensions with two extremes called *poles* (such as easy-difficult, short-long), and that our beliefs and actions are the result of a complex web of such constructs. It is usually very difficult to disentangle this web, but the Repertory Grid Technique aims to make these implicit constructs explicit.

The technique works by asking participants to compare sets of elements (e.g., linear algebra questions) in groups of three, identifying ways in which two are similar and the third different, and then naming the poles of the resulting constructs. This process is repeated as often as necessary until no new constructs emerge. Participants then rate each element on a scale of 1-5 for each construct, with a score of 1 representing strong agreement with the left-hand pole of the construct and a score of 5 indicating strong agreement with the right-hand one. Thus, at the end of the interview the participants will have created a set of constructs (along with their poles) and have rated a subset of the elements on scales derived from these constructs.

METHOD

In a previous study (Docherty et al., 2024), we employed the Repertory Grid Technique with a group of university mathematicians who had experience of teaching linear algebra and/or peer instruction. This group considered the question bank tasks and produced a list of 20 constructs. When the experts scored the presence of these constructs in a set of 20 questions, we found that scores for some constructs were in complete agreement, while for other constructs there was some disagreement. This raises the question of how reliably the constructs can be identified. Nevertheless, the mean scores produced by the experts and by the researchers on this set of sample questions were in broad agreement, giving confidence that we as researchers could interpret the constructs in the way that the experts had intended.

To score the features of the whole collection of 132 questions, we developed an agreed approach to scoring the 20 constructs, which we documented in a scoring handbook (available at osf.io/cyq4b). To do this, we first scored 20 questions independently then discussed and resolved disagreements; we then repeated this process with a further 20 questions. For 9 of the constructs, our inter-rater reliability was sufficiently high (Krippendorf's alpha > .7) that we proceeded with a single scorer, dividing the 132

questions roughly evenly between all four authors. For the other constructs, two authors independently generated scores, then discussed and resolved any disagreements.

To analyse the scores, we carried out a factor analysis. This enabled us to identify a smaller set of underlying factors that can be used to describe variation in the questions, rather than using all 20 constructs. The factor analysis also provides "loadings" for each construct onto each of the underlying factors; these loadings highlight relationships between constructs, by identifying constructs that load strongly onto the same factor. To interpret the meaning of the underlying factors, we examined the constructs that loaded strongly onto each factor, as well as questions with the highest and lowest loadings on that factor, to develop a plausible interpretation.

RESULTS

We first examine how the construct scores were distributed across the question bank and highlight some observations arising from our coding. Then we report on a factor analysis carried out to identify structure in our set of constructs.

Variation of features: distribution of scores

The overall distribution of scores on each construct is shown in Figure 1.

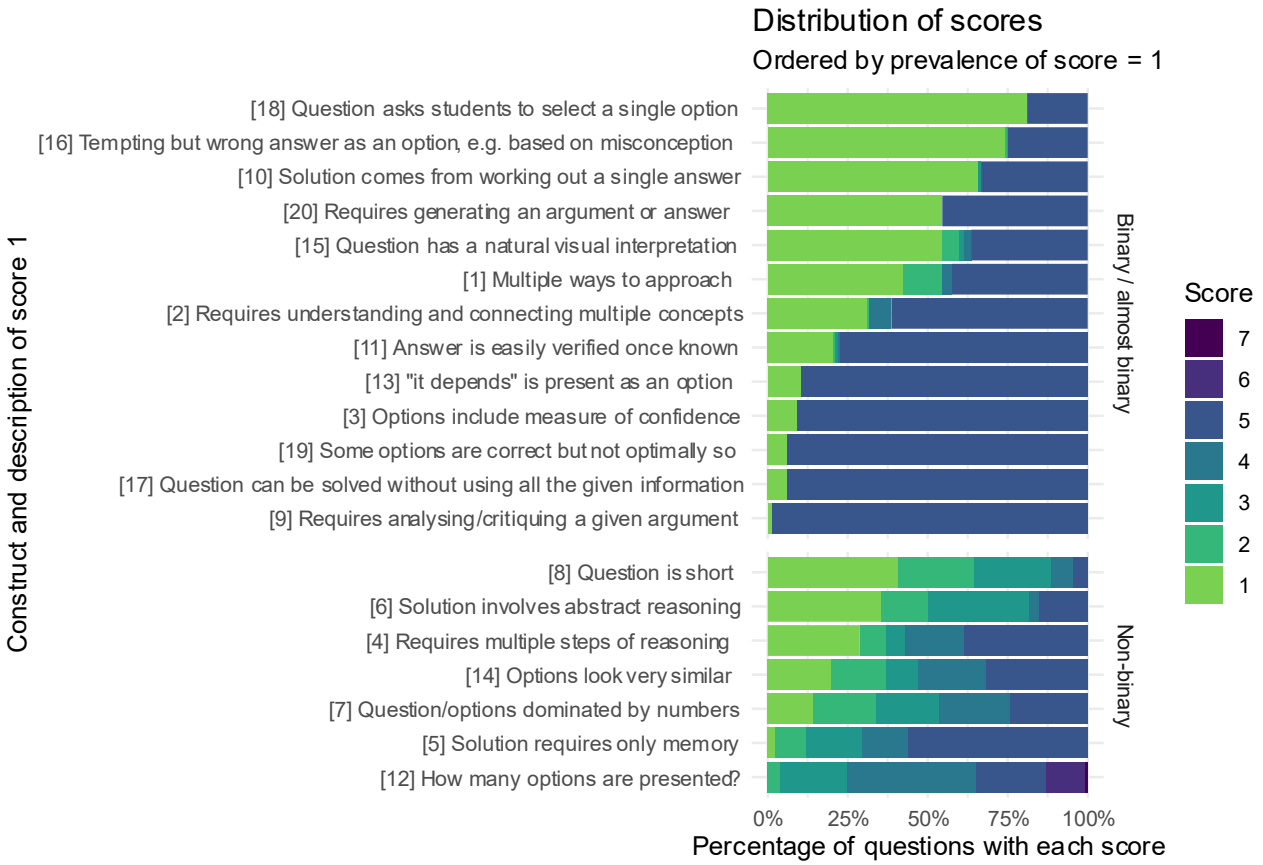
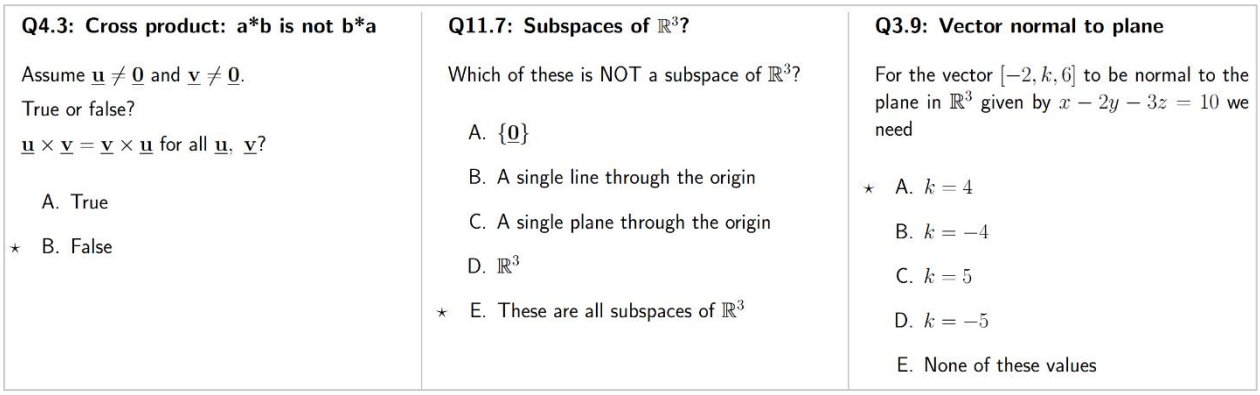


Figure 1: Distribution of scores for the 132 questions on each construct.

In line with our agreed approach to coding, many of the constructs were binaries; however, in practice, some other constructs were "almost binary", in the sense that scores of 1 and 5 were used in over 80% of cases. The distribution of scores for

constructs 6 and 8 (at the top of the "non-binary" group of constructs) shows that the questions tend to be short and to involve abstract reasoning (rather than computations). This is consistent with the advice from Mazur (1997) that PI should focus on conceptual rather than procedural questions, and also reflects the practical constraint that questions should be short enough that students can meaningfully engage with them during a few minutes of lecture time.

Other features were more balanced. For instance, the binary construct 20 showed a roughly even split between questions that require generating an argument or answer (such as Q3.9 shown in Figure 2) and questions that require checking or verifying something that is given (such as the stem in Q4.3 and the answer options in Q11.7 shown in Figure 2). The "checking or verifying" type of questions clearly fit the multiple-choice format, since students are provided with statements to check; in contrast, the "generating" questions could arguably be presented in open response format. However, the benefit of the multiple-choice format for PI is that the distractors can highlight possible misconceptions. Indeed, the distribution of scores for construct 16 shows that a majority of questions feature a "tempting but wrong answer as an option, e.g. based on misconception".



Q4.3: Cross product: a*b is not b*a

Assume $\underline{\mathbf{u}} \neq \underline{\mathbf{0}}$ and $\underline{\mathbf{v}} \neq \underline{\mathbf{0}}$.
True or false?
$\underline{\mathbf{u}} \times \underline{\mathbf{v}} = \underline{\mathbf{v}} \times \underline{\mathbf{u}}$ for all $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$?

A. True
⋆ B. False

Q11.7: Subspaces of $\mathbb{R}^3$?

Which of these is NOT a subspace of $\mathbb{R}^3$?

A. $\{\underline{\mathbf{0}}\}$
B. A single line through the origin
C. A single plane through the origin
D. $\mathbb{R}^3$
⋆ E. These are all subspaces of $\mathbb{R}^3$

Q3.9: Vector normal to plane

For the vector $[-2, k, 6]$ to be normal to the plane in $\mathbb{R}^3$ given by $x - 2y - 3z = 10$ we need

⋆ A. $k = 4$
B. $k = -4$
C. $k = 5$
D. $k = -5$
E. None of these values

Figure 2: Questions involving checking (Q4.3, Q11.7) or generating (Q3.9).

The distributions in Figure 1 also show that some of the features were very rare; for instance, a score of 1 for construct 9 ("Requires analysing/critiquing a given argument") occurred only twice in the collection of questions. Both of these questions asked students to validate a short proof given in the stem of the question (e.g., Q12.5 had a proof that "If T is a linear transformation then $T(\underline{\mathbf{0}}) = \underline{\mathbf{0}}$"). This is perhaps not surprising, given that proof was not a main aim of the course.

While scoring the questions, we noticed that there were two different ways that misconceptions could be addressed by a question. One approach was for the question to directly address a misconception. For instance, Q9.4 (shown in Figure 3) focused on the common misconception that matrix algebra obeys the same rules as for real numbers. Another approach was to include misconception-based answer options in a more incidental way; for instance, Q16.8 is a question about eigenvalues and eigenvectors, but includes answer option A which would result from the misconception

about matrix algebra that they should "square both sides" of the equation $Av = \lambda v$ to get $A^2v^2 = \lambda^2 v^2$.

Q9.4: AB = CB implies A= C

Suppose A, B, C are matrices (with $B \neq 0$) and $AB = CB$. Then $A = C$.

A. True and I am certain
B. True but I am not sure
C. False but I am not sure
⋆ D. False and I am certain

Q16.8: Eigenvalue of Matrix Squared

Let A be a matrix with eigenvector $\mathbf{v}$ and corresponding eigenvalue λ. Then

A. $\mathbf{v}^2$ is an eigenvector of A^2 with eigenvalue λ^2
B. $\mathbf{v}^2$ is an eigenvector of A^2 with eigenvalue λ
⋆ C. $\mathbf{v}$ is an eigenvector of A^2 with eigenvalue λ^2
D. $\mathbf{v}$ is an eigenvector of A^2 with eigenvalue λ
E. We can't say anything in general about the eigenvectors and eigenvalues of A^2

Figure 3: Examples of questions that address misconceptions in different ways.

The questions in Figure 3 illustrate two other features that we saw in the question bank. First, Q9.4 includes confidence-based answer options; such questions were given a score of 1 on construct 3, and the distribution shown in Figure 1 confirms that such questions were rare in the question bank. Second, Q16.8 has a feature that we have named an "implicit stem"; that is, an incomplete question stem that must be inferred from the answer options (in this case, a possible completion of the stem would be "what can we say about the eigenvalues and eigenvectors of A^2?"). The presence of an implicit stem was particularly relevant to scoring construct 20 (generating versus checking), since we determined that the question author intended for students to approach the question by generating an answer to the (implicit) stem rather than by checking the provided options.

Factor analysis: explaining the constructs in fewer dimensions

Recall that the goal of factor analysis is to find a small set of underlying factors that explain the variation in a high-dimensional dataset. Our set of scores was suitable for factor analysis (KMO = 0.52, Bartlett's test of sphericity having $p < 0.001$), and our inspection of the scree plot suggested a 6-factor solution, which explains 42% of the variance in the set of scores. We extracted the 6-factor solution using a varimax rotation to aid interpretation of the factors, since this orthogonal rotation method constrains the factors to be uncorrelated.

For each of the 6 factors, we obtained a loading for each of the constructs, showing how strongly that construct was related to the factor. We have summarised these loadings in Table 1 by showing only those constructs that loaded most strongly onto each of the factors (whether positively or negatively, since the ordering of the poles for the constructs is arbitrary). The full table, showing all factor loadings for each construct, is available in the supplementary material at osf.io/cyq4b.

The names that we have given to each of the 6 factors reflect our interpretation of their meaning, informed by the constructs that load most strongly onto the factor, and our examination of questions with especially high or low loadings onto the factor. For

instance, we labelled Factor 1 as "Checking" because the constructs loading strongly on this factor were all related to questions involving students in checking given options or statements (indeed, Q11.7 in Figure 2 was discussed earlier as an example of a question that requires checking the given options, and it loads strongly on Factor 1). We found that most of the factors could be readily interpreted in this way, bearing in mind that the negative loadings point to an association with the "score=1" pole of the construct. For instance, we named Factor 5 "Sophisticated solution approach" because the construct loadings show that it is associated with questions where the solution requires processing (construct 5), involves multiple steps of reasoning (construct 4) and can be approached in multiple ways (construct 1).

Factor	Constructs loading most strongly on this factor		Loading
1: Checking	18	Question asks students to identify all correct options	0.820
	10	Solution comes from checking the provided options	0.791
	20	Requires checking or verifying something that is given	0.618
2: Computational	6	Solution involves computation	0.826
	16	No obvious misconceptions underlying the wrong options	0.375
	2	No need to make connections between concepts	0.290
3: Answer only	3	Options are answers only	0.992
	19	Options are either correct or incorrect	0.658
4: Appears long	8	Question is long	0.640
	9	Does not require analysis/critique of a given argument	-0.569
	12	Number of options	0.467
	17	Solution requires all information in the question	-0.307
5: Sophisticated solution approach	5	Solution requires processing	0.557
	4	Single step of reasoning (at most)	-0.432
	1	Only one way to approach	-0.408
6: Conceptual understanding of definitions	7	Question/options dominated by text	0.644
	14	Options look different	0.558
	13	"it depends" is not present as an option	-0.401
	11	Answer cannot be easily verified	0.361
	15	No natural visual interpretation	-0.334

Table 1: Factors and the constructs that load most strongly onto them.

However, Factor 6 was more difficult to interpret. By examining the top/bottom 10 questions loading on this factor (including those shown in Figure 4), we concluded that this factor was associated with questions that probed understanding of definitions. Strong positive loadings on this factor seemed to be associated with "wordier"

questions that probed conceptual understanding, while strong negative loadings were associated with more concise questions that were often focused on mathematical expressions and more basic/procedural interpretation of definitions.

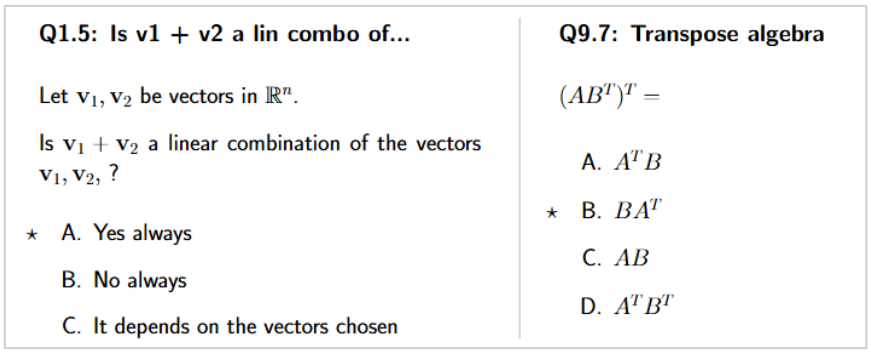


Figure 4: Questions with strong positive (Q1.5) and negative (Q9.7) loading on Factor 6.

Overall, the factor analysis revealed that the set of 20 constructs could be explained by a smaller set of underlying dimensions, each with a meaningful interpretation.

DISCUSSION

In this paper, we have presented the analysis of a large bank of linear algebra PI questions. We scored 132 questions on a set of 20 constructs that were previously identified by a group of experts using the Repertory Grid Technique (Docherty et al., 2024). Our analysis identified a wide range of question types and features in the question bank. We saw that there was a good spread of mathematical thinking skills required to solve these tasks; for example, questions ranged from the more abstract to the more computational (Construct 6), and from requiring multiple steps in the reasoning to simply a single step (Construct 4). Many of the questions had a natural visual interpretation (Construct 15) and could be solved in a variety of ways (Construct 1). As might be expected in a multiple-choice format, many of the questions involved just one concept but there was a sizeable minority which required students to link two or more concepts (Construct 2). However, some constructs were very rare, for example only two questions involved analysing or critiquing an argument (Construct 9). For a course where this learning outcome is crucial, we would suggest that more questions of this type are included. Thus, the list of features or constructs presented here might be useful to lecturers when designing a set of tasks. This type of analysis could be helpful to identify any gaps in a set of questions and may help to align PI questions with course learning goals.

We carried out a factor analysis to determine if there was any underlying structure in the set of constructs. Our analysis found six factors in the types of question: checking, computational, answer only, appears long, sophisticated, and conceptual understanding of a definition. These factors capture aspects of PI question design that have been highlighted in previous work. For instance, the “computational” factor contrasts with the advice from Mazur (1997) that PI questions should have a conceptual rather than procedural emphasis; in practice, many of the questions in this linear algebra question bank involved some computation, which reflected the goals of the course (Docherty,

2023). The "answer only" factor highlights design choices about the answer options; while a majority of the questions in this bank did not include any confidence-based answer options, this has been recommended in previous work (e.g., Stewart et al., 2013). The distribution of factor scores therefore has potential to highlight imbalances that could be addressed when (re-)designing questions. Moreover, the relatively short list of factors could provide task designers with a useful checklist to stimulate ideas for varying the design of tasks (for instance, by seeking to deliberately vary the extent to which the question requires a "sophisticated solution approach").

There are some limitations in our study. When scoring questions, we sometimes faced uncertainty due to the absence of knowledge about the context of the course. For example, we might be unsure as to how the topics in the course were organised and whether students had seen a specific result or not. We often worried about the likely level of understanding or ways of thinking of the student cohort. However, we mitigated these difficulties by referring to the course textbook. While conducting our individual coding, we sometimes missed features of a question such as multiple ways to approach it (Construct 1) but since all questions were coded by at least two of the authors, we are confident that we were able to overcome most of these problems. A further limitation is the fact that we have not yet used classroom data to link the identified constructs and factors to the generation of productive discussions.

We acknowledge that the constructs and the factor structure identified in our work is particular to the set of linear algebra questions that we worked with. However, it seems likely that many, if not all, of the features identified would be relevant to other mathematical topics. To capture extra constructs, we plan to conduct another data collection exercise early in 2026. We will use the Repertory Grid Technique again to ask a group of experts to identify features in a set of PI questions used in an Analysis course. The original aim of this study was to use student voting data to identify constructs and factors that encourage productive classroom discussions. Once we have a full set of features, we will conduct this analysis on the data arising from both linear algebra and introductory analysis classrooms.

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Instrumental and relational understanding of diagonalization of a matrix

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Keywords: linear algebra, relational vs instrumental understanding, diagonalization of a matrix.

INTRODUCTION

Linear algebra is a compulsory first-year mathematics course in National Technical University of Athens. It is the first time when students make the transition from two or three dimensions to n-dimensions. Hillel and Sierpinska (1993) have already mentioned students' difficulties to understand abstract linear algebra's concepts. Students tend to think in practical rather than theoretical ways (Sierpinska, 2000, p.209). We also observe that many students memorize some formulas and/or the method for calculating a mathematical object of linear algebra without understanding the underlying notion. They use it as a tool for the demands of problem solving and they cannot use it as a theoretical notion. Skemp's distinction between *instrumental* and *relational* (or *conceptual*) *understanding* has not lost its interest (Skemp, 1978). Also, Sfard (1987) formulates Skemp's distinction in a slightly different way as *operational (instead of instrumental) vs structural (instead of relational) thinking*. Our research concerns students' relational and instrumental understanding of the process of diagonalization of a square matrix and the related notions of eigenvalues and eigenvectors. In our case "*relational understanding*" helps the students to acquire a global image of the process of diagonalization. Thus students are helped to extend this process from two or three dimensions to n-dimensional Euclidean Space.

RESEARCH QUESTIONS AND METHOD

Our research took place in September 2025 with a sample of 57 students from various Schools and various years of studies in the National Technical University of Athens. The biggest part of the sample consists of 2nd year students in the School of Applied Mathematics and Physical Sciences (37 students). We created a questionnaire in Google Forms platform, allowing participants to respond voluntarily at their own place. The choice of open-ended questions aimed to obtain deeper information about the students' way of thinking, perceptions and conceptual difficulties. The three first questions concern students' perceptions of the process of diagonalization and the two basic concepts that appear in it, eigenvalues and eigenvectors. More specifically, we asked the students: (1) *What does the expression "diagonalization of a matrix" remind you of*?, (2) *What does the expression "eigenvector" remind you of?* and (3) *What does the expression "eigenvalue" remind you of?* With the following two questions we

intended to help students to express a relational understanding, according Skemp (1978), if it exists. So, we asked them (4) *Could you give a geometric interpretation of an eigenvector and its corresponding eigenvalue?* and (5) *Could you give a geometric interpretation of the diagonalization of a matrix?* The appropriate geometric interpretation mainly demonstrates a conceptual understanding. The last question aims to explore the connection between the theory and its applications: (6) *Have you already met any applications of the diagonalization of a matrix during your studies?*

ANALYSIS

Our qualitative analysis of students' responses showed that in the first question most of the students mentioned either the formula $A = P \cdot D \cdot P^{-1}$ or they refer to the notions of eigenvalues and eigenvectors as *intermediate tools* for the diagonalization. More precisely we consider as "*instrumental understanding*" a "neatly operational" thinking of the subject, that is conceiving the whole subject as a procedure of fragmentary steps. On the other hand we consider as "*relational understanding"* a geometrical interpretation of the process of diagonalization to n-dimensional Euclidean Space. In this direction, due to the fifth question, we observe a big raise of responses: "Geometrically, the diagonalization of a matrix means finding a system of coordinates in which the axes follow the directions of the eigenvectors, so that the matrix operates simply as an elongation or a shrinking of the vectors on these directions.", "… this is equivalent with changing the system of coordinates in such a way that the transformation appears as creating a new scale of measurement on the same axes.". In students' answers we can observe a correct geometrical conception described by using natural language but no geometrical figures. There are also some responses, mainly in questions (2), (3), (6) showing a knowledge and interest about applications in *physics* and especially *quantum mechanics*: "Since I recently got involved with quantum mechanics, eigenvectors remind me of characteristic situations expressed in a Hilbert Space.", "Mainly physical quantities as the energy of a system in quantum mechanics". The previous answers come from last year students, whose interests do not seem to be encouraged by teaching, since linear algebra and physics (especially quantum mechanics) are not organically related in the university curriculum.

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Ostensive Nature of Matrices: Toward an Epistemological Analysis of Linear Algebra

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Keywords: Linear algebra, epistemological difficulty, ATD, ostensive objects.

INTRODUCTION

For several decades, many studies have shown that students continue to experience difficulties when learning linear algebra (Stewart et al., 2018). Some studies investigated the epistemological difficulties arising from the characteristics of the concepts dealt with in linear algebra. Larson and Zandieh (2013) showed that various interpretations of a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ bring about one of these difficulties. The diversity of interpretations regarding matrix expressions and operations arises from the symbolic nature of matrices. One theoretical notion that highlights the importance of manipulating symbols and their meanings is the *ostensive dimension in mathematical activity* (cf. Bosch & Chevallard, 1999). In this poster, I examine the ostensive dimension of students' activity with matrices and discuss what this perspective can reveal about learning in linear algebra.

THEORETICAL PERSPECTIVE

This study draws on the notion of ostensive objects developed in the anthropological theory of the didactic (Bosch & Chevallard, 1999). From this perspective, mathematical activity is understood as highly dependent on the visible and material elements that support it, such as symbols, notations, diagrams, and written expressions. These ostensive objects do not simply represent mathematical ideas; they also make particular techniques possible. However, when the institutional context does not provide the related techniques or the explanations that justify their use, students may treat these notations as meaningless symbols. This illustrates how the ostensive dimension influences what kinds of mathematical work can be performed in practice. Guided by this perspective, the present study examines the following research question: How do the ostensive features of matrix notation enable and constrain students' mathematical activity in linear algebra?

EPISTEMOLOGICAL DIFICULTY OF MATIRIX.

Matrix notation offers a rich set of possible interpretations, each of which supports different kinds of mathematical activity. A matrix A can be seen as a single unified object (A), as a collection of individual entries (a_{ij}), as a horizontal array of column vectors ($\boldsymbol{a}_j$), or as a vertical array of row vectors ($\boldsymbol{a}_i^T$). These different ways of viewing a matrix are especially important for understanding matrix multiplication. When we consider a product $C = AB$, where each matrix is treated as a single object, the expression can be interpreted in at least three ways: 1) $c_{ij} = \sum_k a_{ik} b_{kj}$; 2) $\boldsymbol{c}_j = A\boldsymbol{b}_j = \sum_k b_{kj} \boldsymbol{a}_k$; 3) $\boldsymbol{c}_i^T = \boldsymbol{a}_i^T B = \sum_k a_{ik} \boldsymbol{b}_i^T$.

Each of these three interpretations gives rise to different mathematical activities. The entry-wise interpretation, most often used when defining matrix multiplication, serves as the main computational tool. In this sense, it supports tasks in which the goal is to determine the product of two matrices. The column-wise interpretation allows the product to be understood as a linear combination of the columns of A. This perspective is central to developing an understanding of linear mappings: it clarifies how a matrix represents a mapping by showing the images of basis vectors, and it plays a crucial role in reasoning about column spaces and null spaces. The row-wise interpretation is also important: left multiplication by an elementary matrix can be understood through viewing the product as a linear combination of the original matrix's rows. This understanding is closely related to invertibility of elementary matrices and highlights the fundamental idea behind the uniqueness of matrix rank. These examples illustrate that the *instrumentality* of matrix notation—its function as a tool within techniques for accomplishing tasks—is inseparable from conceptual understanding. However, many students struggle to move among these interpretations and have few opportunities to use this instrumentality. The difficulty of linear algebra arises not only from its abstract and axiomatic nature but also from the ostensive features of matrix notation.

These different ways of interpreting a matrix are grounded in the ways a matrix can be partitioned or blocked. Although such partitions are often introduced in textbooks simply as techniques for simplifying computations, they in fact play a much deeper theoretical role: they give rise to the various instrumental uses and conceptual understandings described above. They are the ostensive structures that make it possible to shift among multiple interpretations of a matrix and to perform the different mathematical activities associated with each one. At the same time, these structures also contribute to the difficulties that students encounter. Moving flexibly across several interpretations, and recognizing how each partition supports a different conceptual view or operation, is not straightforward. Thus, both the potential and the challenges involved in learning linear algebra are closely linked to the ostensive nature of matrix notation. This observation highlights the importance of developing further epistemological research on matrices and other fundamental objects in linear algebra, as such analyses can inform more effective curriculum design and instruction.

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Symbolizing in an Inquiry-Oriented Sequence for Least Squares

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In this paper, we analyse students' progressive mathematization as they engage in an inquiry-oriented linear algebra task sequence for least squares approximation in an introductory linear algebra course. We use the mathematical activity of symbolizing as a lens through which to document the emergence of the least squares equation. In particular, we highlight students' interactions with three key equation types that evolve through the emergent modelling process.

Keywords: linear algebra, least squares equation, Realistic Mathematics Education, emergent models, design-based research.

INTRODUCTION

In this paper, we outline students' progressive mathematization as they engage in an inquiry-oriented linear algebra (IOLA) task sequence for least squares (Wawro et al., 2025b). By inquiry we refer to both the inquiry of students into the mathematical ideas and the inquiry of the instructor into student thinking to aid in their mathematical progression (Wawro et al., 2022). In the task sequence, which leverages Realistic Mathematics Education (RME) design principles (Freudenthal, 1991), students first develop relationships that structure an experientially real starting point through various symbolizations, culminating in the least squares equation $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$. We frame student activity through the lens of *symbolizing* with the research question: What symbolizing occurs in students' progressive mathematization towards the guided reinvention of $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$?

LITERATURE REVIEW

Least squares approximation is recommended as a topic appropriate for a first linear algebra course in the United States (Stewart et al., 2022); a survey of linear algebra instructors from institutions of various types across the US and Canada found that 50% of respondents include least squares in their first courses in linear algebra (Andrews-Larson et al., 2022). However, in a review of linear algebra research, Stewart et al. (2019) found that relatively few empirical papers focused on the teaching and learning of least squares. Talbert's (2014) study had students watch a context-free instructional video on least squares outside of class and then work together on context examples in class; he found that students performed well on the final exam solving an application problem using the normal equation. In early studies of student thinking related to this paper, we found that students leverage intuitive notions of shortest distance and the span of travel vectors towards a formal symbolization using the dot product (Lee et al., 2022), and that students flexibly leverage equation types, symbol use, and solution strategies from throughout the semester as they construct successful solution methods

(Wawro et al., 2023). In this paper, we highlight students' symbolizations that emerge across the task sequence throughout the reinvention process for $A^T A\hat{\boldsymbol{x}} = A^T \boldsymbol{b}$.

THEORETICAL FRAMEWORK

Freudenthal (1973, 1991) refers to mathematics as a human activity and as an activity of organizing a subject matter. This activity of organizing reality into mathematical patterns or organizing mathematical ideas in new ways is referred to as mathematizing (Freudenthal, 1973). One type of such activity is *symbolizing*. By symbolizing, we refer to the creation and use of mathematical symbols, which may be numbers, letters, vectors, matrices, and equations or expressions using these, as well as the creation and use of other types of inscriptions such as graphs and diagrams. Treffers (1987) distinguished two kinds of mathematization: horizontal and vertical. *Horizontal mathematizing* is the activity of organizing a situation in a way that makes it more amenable to further mathematizing activity, such as creating symbolic expressions that represent aspects of a task setting, including patterns and relationships. *Vertical mathematizing* takes the results of horizontal mathematizing and leverages them to generalize, make new connections, and create new ideas. Rasmussen et al. (2005) explain that "horizontal and vertical mathematizing activities do not occur in isolation, but comprise a duality referred to as progressive mathematizing" (p. 69).

Based on the work of Freudenthal (1991), Gravemeijer (1999) discusses three instructional design heuristics of RME that encourage students to develop more formal mathematics through mathematizing their own mathematical activity. *Didactical phenomenology* is finding an appropriate phenomenon, an experiential real setting, in which students can immediately engage that will allow for the development of the mathematical ideas that curriculum designers want the students to understand. *Guided reinvention* is the heuristic that students will reinvent the mathematical ideas with the guidance of their instructor. The *emergent models* heuristic describes the process of students building *models-of* their activity that can be used as *models-for* further reasoning. Gravemeijer (1999) provides additional structure for the model-of model-for transition by describing four levels of activity. We explain the levels with a focus on symbolizing, characteristic of Gravemeijer et al. (2000). *Situational activity* is activity in a task setting chosen so that students can immediately engage in the task and begin to organize their ideas mathematically, in our case, by symbolizing. *Referential activity* is activity in which students begin to create models-of their activity in the task setting; thus, "At the referential level, models are grounded in students' understanding of paradigmatic, experientially real settings" (p. 243). In our paper, models take the form of symbolizations that students create and use to organize their mathematical activity. *General activity* is activity that uses previously made models (symbolizations) in ways that lose their dependence on situation-specific imagery and are now used as models-for creating new mathematical relationships. "In the process, the model becomes an entity in its own right and serves more as a means of mathematical reasoning than as a way of symbolizing mathematical activity grounded in particular settings" (p. 243). *Formal activity* occurs when the models evolve into conventional

symbolizations that students can leverage future mathematical situations. We find it helpful to illustrate the four levels of activity through the stairstep diagram in Figure 1.

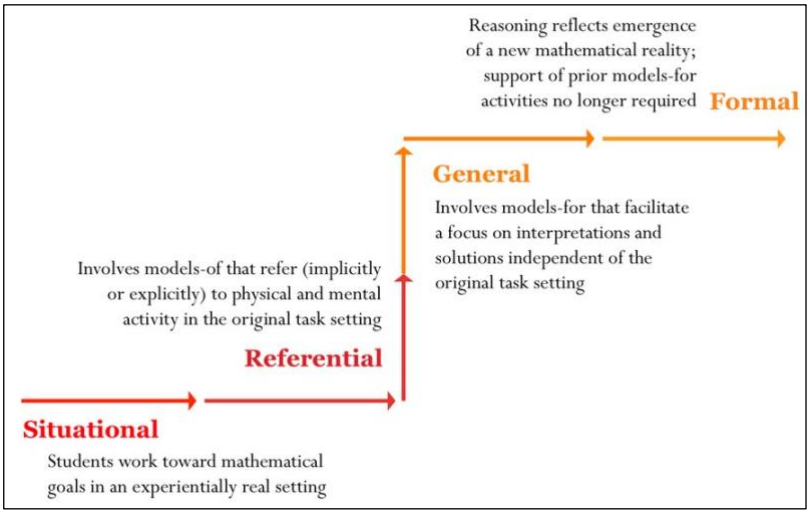
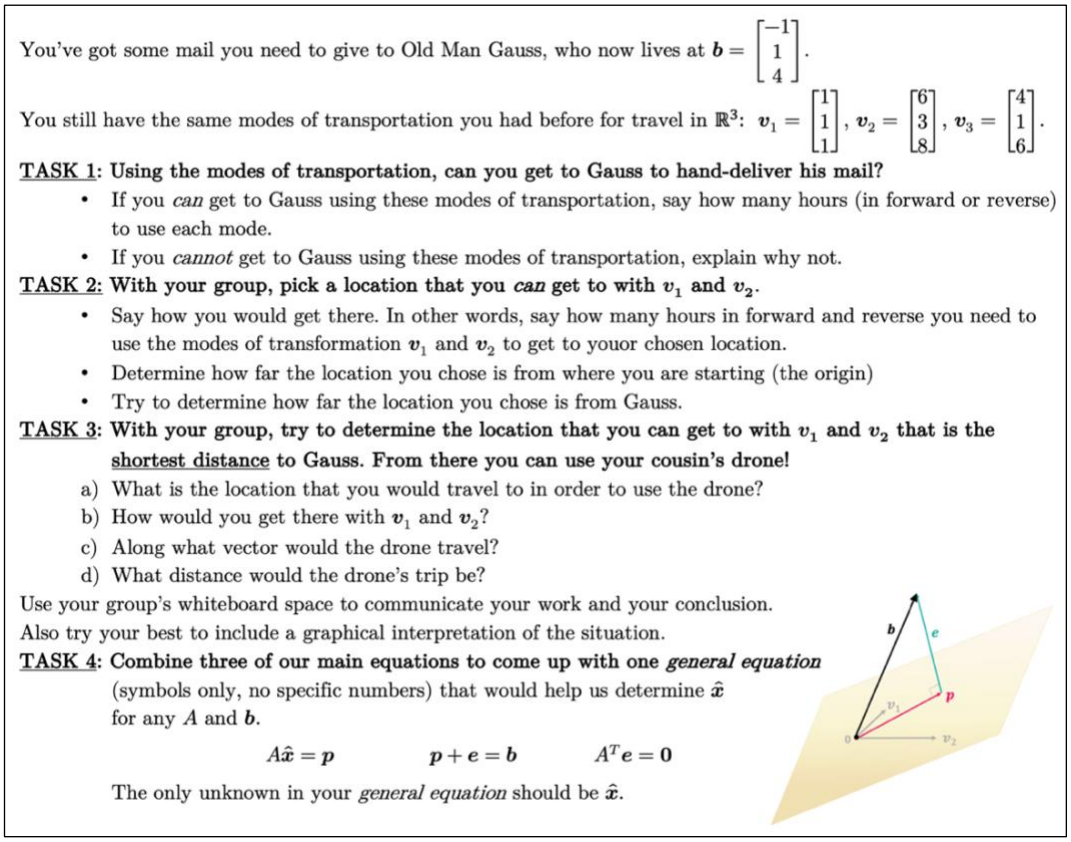


Figure 1: Diagram of the four levels of activity, highlighting the model-of/for transition.

METHODS

To create and refine the tasks of the IOLA Least Squares unit (see Figure 2), we used the RME design principles (Gravemeijer, 1999) and the Design Research Spiral (DRS) (Wawro, et al., 2022), which has five phases: Design, Paired Teaching Experiments (PTE), Classroom Teaching Experiments (CTE), Online Working Groups (OWG), and Web (see Wawro et al. (2025a) for more about the unit and the Design Research Spiral).


You've got some mail you need to give to Old Man Gauss, who now lives at $\boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

You still have the same modes of transportation you had before for travel in $\mathbb{R}^3$: $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$.

TASK 1: Using the modes of transportation, can you get to Gauss to hand-deliver his mail?

- If you *can* get to Gauss using these modes of transportation, say how many hours (in forward or reverse) to use each mode.
- If you *cannot* get to Gauss using these modes of transportation, explain why not.

TASK 2: With your group, pick a location that you *can* get to with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

- Say how you would get there. In other words, say how many hours in forward and reverse you need to use the modes of transformation $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to get to youor chosen location.
- Determine how far the location you chose is from where you are starting (the origin)
- Try to determine how far the location you chose is from Gauss.

TASK 3: With your group, try to determine the location that you can get to with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ that is the shortest distance to Gauss. From there you can use your cousin's drone!

a) What is the location that you would travel to in order to use the drone?
b) How would you get there with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?
c) Along what vector would the drone travel?
d) What distance would the drone's trip be?

Use your group's whiteboard space to communicate your work and your conclusion.
Also try your best to include a graphical interpretation of the situation.

TASK 4: Combine three of our main equations to come up with one *general equation* (symbols only, no specific numbers) that would help us determine $\hat{\boldsymbol{x}}$ for any A and $\boldsymbol{b}$.

$$A\hat{\boldsymbol{x}} = \boldsymbol{p} \qquad \boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b} \qquad A^T\boldsymbol{e} = \boldsymbol{0}$$

The only unknown in your *general equation* should be $\hat{\boldsymbol{x}}$.

Figure 2: Abbreviated task statements for the IOLA Least Squares instructional unit.

Wawro et al. (2025a) provides considerable detail about the learning goals of these tasks. Here we summarize an a priori analysis. Task 1 asks students to refresh their understanding of linear independence and span. Task 2 discusses the length of a vector and the distance between vectors. Task 3 engages students in the important mathematical relationships involved in understanding least squares, while Task 4 helps students consolidate these in the formal symbolization $A^T A\hat{\boldsymbol{x}} = A^T \boldsymbol{b}$.

For this paper, we analysed data collected from four classes during various DRS phases to inform implementations and revisions of the task sequence. In the results, we note the instructor, institution location, and DRS phase of the three classes: Class A: Instructor 1 at Institution 1 in Eastern US, CTE 1 in 2021; Class B: Instructor 1 at Institution 1, CTE 2 in 2022; and Class C: Instructor 2 at Institution 1, OWG in 2023.

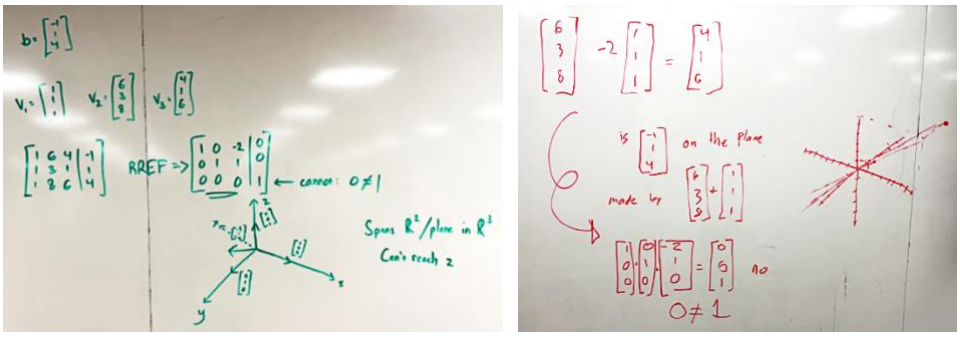
In previous analyses of the data, we focused on overall themes in student thinking (CTE 1; Lee et al., 2022) and students' specific uses of symbols (CTE 2; Wawro et al., 2023). In this paper, we revisit the data to illuminate important aspects of the progressive mathematization of symbolizing that occur as students move through the task sequence. We document how student mathematizing resulted in a chain of symbolizations that evolved from partial solutions to an initial task setting and led to the guided reinvention of the least squares equation (LSE) $A^T A\hat{\boldsymbol{x}} = A^T \boldsymbol{b}$. We examined the data, focusing on CTE 1 and 2, for how students used symbols to organize their activity and how instructors used symbols to record student mathematizing. Based on RME design heuristics, our analytical focus describes the chain of symbolizations that evolved and highlight important transitions that occurred as student mathematizing progressed through the task sequence.

RESULTS

We highlight three important transitions in the stairstep diagram: horizontal to vertical mathematizing, model-of to model-for, and vertical to horizontal mathematizing.

Horizontal to Vertical Mathematizing

Students' initial mathematizing in Task 1 is seen in Figure 3a and 3b.



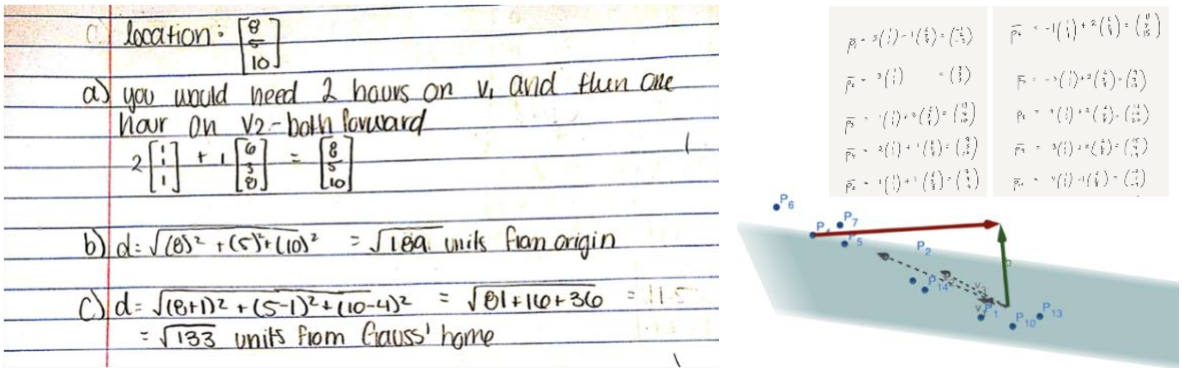
(a) **(b)**

Figure 3: Examples of students' situational symbolizing activity in Task 1 (Class B).

This activity is primarily horizontal in that it organizes the task setting in ways that are familiar to the students. This initial activity is formal to the extent that it involves using

previously mathematized symbolizations, such as augmented matrices. For example, the row reduction in Figure 3a. Students' activity is also situational as it involves organizing information in the task setting. For example, in Figure 3b students discussed that the travel vectors could not reach Gauss because he is not on that plane. Additional examples of students' initial symbolizing activity include sketches of coordinate axes, vectors, and planes tied to the travel situation.

In Task 2, students continue to engage in horizontal mathematizing. For example, in Figure 4a, a group generates the vector ⟨8,5,10⟩ using a linear combination of the travel vectors and determines the distance between this location and Gauss. As students explore locations they can reach, instructors may collect multiple student responses and graph them to consider how close they are to Gauss (Figure 4b). Note that the linear combinations of vectors in the plane and the triangle hinted at in the GeoGebra figure are symbolizations that foreshadow what will be called ① and ② in Figure 5.



(a) (b)

Figure 4: (a) A student's reflection from Class A (b) a GeoGebra app from Class B.

As students begin Task 3, they consider what point on the plane is the shortest distance to Gauss. This shift in student activity involves organizing mathematical relationships between the closest location in the plane to Gauss and the path the drone travels to reach him. This organizing activity provides a foundation for the symbolizations for ③ in Figure 5, the relationship between the path of the drone and any vector in the plane. Student explorations are often like those of Liam and Luka's from Class B:

Liam: "If I had to imagine, Gauss would travel in a straight line orthogonal to the plane formed by the three vectors. In this case, Gauss would have to move minimally as you both would meet where his line and your plane intersect."

Luka: "I'm not sure how to find the point where the transportation gets closest, but the vector he travels along should be perpendicular to the final mode of transportation's vector."

In this section students primarily engage in horizontal mathematizing as they explore relationships within the task setting (situational activity). At times the symbolizations describe ideas framed outside the immediate task setting, but students refer to the task setting as part of their analysis (referential activity). As students make new realizations

about the mathematical situation (e.g., the path with shortest distance is perpendicular), their organizing activity begins to be more vertical than horizontal.

The Model-of to Model-for Transition

In the discussions of Tasks 1 and 2, the instructor may highlight important relationships the students mention during class. After students' initial work on Task 3, the instructor may summarize student ideas in a manner similar to Figure 5. Rasmussen and Marrongelle (2006) point out that an important part of the RME heuristic guided reinvention is the instructor creating a transformational record, i.e., a symbolization that both records students' ideas and makes a connection to important mathematical ideas that the task sequence is meant to teach. The symbolizations in Figure 5 remind the students of ① the relationships involved in travelling to points on the plane and ② the relationships between the vectors in the triangle formed by $\boldsymbol{p}$ (a point reached on the plane), $\boldsymbol{b}$ (Gauss's location vector) and the difference between these. In Task 3, $\boldsymbol{e}$ also represents the path of the drone. Note that the equations in ③ in Figure 5 follow from the ideas of students like Liam and Luka and are contingent on the class having discussed that the dot product of two vectors is 0 when the vectors are orthogonal. Using this, ③ shows the dot product of $\boldsymbol{e}$ with each of $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{p}$ as equal to 0.

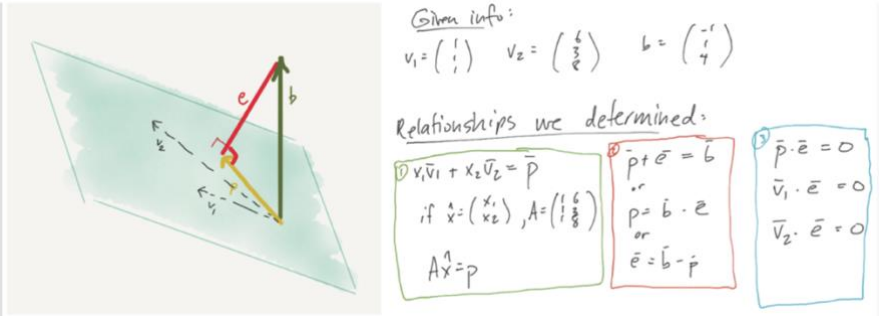


Figure 5: Instructor's transformational record of student symbolizing.

We think of the symbolizing in Figure 5 as a model-of student activity on Tasks 1, 2, and the beginning of Task 3. This symbolizing by the instructor did not occur in isolation. Rather it is the result of a gradual refinement and distilling of these ideas as they occurred in the various symbolizations students used during situation and referential activity. As students' horizontal mathematizing becomes distilled into an explicit symbolic tool, we think of this as rounding the bottom corner of the stairstep diagram in Figure 1 (horizontal to vertical mathematizing).

Students transition to using the symbolizations that they and the instructor previously recorded (the models-of their previous activity) as models-for further reasoning to complete Task 3. To illustrate in some detail how the students use the symbolizations as models-for their reasoning, in Figure 6 we label variations of equations that we informally think of as ① travel on the plane, ② the triangle equations, and ③ the shortest distance is perpendicular to the plane. In Figures 7-8, we provide student examples from Class B that illustrate different ways of combining the ① ② ③ equations to solve Task 3. This is general activity (and vertical mathematizing) because

students create new symbolizations that represent new relationships between quantities. Their symbolizing folds back to referential activity when they refer to numerical results as representing particular information in the travel task setting.

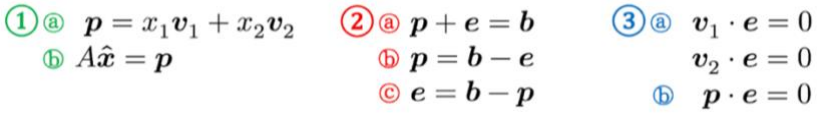


Figure 6: Equations that emerge as part of the model-of to model-for transition.

There are many ways students can make connections between the symbolizations presented in Figure 6. In the first example, see Figure 7, the student uses ①a and ②a to create a system of three equations with five unknowns. The student then uses ③a to find e_y and e_z in terms of e_x. The student uses these equations as a model-of their initial activity which became symbolizations for creating a new system of equations whose solution allowed them to answer all four parts of Task 3. The student creates a new symbolization whose column vectors span all of $\mathbb{R}^3$.

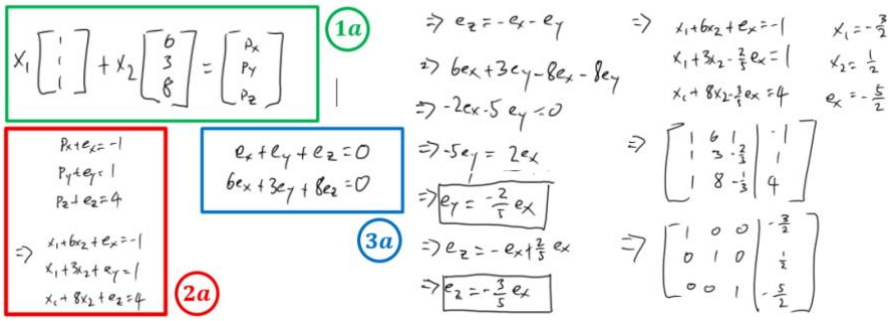


Figure 7: A student uses 3a to find the direction of $\boldsymbol{e}$ and solves for $\hat{\boldsymbol{x}}$ and $\boldsymbol{e}$ together.

In contrast, the student work in Figure 8 uses ①a and ②c to determine the entries of $\boldsymbol{e}$ in terms of x_1 and x_2. Using ③a, the student creates a system of equations in terms of x_1 and x_2 that experts may recognize as the coefficients of the A^TA matrix and the entries of $A^T\boldsymbol{b}$, symbolized as $4 - 3x_1 - 17x_2 = 0$ and $29 - 17x_1 - 109x_2 = 0$ in Figure 8 (after fixing a small arithmetic mistake).

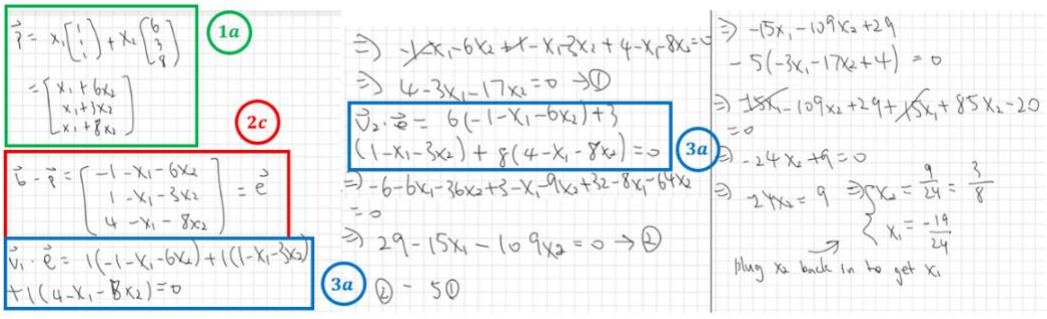


Figure 8: A student combines 1a, 2c, and 3a to eliminate $\boldsymbol{e}$ and solve directly for $\hat{\boldsymbol{x}}$.

The student work in Figures 7-8 illustrate two quite different ways of symbolizing with similar equations. In addition to allowing for finding values for the travel situation of Task 3 (referential activity), the symbolizing provides the opportunity for generalizing. For example, the ③a in Figure 7 is equivalent to $A^T\boldsymbol{e} = \boldsymbol{0}$. This is leveraged in Task 4.

The student solutions here use the dot product equations to incorporate the idea that the shortest distance is perpendicular. Another student solution that we have seen uses the cross product of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to find a vector perpendicular to the plane. This method results in solving an augmented matrix similar to that in Figure 7, but with integers for the $\boldsymbol{e}$ direction. An additional option for ③ would be for students to use partial derivatives from calculus to minimize the distance.

Vertical to Horizontal Mathematizing

Student symbolizing on Task 3 allows for the creation of new mathematical relationships (vertical mathematizing) in ways that were both general activity (focused on the relationships between the abstract symbols without reference to the original task situation), and also referential activity (when interpreting results in the task setting). With the addition of $A^T\boldsymbol{e} = \boldsymbol{0}$ to the symbolizations of the orthogonal relationship ③, the students can engage in Task 4 (see Figure 2) which begins with a version of the ① ② ③ equations now written with an emphasis on the more abstract matrix versions of the equations. Task 4 is fully stated as requesting general activity in that students should find an equation that determines $\hat{\boldsymbol{x}}$ for any matrix A and vector $\boldsymbol{p}$.

Figure 9 shows student symbolizing on Task 4. In Figure 9a, the student combines the two equations of Task 4 (①b and ②a from Figure 6), applies A^T to both sides of the equation, and uses the third equation of Task 4 (a version of ③) to create the LSE. This student also applies the new equation to the situation in Task 3. In Figure 9b, using $\boldsymbol{g}$ instead of $\boldsymbol{b}$ for the vector, students combine the second two equations of Task 4 (②a is converted to ②c and then combined with a version of ③). After distribution of the of A^T, the first equation 1 of Task 4 (①b) is substituted to create a version of the LSE.

Task 4 completes the guided reinvention of the LSE. We characterize student symbolizing activity in Task 4 as moving from vertical to horizontal and thus rounding the top corner of Figure 1. Creating the LSE is vertical mathematizing in that this symbolization is a new distillation of the ① ② ③ equations that allows for direct solving without having to determine $\boldsymbol{e}$. This equation is important not only for its efficiency but also because it connects students' less formal reasoning to an important symbolization recognized by the mathematical community. From this point, the students can now apply this equation using formal activity as was done in Figure 9a. The formal activity is characterized as horizontal in that it organizes the mathematical situation by applying a known symbolization as opposed to creating a new symbolization representing new mathematical connections.

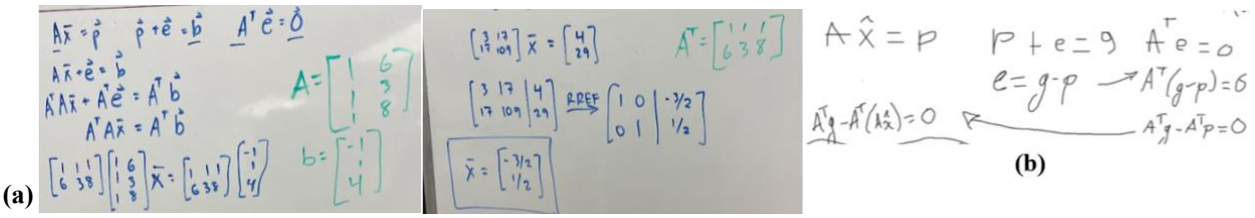


Figure 9: Student work on Task 4 from (a) Class C and (b) Class A.

DISCUSSION AND CONCLUSION

As illustrated through the three Results subsections, this paper highlights student symbolizing that reconstructs the LSE through progressive mathematizing. We overlay two ways to describe progressive mathematizing. One describes how horizontal moves to vertical and back to horizontal through the stairstep diagram (Figure 1). The other describes the progression of the four levels of activity with an emphasis on the model-of to model-for transition as the heart of the vertical mathematizing. We see the overlay of the two as applying to any type of progressive mathematizing, but in this paper we focus on symbolizing and the guided reinvention of the Least Squares Equation. The key to the model-of to model-for transition, and thus the progressive symbolizing, of the LSE lies in the set of three ideas that we have called the ① ② ③ equations (Figure 6). The precursor ideas for these equations occur initially as horizontal mathematizing in situational activity and referential activity of Tasks 1, 2, and the beginning of Task 3. As these ideas are refined, including with the instructor's transformational record (Figure 5), the symbolizations become a model-of student activity that can be used as a model-for further reasoning. The general activity with the ① ② ③ equations initially creates an abundance of symbolizing, as exemplified in Figures 7-8. This then is refined and organized, again with the help of the instructor, into the versions of the ① ② ③ equations stated in Task 4 that can then be combined by students into the LSE.

We see this focus on the overlay of ideas highlighted in Figure 6 and the model-of to model-for transition as helpful to RME curriculum designers, instructors using an RME designed curriculum, and researchers. In each of these cases, it is helpful not only to focus on the end result (here the LSE), but also on the emergent model that functions as the fulcrum for students' symbolizing activity.

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Students' proof styles as individualizations of the genre of proof

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This conceptual paper advances the genre perspective in the area of proof. The paper introduces individualized proof styles—cumulative networks of personally adapted rules of proof that underpin one's proof productions. Grounded in genre theory and the commognitive framework, the construct illuminates the conventions of the proof genre that students individualize and enact in their proof productions. This is how proof styles shape whether students' productions will be accepted as instances of the proof genre by a wider community. The construct is illustrated with a third-year topology student, who produced textbook-like proofs with minimal substantiation gaps.

Keywords: commognitive framework, genre of proof, proof and proving, proof style

RATIONALE

In their literature review, Stylianides et al. (2017) note that "there is no disagreement that proof is indispensable to the work of mathematicians […]. There is also wide recognition that proof should play an important role in all students' mathematical education" (p. 237). Yet, research repeatedly demonstrates the challenges with proof that students face (e.g., Weber, 2005). This may explain why our field is constantly seeking new approaches to proof and its learning (e.g., Stylianides et al., 2017).

Selden and Selden (2014) were the first mathematics educators to position proof as a writing *genre*. Their argument was grounded in common features of proof, such as "proofs are not reports of the proving process" and "they contain little redundancy […] and definitions that are available elsewhere." The researchers supported these features through a literature review and interviews with mathematicians.

References to "the genre of proof" feature in many mathematics education papers (e.g., Bowers & Kühle, 2019). Dawkins and Vroom (2025) argue that "a lens on proof as a genre of text is a worthwhile area to study that can provide important insights into the teaching and learning of proof" (p. 293). However, the lens's potency has not yet been mobilized for empirical purposes. Hence, this paper aims to introduce the construct of *individualized proof styles* as a tool for exploring students' learning of the proof genre. The construct stems from the genre research and the commognitive framework.

The construct aspires to address a lacuna in the didactical literature on proof: while foregrounding interpersonal commonalities of students' proofs (e.g., Weber, 2005), research appears to overlook students' individual styles, that is, idiosyncratic fusions of substance, form, and other features that are inseparable from one's partaking in proof as a genre. A style can leave a distinctive footprint on an individual's proof productions. As such, a proof style can play a role in whether a broader mathematics community will endorse a student's productions as instances of the proof genre. I illustrate these ideas with a proof style of a third-year student in a topology course.

GENRE RESEARCH

Etymologically, ‘genre’ comes from the French (and originally Latin), where it stands for a certain ‘type’ or ‘kind.’ As a subcategory of critical theories, genre studies focus on important social constructs and investigate their features with the aim of delineating common aspects that might otherwise go unnoticed (Bowers & Kühle, 2020).

Theorists distinguish between two developmental phases in genre research. The first phase considered genres as combinations of substance and form (Chandler, 1997). This phase is referred to as conventional since it can be traced back to Aristotle. A central criticism of this phase pertains to its attempts to define a genre through a list of necessary and sufficient characteristics (Miller, 1984). However, genre characteristics are not unique, and it is their prominence, combination, and functions that make them distinct (Chandler, 1997). Furthermore, the differences within a genre cannot be underestimated. Neale (1980) contends that “difference […] is absolutely essential to the economy of genre” (pp. 50, 48).

The modern phase of genre research is often linked to Miller (1984). She argues that genres are socially constructed as intentional responses to certain sets of societal situations. In the words of Bitzer (1968), “from day to day, year to year, comparable situations occur prompting comparable responses” (p. 13). This explains the emergence of conventional forms, characterized by common structures and elements. Miller (1984) conceptualizes a genre as a social action and argues that its definition “must be centered not on the substance or the form of discourse but on the action it is used to accomplish” (p. 151). Consistently, contemporary genre studies highlight the intentions that rhetors pursue through a genre.

Within the modern approach, genres are construed as mediating frameworks between texts, their producers, and readers, which affords the communication of content (Chandler, 1997). Specifically, producers mobilize conventional genre assumptions to enhance communication efficiency. In turn, familiarity with the genre equips the interpreters with appropriate expectations regarding the text in hand (Neale, 1980).

GENRE OF PROOF

Consistent with genre studies, attempts to define proof in terms of necessary and sufficient conditions have been declared unsuccessful (e.g., Davis & Hersh, 1981). Building on Wittgenstein, Czocher and Weber (2020) argue that proof can be productively considered as a cluster account, that is, as a set of characteristics that “count towards” category membership, without any of them being sufficient or necessary for it. Specifically, Czocher and Weber render a proof as a deductive justification “that has been sanctioned by the mathematics community [and that is convincing, perspicuous, and transparent] for a knowledgeable mathematician” (p. 61). This perspective is consistent with Miller’s (1984) approach to a genre that foregrounds provers’ intentions (e.g., to be convincing and transparent) and social exigence (e.g., to establish the veracity of a theorem). Textual regularities (e.g., Bowers & Kühle, 2020) emerge as attributes of the proof genre, supporting efficient communication

between proof producers and readers. For instance, Csiszar (2003) argues that these attributes feed into readers' sense of the inevitability of proof. Selden and Selden (2014) add that the attributes simplify proof validation and make it more reliable.

To summarize, I concur with Selden and Selden's (2014) proposal to construe proof as a genre. However, while the researchers based their argument on textual regularities, Czocher and Weber's (2020) perspective enables considering proof as a genre, in the sense that the construct is used in modern genre research (cf. Miller, 1984).

INDIVIDUALIZED PROOF STYLES

Conceptual links between genre and discourse

So, how can the lens of a genre be applied to research on the learning of proof? Some may propose that this question instantiates a broader phenomenon of mastering a genre. However, a quick literature review reveals that genre studies tend to focus more on teaching than on learning (e.g., Chandler, 1997). Furthermore, given the specificity of mathematical proof, it is only reasonable to turn to theories developed within the field of mathematics education.

I propose that the commognitive framework (Sfard, 2008) is a viable candidate for addressing the above question. Indeed, commognition theorizes how newcomers become participants in historically established *discourses*. The conceptualizations of genre and discourse are inextricably interwoven. For instance, Miller (1984) refers to a genre as "certain discourses [that] constitute a distinctive class" (p. 151). Overall, both constructs are socio-cultural, communication-based, and attentive to textual patterns. More specifically, and as mentioned earlier, substance, form, situations, and rhetors' intentions are key elements of a genre (Miller, 1984). The first three elements can be discerned in Sfard's (2008) definition of a discourse, which is "set apart by their objects, the kinds of mediators used, and the rules followed by [discourse] participants" (p. 93). Regarding intentions, Sfard (2008) construes discourse as a type of communication that "enables interpersonal coordination needed for the collective implementation of complex [practical and rhetorical] activities" (p. 87).

Furthermore, Karavi et al. (2022) argue that commognitive studies have the potential to advance research on proof in university mathematics. This is another reason for leveraging the conceptual and analytical apparatuses of the framework for studying students' learning of the genre of proof.

Commognitive theoretization

Kontorovich et al. (2024) introduced proof-based mathematics as a discourse, in which the generation of mathematical narratives is separated from the process of their substantiation. Indeed, proof-less mathematics, the one that is often studied in secondary school and first-year university courses in some countries, is replete with standardized algorithms and formulas. Their application not only produces mathematical narratives, but it also warrants their validity. The request to substantiate these narratives is tantamount to questioning the conventional mathematical machinery

that engendered them. Kontorovich (2019) hypothesizes that this may explain why students are reluctant to check their answers to routine mathematical problems. In turn, narrative substantiation is a distinct discursive activity in proof-based mathematics.

Sfard (2008) maintains that this activity is carried out "according to well-defined rules" (p. 224). And while the mathematics community has been revisiting these rules throughout history (e.g., Kleiner, 1991), Sfard (2008) argues that "for today's mathematicians, the only admissible type of substantiation consists in manipulation on narratives, and it is thus purely intradiscursive" (p. 232). Accordingly, Kontorovich et al. (2024) introduce the notion of *rules of proof* (RoPs) to capture intradiscursive principles that underpin proof production.

RoPs are a metadiscursive construct in the sense that they "define patterns in the activity of the discursants trying to produce and substantiate [mathematical claims]" (Sfard, 2008, p. 201). In our case, RoPs shape what proofs should look like, their structure and layout, argumentation and formulations employed, and discursive routines to use and avoid.

Sfard (2008) maintains that metadiscursive rules (or *metarules*) are evolving, tacit, value-laden, constraining, and contingent. The first characteristic implies that these rules are sensitive to time and place. This is especially evident in the case of proof, the nature of which has been transforming throughout mathematical history (e.g., Kleiner, 1991). Tacitness highlights that the rules are not necessarily followed deliberately by the discourse participants. In the words of Bourdieu (1999), the patterns spotted by an observer are not "in any way the product of obedience to rules, they can be collectively orchestrated without being the product of the organizing action of a conductor" (p. 109). Consistently, commognition distinguishes between the rules one *enacts* through their actions and the rules they *endorse* by capturing them in words (Kontorovich et al., 2024). Sfard (2008) links the value of some metarules to their broad dissemination. The constraining nature of metarules limits the possibilities for action in certain circumstances without necessarily prescribing to the prover what to do.

Lastly, metarules are contingent in the sense that they "are the result of custom-sanctioned associations rather than a matter of externally imposed necessity [which] does not mean there are no reasons for their existence" (Sfard, 2008, pp. 206–207). Indeed, from a commognitive standpoint, even the most "objective" rules, which appear to be governed by logical inevitability, are products of community-wide choices that have withstood the test of time. The contingency of other RoPs is more apparent. For instance, traditions appear to be the only reason today's mathematicians conclude their proofs with the Halmos symbol or "Q.E.D."

Having discussed RoPs as a social endeavor, I now shift to their individual versions. As a participationist framework, commognition assumes that collective forms of human communication and actions are developmentally prior to the activities of an individual (Sfard, 2008). In other words, participationists believe that learning occurs through one's transition from a peripheral role in collective activities to the ability to

perform the activity independently. Sfard (2008) refers to this process as individualization of the collective. In our case, the collective pertains to RoPs practiced by a broader mathematical community.

In the above terms, individualization produces personally modified versions of RoPs. Their constraining rather than prescriptive nature necessitates choosing a certain way to implement a rule. For instance, does one prioritize verbal statements over notation or vice versa? Recurrent enactment and endorsement solidify certain versions of RoPs, where they acquire permanence, distinctive characteristics, and personal values. Accordingly, I use the term *individualized proof style* (or a proof style, for short) to refer to a cumulative network of individualized versions of RoPs to which one regularly adheres when producing proofs. With proof production, I refer to both the patterns one exhibits in a proving process (cf. routines) and the proofs, that is, the eventual narratives that the prover shares with a broader community.

Five clarifying notes on proof styles

Let me make some clarifying notes on proof styles. First, while previous research has been mostly attentive to students' proof-production patterns (e.g., Mejía-Ramos et al., 2015; Weber, 2005), proof styles center around proof-narratives as carriers of stylistic decisions. Hereby, I contend that accounting for both proving and the eventual proofs is imperative for unpacking students' learning of the proof genre. Indeed, students' vision of the target outcomes is likely to shape the course of their activity.

Second, proof styles are attentive to the rules that the provers *enact* and *endorse*. The former rules are visible through one's actions, whereas the latter emerge when an individual captures them in words (Sfard, 2008). Commognitive research has shown that the two types do not always align (e.g., Kontorovich et al., 2024).

Third, hopefully, readers will not confuse the proposed proof styles with "proving style," a notion used in the literature. Weber (2005) defined it as "students' personal reasoning preferences when it comes to the ways in which they construct proofs" (p. 358). Epistemologically, the notion is an adaptation of Riding and Cheema's (1991) cognitive style—a strategy consistently applied to cope with a specific type of activity. Accordingly, Weber's notion is cognitive, and it views students' styles as a matter of taste, preferences, and propensities (e.g., Mejía-Ramos et al. 2015). In turn, I rooted proof styles in the grounds of discourse and genre, emphasizing that they constitute personalized adaptations of conventions initially residing on a social plane. Methodologically speaking, research on proving styles highlights the similarities across proof productions of different students (e.g., semantic, syntactic). In turn, proof styles open the space for noticing the individualism of each student's proofs.

The remaining notes are informed by genre research. The fourth note emphasizes the potential sensitivity of one's style to the affordances of proof-requiring situations. Mejía-Ramos et al. (2015) demonstrated that students' proving habits can vary across mathematical areas. In a similar vein, Kontorovich and Greenwood (2023) found that a proof constructed by students in a group setting can undergo significant

transformation when shared with the whole class. Accordingly, extra care is needed when analyzing one's proof style, especially when inferring the scope of applicability of the RoPs one appears to follow.

Lastly, operating within a genre comes not only with obligations to follow its rules but also with expectations to capitalize on its features. In the words of Chandler (1997), "from the point of view of the producers of texts within a genre, an advantage of genres is that they can rely on the readers already having knowledge and expectations about works within a genre" (p. 6). In our case, this may mean that when preparing a proof for sharing with a broader community, a prover can assume that its readers are familiar with the assumptions of the proof genre. This aspect draws attention to how students mobilize the conventions of the proof genre in their proof productions. This aspect is the focus of the illustration that follows.

ILLUSTRATION

Proof-based course in topology

The illustration comes from a broader project on topology teaching and learning (for previous reports see Kontorovich, 2021; Kontorovich et al., 2024; Kontorovich & Greenwood, 2023). The data were collected from a proof-based course offered by a mathematics department at a New Zealand university. The course covered standard topics in point-set and algebraic topologies. Seven students in their final year of the mathematics major enrolled in the course. It was taught by an esteemed topologist with decades of teaching experience. As part of the course, students received two homework assignments, which asked them to prove 13 claims in total. These proofs and accompanying interviews with the students enabled characterizing their proof styles.

The illustration focuses on a case of Jonah (pseudonym), whose proofs stood out for their length, structure, and level of detail. Accordingly, I present his case to illustrate what a student's proof style may look like, with special attention to his use of the conventions of the proof genre. For the interpretive purposes that follow, I would like to mention that Jonah aspired to pursue postgraduate studies in mathematics.

I approached Jonah's proofs with the question, "What RoPs can be discerned across his proofs?" The analysis included an iterative comparison of all Jonah's proofs to identify patterns and characteristic features. Initially, dozens of candidates for RoPs were extracted from Jonah's proofs, informed by the literature (e.g., Andersen, 2020; Selden & Selden, 2014). Through an iterative systematic comparison, the analysis converged on the rules that Jonah enacted in most of his proofs.

On Jonah's proof style in topology

This section focuses on *a multicompartmental proof structure* and *a minimal substantiation gap* – networks of metarules that featured in Jonah's proofs. To get a taste of them, let us consider his proofs of two claims: (i) "Suppose $\mathcal{F}$ is a locally finite family of closed subspaces of a topological space X whose union is X, and $f: X \to Y$ is

a function such that $f \restriction F$ is continuous for each $F \in \mathcal{F}$. Prove that f is continuous"; and (ii) "Cantor's ternary set is homeomorphic to $2^{\mathbb{N}}$" (see Figures 1 and 2).

Most of Jonah's proofs involved four types of narratives: *set-up narratives*, *lemmas, corollaries*, and *assembling narratives*. The set-up narratives recapped selected givens from the assigned claim, reformulated them, and introduced mathematical objects used in the later parts of the proof. For instance, in Figure 1, the set-up narrative replaces the original "closed subspaces" with "F_i" and mobilizes it to symbolize some of the original givens (e.g., "$\mathcal{F} = \{F_i \subset X | i \in I\}$" instead of "$\mathcal{F}$ is a locally finite family of closed subspaces of a topological space X" in the original claim). No deductive inferences were made in this part. Lemmas opened with clear declarations of target sub-claims to be proved (e.g., see "f is 1-1" in Figure 2). The corollaries applied lemmas to the objects in the focus of the originally assigned claim or previous lemmas (e.g., see "Corollary 3 and "lemma 2" in Figure 1). In other words, while the target claims of lemma-narratives were declared before being proved, corollary-narratives unfolded as linear deductive chains. Lastly, the assembling narratives concluded the proof by integrating previously obtained results into a single whole that endorsed the initial claim.

The above suggests that Jonah's proofs had a multicompartmental structure, with a narrative in each compartment adhering to its own RoPs. Occasionally, Jonah explicated the rules of assembling narratives by opening them with meta-level utterances about the claim to be proved and the method to do so (e.g., "It suffices to show that $f^{-1}(V) = \overline{f^{-1}(V)}$ in Figure 1). However, he never spelled out the rules of other types of narratives, which raises the question of how Jonah expected the reader to grasp their "inner logic." I contend that Jonah addressed this expectation by employing a repertoire of communication moves that drew on the conventions of the proof genre. Indeed, note that RoPs that Jonah enacted in his "lemmas" and "corollaries" are not very far from how these narratives are structured in mathematics textbooks and journal articles (cf. Dawkins & Vroom, 2005). Hereby, it can be suggested that Jonah, by labeling these narratives, appealed to the reader's familiarity with the conventional rules of narratives of the relevant type. He also numbered them sequentially as if to highlight how an outcome of a certain lemma fed into the corollary that followed. Moreover, Jonah used conspicuous horizontal lines to distinguish different proof compartments. The lines visually mediate (cf. Sfard, 2008) the points at which a certain set of RoPs commences and concludes.

Another aspect of Jonah's style concerns the minimality of substantiation gaps that he left in his proofs (cf. Andersen, 2020). Overall, his proofs mostly followed a RoP that can be formulated as "any claim a proof involves requires a proof of its own." For example, "lemma 2" in Figure 1 shows that some of Jonah's lemmas revolved around general mathematical objects rather than those featured in the formulation of the assigned claims. Most of these lemmas were either proved in the classroom or presented as facts that students could use "as shared." Indeed, "lemma 2" appeared in

the course notes and was proved in the classroom. Nonetheless, Jonah still recreated that proof in his submission.

DISCUSSION

Gilson and Goldberg (2015) maintain that conceptual papers focus on "integration and proposing new relationships among constructs" (p. 127). As such, this paper proposes individualized proof styles as a novel construct in the area of proof, relating a genre perspective and the commognitive framework. The construct is defined as a cumulative network of individualized versions of RoPs to which one adheres in their proof productions. The inclusive character of this definition aims to open the space for future research, while providing access to the commognitive apparatus. Indeed, a proof style accounts for both the routines one implements in the proving process and the metarules that their eventual proof-narratives follow (cf. Lavie et al., 2019). A *task situation* can capture the circumstances in which a proof is generated, whereas a *task* (i.e., an individual's interpretation of the task situation) may regard what Miller (1984) refers to as "rhetorical 'response' to situational 'demands' perceived by the rhetor" (p. 152).

Proof styles contribute to two lines of research. First, Stylianides et al. (2017) distinguish among three broad perspectives in the area of proof: the cognitive – proving as problem solving, the constructivist – proving as convincing, and the social – proving as a social activity. The researchers contend that the social perspective "is still developing [and] it does not have common, widely used constructs" (p. 248). The proposed construct embodies the social perspective, as it illuminates the dialectics of individualized proof productions: as carriers of a style characteristic of an individual and as instances of the community-wide proof genre. In this way, the construct takes a modest step toward strengthening the social perspective.

Second, as a commognitive construct, proof styles advance the framework in the area of proof. Commognition has recently entered the area, with only a handful of studies conducted at the university level so far (e.g., Kontorovich & Greenwood, 2023). Karavi et al. (2022) discuss the potential of the framework to contribute to research on proof.

Overall, I hope that proof styles will encourage educators to appreciate that proof requires "not just to write convincing arguments, but to write in a way acceptable to the community" (Selden et al., 2008, p. 305). The case of Jonah illustrates that this nuanced distinction may not be immediately apparent to students, even those at advanced stages of their mathematics studies. On the one hand, Jonah produced multicompartmental proofs that involved different types of narratives that followed different RoPs. He masterfully capitalized on the conventions of the proof genre to communicate the structure and the "inner logic" of his proofs to the reader. This made his proofs similar to those featured in mathematics textbooks and articles. On the other hand, Jonah left minimal gaps in his proofs, something that is uncharacteristic of the professional mathematics community to which he aspired to belong.

A methodological note is in order here. Mejía-Ramos et al. (2015) caution against inferring students' styles based on their proving in a singular mathematical area.

Accordingly, Jonah's style is inseparable from the topology course, and even from the coursework setting within which his proofs emerged. These observations raise the question of how discipline-specific students' proof styles are.

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Teaching Logic to Support Abstraction of Logical Structure

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Keywords: abstraction, logic, conditionals, Euler diagrams

CLARIFYING THE GOALS OF LOGIC INSTRUCTION

This poster (funded by USA NSF grant #2141925) will share our goals for teaching logic to answer the question: how can logic instruction support abstraction of logical structure? Toward this end, we portray our instructional method by contrasting with alternatives and situating it within relevant literature. In our context in the US, logic is primarily taught as a requisite topic for preparing mathematics students to engage in proving. Students need to understand the intended meaning of mathematical statements and the conditions for verifying (or falsifying) them to productively engage in proving. However, what topics should be covered, and in what way, is not standardized.

Durand-Guerrier et al. (2012) argued that students should come to understand both the syntactic and semantic aspects of logic, meaning both relationships within sign systems and how signs refer to objects. We concur with Deloustal-Jorrand's (2004) argument that students need to coordinate reasoning about implications in terms of formal logic, sets, and deductive reasoning. In this poster, we will contrast three ways that logic may be taught that correspond to these three options. The most common approach uses truth-tables, which emphasizes formal logic. Our research has developed a second approach emphasizing sets (e.g., Dawkins et al., 2023). Third, logic can be developed by engaging students in proving itself, which emphasizes deductive reasoning.

Though Deloustal-Jorrand (2004) recommends that students learn to coordinate these viewpoints on logical implication, introductory logic must treat one as primary. Our poster will argue for the preferability of the set-based approach since it can support students in abstracting logical structure. We use *abstraction* here in the Piagetian sense (Ellis et al., 2024) of projecting some kind of activity onto a higher plane in which it can be reflected on, reorganized, and reasoned about as an object of further activity. We will argue why we find both the truth-table approach and the warranted conditional approach less fitting by this criterion. We claim this framing of the teaching and learning of logic satisfies the recommendations offered both by Durand-Guerrier et al. (2012) and Deloustal-Jorrand (2004).

SETS SUPPORTING ABSTRACTION OF LOGICAL STRUCTURE

Our experiments have helped us see that truth-tables run contrary to the way students interpret mathematical language. This is because students take language to refer to objects, but truth-tables replace reference to objects with mere truth or falsehood (a truth-value). This violates Durand-Guerrier et al.'s (2012) condition that logic entail a semantic aspect. By contrast, set-based truth conditions are logically equivalent, but

they allow statements to refer to mathematical objects. In particular, the conditional "Given any $x \in S$, if $P(x)$, then $Q(x)$" is true whenever the truth-set of P ($A = \{x \in S | P(x)\}$) is a subset of the truth set of Q ($B = \{x \in S | Q(x)\}$). This upholds students' intuition that statements refer to objects. Further, it allows students to represent these sets using Euler diagrams. Sato and Mineshima (2015) showed the power of Euler diagrams to support students in reasoning about syllogism tasks. Further, Alcock (2025) found that mathematicians tended to use Euler diagrams to represent conditional relationships. Our studies show that students can use set language and set diagrams to abstract a generalized structure for conditional statements. True conditionals express subset relations ($A \subseteq B$). Further, students can coordinate these set relations with implications between the properties that define them. Thus, the set-based view affords student reasoning about all three views advocated by Deloustal-Jorrand (2004). Finally, though implication operates in mathematical practice as *provability*, we doubt whether inferring helps students apprehend logical structure. If we only engage students in proving and deductive reasoning, how will they come to see a recurrent structure among drawing inferences in number theory and geometry? There is little to support them in abstracting by representing a shared structure of inference, to reflect on that structure, and be able to treat it as the object of reflection.

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A Cross-Disciplinary Didactic Transposition from the Literature Domain to the Context of Proof

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Proof comprehension has been identified as a key skill for students to learn as they transition into courses where the formal proof becomes a major component of instruction. Yet, curriculum materials rarely address how to read proof explicitly. We suggest that the domain of literature study can provide didactic tasks and technologies that can be transposed to the proof classroom. We report on a design study that has involved the development of a curriculum that centers reading and appreciating mathematics proofs as a human-generated genre of writing.

Keywords: transition to proof, didactic transposition, curriculum development

INTRODUCTION

Over the last decade, there has been increasing attention to the comprehension of mathematical proofs (e.g., Dawkins & Zazkis, 2021; Mejia-Ramos et al., 2012; Yang & Lin, 2008) as a type of task critical for undergraduate mathematics students. This literature has suggested that proof comprehension is a skill that needs to be developed and reflects a multitude of distinct strategies. Yet, undergraduate proof-based courses rarely address comprehension explicitly (Weber, 2012) despite the increased prevalence of transition to proofs and bridging courses whose aim is to expose students to the mathematical practices needed to succeed in proof courses. Rather, these courses are most typically structured such that the goal is to learn how to construct proofs using basic techniques (e.g., David & Zazkis, 2021). That is, the object of learning becomes the proof techniques, and the specific mathematical domain is backgrounded. This structure can be problematic. First, the series of theorems to be proven are often unmotivated, such as proving basic number facts that are intuitive or unsurprising. Blåsjö (2016) argues this "form without substance" can lead to proceduralized views of proof. While a second model of transition to proof in specific contexts (such as algebra and analysis) attempts to counter some of this issue, proof techniques may be developed in quite bounded ways in a specific context (Savic, 2017). We posit a fundamentally different type of transition to proof course, one rooted in the disciplinary models from the teaching and learning of literature, may help alleviate these tensions.

In this report, we share about our work on the *Reading and Appreciating Mathematical Proofs* (funded by USA NSF grant #2141925) project designing and testing a curriculum in which students engage with proofs in ways that parallel how one might engage with great literature. Scholars have posited that proof is a genre (Selden & Selden, 2013) worthy of studying its norms and stylistic features. When learning a new genre in literature or music, students are not simply taught grammar or music theory, but they also are positioned to study great literature or great pieces of music.

These objects of study inform their models of and appreciation for the genre. Further, they learn not just about the great works, but the authors and musicians who create them. Anthology literature textbooks that include study of literature genres, great pieces of literature and author backgrounds are prevalent throughout postsecondary study (e.g., Greenblatt, 2018).

Toward these ends, we performed praxeological didactic-to-didactic transposition that moves from the school literature setting to the school mathematics setting. We address the following research question:

> How might we reconfigure a family of reading practices institutionally developed in literary studies (close/global reading, genre study, author and reader-response work) as a mathematical praxeology of proof reading within the university setting?

THEORETICAL UNDERPINNINGS OF THE RAMP PROJECT

In the Anthropological Theory of the Didactic (ATD), all human activity occurs within institutions and can be broken in constituent parts within two related blocks: praxis (the practical block) and logos (the theoretical block) (Chevallard, 2019). Praxis refers to tasks (T) and the techniques to perform the tasks (τ). The logos serves to justify the praxis where a technology (θ) serves as the argument and organization of the praxis, while a theory provides the basis for the technology. Praxeologies are then products of institutional conditions and constraints stemming from different levels. A discipline (e.g., mathematics) contains various domains (e.g., calculus, modern algebra) which contain sectors (e.g., single variable calculus, group theory) which then contain themes (e.g., differentiation, identifying finite groups), that ultimately contain specific questions that align with types of tasks (e.g., How does one find all the groups of prime order?). A theme is unified by technology, and a sector is unified by theory (Winsløw et al., 2014).

A key process in ATD is didactical transposition (Chevallard, 2019). This occurs when knowledge from a scholarly institution is transposed to the school institution for students to encounter and learn. That is, the praxeologies found in expert activity are not identical to their counterparts in learning settings. Adaptations or distortions are necessary to meet the needs of the differing institutions (Gascón & Nicolás, 2019). Transpositions are not limited to didactic and the movement of a praxeology between institutions can be more broadly thought of as institutional transpositions (when neither institution is didactic). Transpositions between institutions are considered external whereas those within an institution (e.g., from curriculum to instruction) are internal. We treat literary reading and university mathematics as two different didactic institutions reflecting two disciplines. Our work is therefore not only a case of external didactic transposition (Chevallard & Bosch, 2014) but specifically a transposition between didactic settings. Thus, we position our work as extending transposition analyses to cross-disciplinary didactic transpositions. Since this is a transposition between disciplines rather than from a scholarly to didactic instruction within the same

discipline, the connecting praxeological elements are less rooted in a common logos, but the commonality is in praxis where common tasks and techniques available.

Praxeologies in Literature Education

To situate our work, we begin by outlining a sample of praxis in literature and reading comprehension education. We do not intend to present an exhaustive view on this literature base, but rather we focus on outlining a few key point/local praxeologies that have been well-established within that institution (e.g., Pearson & Cervetti, 2017) that are reflected on our work. We focus on the tasks: reading comprehension (local), reading comprehension (global), genre study, and author/reader response work.

Task family	Types of tasks (T)	Example Techniques (τ)
Local reading comprehension (close reading) Pearson, 2014; Channa et al. 2015; Song et al., 2020	**T_1a**: Understand the meaning of individual sentences; **T_1b**: Clarify vocabulary, pronoun reference, and local relationships (cause, contrast, example); **T_1c**: Maintain local coherence between adjacent sentences.	Use context to determine the meaning of an unfamiliar word. Questioning ("What is happening here?" "Who is this about?")
Global reading comprehension (whole-text understanding) Channa et al. 2015; Pearson, 2014; Stevens and Vaughn, 2020	**T_2a**: Identify the main idea or theme of a text; **T_2b**: Describe overall text structure (e.g., narrative arc; problem–solution); **T_2c**: Relate parts (chapters, scenes, stanzas) to the whole.	Write or orally share 1–2 sentence summaries. Use graphic organizers (story maps, sequence charts, theme webs). Check headings and then write the "gist" after rereading section
Genre study Applebee, 1996; Bogaerds-Hazenberg et al., 2021	**T_3a**: Identify a text's genre (e.g., persuasive essay, sonnet); **T_3b**: Describe characteristic features of the genre; **T_3c**: Compare texts within a genre to see common patterns and variations.	Use anchor charts / lists of genre features. Make comparison charts for two texts in the same genre. Highlight passages that show genre features (e.g., stage directions in drama, rhyme and meter in poetry).
Author / reader-response work Applebee, 1996; Berger, 1996;	**T_4a**: Learn about an author's life and historical/cultural context; **T_4b**: Articulate one's personal response and reflect on how the text resonates with or challenges	Read and discuss brief author biographies or context notes.

Gabrielsen et al. 2019	one's own experiences, beliefs, or identity.	Keep response journals ("What surprised me?", "What I related to…").

Table 1: Sample of Praxeology Components from Didactics of Literature

Transposing Literature Education Praxeologies to the Undergraduate Proof Setting and Developing the Didactic Milieu

To bring praxeologies from the didactics of literature into the undergraduate proof setting, we undertook a didactic transposition, adapting and distorting tasks and techniques to function within the epistemic and institutional constraints of proof courses. In the hierarchy of discipline, domain, sector, and theme, our domain is an introductory proof course where students encounter proofs from several content sectors (e.g., set theory and functions, number theory). Such courses are common in the United States undergraduate system where they serve to bridge students from computational-based courses to proof-based courses. The goal of such courses is not specific to a sector, but rather the overarching aim is for students to learn to write proofs broadly (Schumacher et al., 2013). Our proof-comprehension task families (local and global proof reading, proof genre study, and reader/author-oriented work) function as transversal themes: they are not tied to any single sector but instead provide praxeologies for engaging with proofs across sectors in this domain.

Our transposition of techniques between the literature and proof context was informed not just by existing models for studying literature, but also by the research literature and by our own studies of how students learn to engage in proving activity. For example, we made use of the growing body of literature on proof comprehension strategies and components (e.g., Mejia-Ramos et al., 2012). We suggest that the logos supporting proof reading and appreciation must be rooted in logos related to the theory of deductive formal proving broadly in pairing with sector theory. For example, a task "why does this line follow from the previous" will be answerable by appealing to the deductive system that includes a specific, logically expectable way a statement may follow another in a proof. However, it also draws on theory from the relevant sector such as number theory, algebra, or other areas.

Finally, we also consider other constraints and conditions that differ in a transition-to-proof course when compared to the study of literature. These courses tend to serve as gateway courses to upper-division mathematics. There is an expectation that students who complete the course can produce individual proofs in a recognizable format. In contrast, literature courses rarely expect students to recreate the works they study, although there is often an expectation that they can write in the relevant genre. There are also differing time constraints and necessary topic compression. Studying literature covers many years, courses, and foci. In contrast, we are operating in a single

undergraduate course, meaning that the depth and breadth of study are necessarily different in our context.

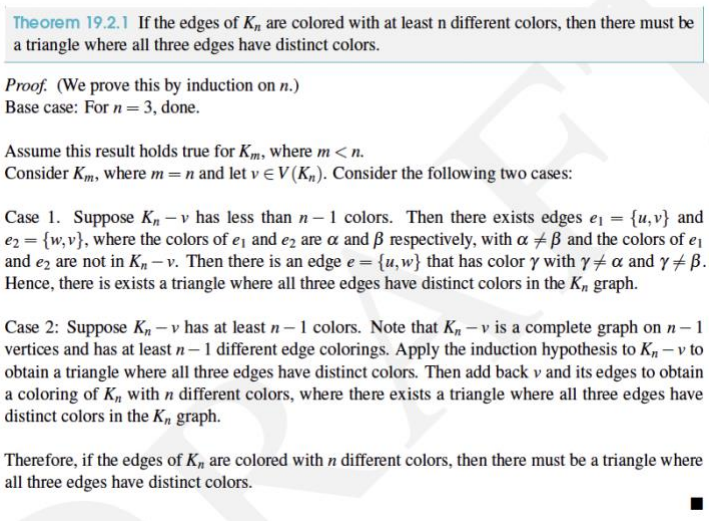
THE TRANSPOSITION AND ENGINEERING OF DIDACTIC MILIEU FOR A READING-BASED TRANSITION TO PROOF COURSE

The RAMP curriculum was developed to elicit the kinds of reading, sense-making, and reflective work we sought to cultivate. The sequencing of statements, proofs, comprehension tasks, and mathematician stories functioned as a milieu that calls for particular techniques (e.g., identifying difficult lines, summarizing structure, noticing genre structures, reflecting on narratives) and supports students in developing the associated technologies. We began with a basic goal: "How might students learn about proofs via reading and appreciation of the genre?" This led to several sub-questions including: "How might a student learn to comprehend a specific proof?", "How might a student come to know the norms of the genre?" and "How might a student learn to situate proofs as works of human activity?" Our design began by using existing curriculum to have students reinvent logic for proving activities by leveraging set-based reasoning (see Dawkins et al., 2023). Students engaged in core tasks, including comparing multiple proofs of the same theorem, comparing proofs of similar statements, identifying proof methods, and analyzing features of proofs. In ATD terms, these activities constitute types of tasks (T), supported by techniques (τ), such as attending to the first and last lines of a proof, that help students determine whether they prove a given statement. We situate this activity in genre study (e.g., Fountas & Pinnell, 2012) where we can see reflections of tasks from literature such as comparing versions of the same story (T_3c) and identifying genre conventions (T_3b). However, the earlier parts of this curriculum were not intentionally based on literature genre study, but natural parallels nevertheless exist.

Where we engaged in targeted cross-disciplinary didactic transposition was in the next component of the curriculum where a literature-style textbook of proofs was developed. To create this component of the didactic milieu, we had to consider how we might reconstitute praxeologies related to authors and literature reading. Texts serving to study literature contain regular elements of author backgrounds, introductions to works, and the literary work (Leitch, 2000). While there is criticism to be had at the superficial nature of such a structure, it does provide a basic template for literature-based approach to proof. To engage this work, we recognized a need to have undergraduate accessible proof texts and corresponding author materials. No such pairing existed beyond the standard textbook approach of sharing a few notable historic figures in certain domains without corresponding tasks to engage (e.g., sharing information about Galois in relation to modern algebra).

To develop the materials, we engaged a group of eight diverse, active mathematicians in the United States (Contreras et al., 2025) to author proofs and stories. They were tasked with writing three proofs that are accessible, yet interesting, for students transitioning to proofs. These proofs were then refined and further developed in collaborative workshops between the mathematicians and the research project team. At

this point, we also engaged in developing reading support materials in the form of a series of questions that students could ask of a proof. To promote transposition to this didactic institution, we found further guidance from the works of Mejia-Ramos et al. (2012) and Yang and Lin (2008) who had developed a series of comprehension strategies and question types based on literature, mathematician and mathematics teacher interviews. Yang and Lin outlined the ways that proof comprehension shares common families of tasks as language learning reading comprehension research citing Gagné et al. (1993) and organizing their types of tasks into literal and semantic comprehension, inferential comprehension, and monitoring comprehension. Mejía-Ramos et al. (2012) drew upon this framework to create a framework for assessment items organized into local and holistic types focused on local meaning, implications, and justification and holistic understandings evidenced by identifying substructures, key ideas and summarizing. From an ATD point of view, these scholars had already transposed a substantial amount of knowledge from the scholarly institution to the didactic institution. Using the Mejia-Ramos framework and the goal of developing a literature study style class, the project team developed a set of questions to pair with each proof and a shared understanding of what might constitute an author story. These stories were ultimately written with a set of guiding questions in which the authors shared how they arrived at their current positions in mathematics and connected to their proof choice when relevant.



Theorem 19.2.1 If the edges of K_n are colored with at least n different colors, then there must be a triangle where all three edges have distinct colors.

Proof. (We prove this by induction on n.)
Base case: For $n = 3$, done.

Assume this result holds true for K_m, where $m < n$.
Consider K_m, where $m = n$ and let $v \in V(K_n)$. Consider the following two cases:

Case 1. Suppose $K_n - v$ has less than $n-1$ colors. Then there exists edges $e_1 = \{u, v\}$ and $e_2 = \{w, v\}$, where the colors of e_1 and e_2 are α and β respectively, with $\alpha \neq \beta$ and the colors of e_1 and e_2 are not in $K_n - v$. Then there is an edge $e = \{u, w\}$ that has color γ with $\gamma \neq \alpha$ and $\gamma \neq \beta$. Hence, there is exists a triangle where all three edges have distinct colors in the K_n graph.

Case 2: Suppose $K_n - v$ has at least $n-1$ colors. Note that $K_n - v$ is a complete graph on $n-1$ vertices and has at least $n-1$ different edge colorings. Apply the induction hypothesis to $K_n - v$ to obtain a triangle where all three edges have distinct colors. Then add back v and its edges to obtain a coloring of K_n with n different colors, where there exists a triangle where all three edges have distinct colors in the K_n graph.

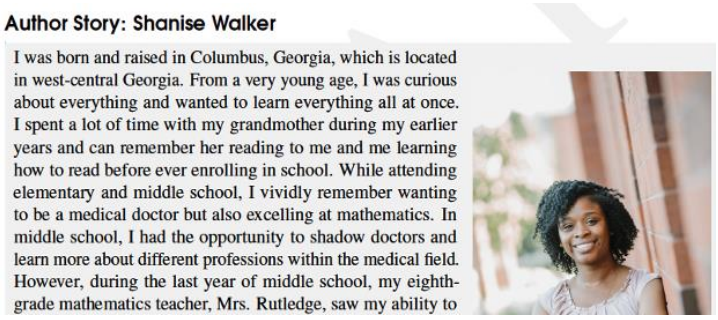
Therefore, if the edges of K_n are colored with n different colors, then there must be a triangle where all three edges have distinct colors.

■

Figure 1: Proof from an Early Draft of the Resulting Book

The proofs paired with extensive sets of comprehension questions were tested with students in cognitive interviews. Members of the research team interviewed pairs of students currently enrolled in transition-to-proof courses at two institutions in the United States. They were asked to read the proofs and then to address the questions about the proofs. Many of the questions were specific to a certain proof such as asking for the meaning of a particular term (T_1b) or how one line followed from the prior (T_1c). Additionally, all students were asked holistic questions such as prompting to summarize the proof. From this process, we made edits to questions and identified which seemed most crucial to engage students in the intended comprehension activity.

These ultimately were preserved and became part of the resulting textbook. Figure 1 and Figure 2 illustrate some of the textbook.



Author Story: Shanise Walker

I was born and raised in Columbus, Georgia, which is located in west-central Georgia. From a very young age, I was curious about everything and wanted to learn everything all at once. I spent a lot of time with my grandmother during my earlier years and can remember her reading to me and me learning how to read before ever enrolling in school. While attending elementary and middle school, I vividly remember wanting to be a medical doctor but also excelling at mathematics. In middle school, I had the opportunity to shadow doctors and learn more about different professions within the medical field. However, during the last year of middle school, my eighth-grade mathematics teacher, Mrs. Rutledge, saw my ability to

Figure 2: Author Story Snippet from an Early Draft of the Resulting Book

Using the proof in Figure 1 to illustrate, we can consider ways literary tasks and techniques can be reconstituted in mathematics. At the global level, students can engage in the task of summarizing the proof (T_2a) or address specific questions such as, "What is the purpose of each of the part of the proof?" (T_2b). Our prompts were initially inspired by Mejia-Ramos et al.'s (2012) proof comprehension framework, which itself draws Yang and Lin (2008)'s transposition from the reading comprehension domain. In contrast, the "purpose of each part" task (T_2b) reflects more substantial transpositional change. At a surface level, checking headings (τ_2a_3) becomes identifying key indicators in proofs, such as implication symbols or labeled cases (τ_2a_{p3}). At a deeper level, whereas logos in literary contexts reflects the norms of a given genre, the goal of proof is uniform: to provide a valid argument for the truth of a statement. We suggest the following praxeology: T: Identify the purpose of each part of the proof; τ: Segment (if needed using key proof indicators); condense each part to one paraphrased idea, state what the step accomplishes, relate each step back to the overall theorem being proved. The logos block [θ/ Θ] is then rooted in ideas that proofs are structured such that each part is necessary to support the truth of an assertion and therefore each individual section serves a role in the argument, reflecting norms of mathematical proofs and the theory of proof as a deductive, structured argument.

At the local reading level, students may engage in the task of clarifying vocabulary (T_1a). In the literary setting, this is paired with the techniques like rereading (τ_1a_1) and looking for context clues (τ_1a_2). In mathematics, definitions are stipulated rather than descriptive. So, the task of clarifying vocabulary (T_1a) might better pair with *finding and reading formal definitions* (τ_1a_{p2}) as a technique. This category of task can be further transposed to mathematics to include clarifying *symbols* (What does K_n-v represent?) where the technology of context clues (τ_1a_1) or finding definitions may be appropriate (τ_1a_{p2}). The task of maintain local coherence between adjacent sentences (T_1c) transposes into the proof setting where a student might address a question like "Why do we know there are colors α and β that are not in K_n-v?" where techniques such as questioning what is going (τ_1c_1) remaining consistent but logos is fundamentally different where technology and theory rely on deductive logic.

In contrast to the prior types of tasks, we did not find existing literature related to story engagement in author stories paired with mathematical texts beyond advocating for using stories to broaden who is seen as a mathematician (e.g., Mayes-Tang, 2019). This required a transposition with less direct knowledge of the mathematics domain conditions. Ultimately, we designed for the author stories to serve the contextualizing feature of literature anthologies (T_4a), but also with the additional didactic aim of introducing students to multiple ways of being a mathematician which was better supported with personal response tasks (T_4b) such as responding to questions like, "What do you relate to?" where targeted techniques include making connections between their lives and the mathematicians' lives and reflecting on how their understanding of mathematicians and/or mathematics changed. An initial set of prompts was developed to elicit these techniques and tested in two classes. We subsequently modified the prompts based on student reflections indicating decreasing engagement due to repetitive questioning.

DISCUSSION, EFFICACY, AND FUTURE DIRECTIONS

In this report, we shared some of our work and insights from engaging in a cross-disciplinary didactic transposition. The development of our materials stemmed from our desire to position students to not just construct but comprehend and appreciate proofs as both genre texts and human products. A more thorough articulation of transpositions would reflect a more complex network of transpositions capturing not just this cross-disciplinary transposition, but other external and internal transpositions. At the present stage of our work, we have found some productive adaptation of tasks and techniques of the literary discipline. However, we have not developed a full reference epistemological model and future work could include a more comprehensive and theoretically grounded version of this work. There are many areas we have not fully made use of praxeologies from the literature discipline, but due to the similarity of our reading texts, we can further consider transpositions. For example, as the text contains proofs from differing authors, we have provided opportunity for a type of task that is not possible in a standard approach to transition to proof: comparing different approaches to writing the formal proof. This could provide opportunity to engage with genre to better attend to both conventions and variations.

Currently, we are in the process of examining the efficacy of our approach in achieving didactic goals related to students' capacity to read proofs and increasing their sense of belonging in the mathematical community. We have gone through several rounds of engineering and testing the materials which have now been implemented in four instructors' classes across three undergraduate institutions in the United States. We have analysed students' written reflections on the stories, conducted interviews with students about their experiences in the classes, and collected survey data. Our next stages will include a more holistic evaluation of the efficacy of this approach with the aim to continue to improve our approach through both adaptation of literary praxeologies and data about how this engineering operates in the context of undergraduate students' first encounters with formal proof.

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An intersection of linear algebra, complex numbers, and quantum computing: The case of the Bloch sphere

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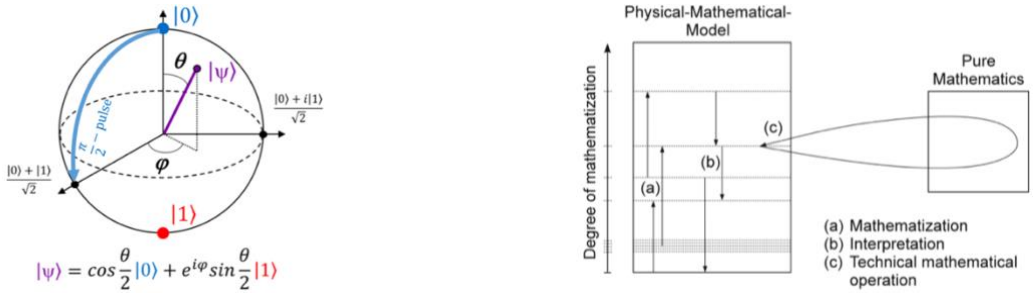
Keywords: linear algebra, quantum information science, graphical reasoning, physical-mathematical model, qualitative analysis.

Quantum information science (QIS) is an interdisciplinary field emerging from key concepts and techniques in a variety of traditional areas such as physics, computer science, and linear algebra. With the fast, global rise of QIS across a spectrum of academic and industry forefronts, important research goals are understanding how university students make sense of the foundational interconnectedness of mathematics in QIS, and how professors can facilitate mathematical and physical understanding in quantum instruction to promote student success and access to the quantum workforce.

In our poster, we will analyse a compelling intersection of linear algebra, complex numbers, and quantum computing: a graphical representation known as the Bloch sphere. The Bloch sphere is a mapping of elements from a two-dimensional complex Hilbert space onto the surface of a unit sphere in three-dimensional real space, and it is used to visualize qubit states (see Figure 1a) and the effect that various quantum gates have on them (Hu et al., 2024). The qubit is the fundamental unit of information in quantum computing, a bit that can be in a quantum state via a superposition of states that correspond to the classical bits 0 and 1 (Watrous, 2025). Qubit states are associated with elements of $\mathbb{C}^2$ and are often symbolized with Dirac notation. For example, any qubit state $|\psi\rangle$ can be expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for $\alpha, \beta \in \mathbb{C}$ with normalization condition $|\alpha|^2 + |\beta|^2 = 1$, where $|0\rangle \doteq \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle \doteq \begin{bmatrix}0\\1\end{bmatrix}$. The states $\{|0\rangle, |1\rangle\}$ form what is referred to as the computational basis; a different commonly used orthonormal basis is $\{|+\rangle, |-\rangle\}$, where $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$. Finally, another valid and useful expression for any qubit state is $|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle$ for $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$. Note that this expression relies only on two real parameters, θ and φ, which facilitate the visualization of any qubit state as an element of a unit 2-sphere (Figure 1a).

Research on student sensemaking in QIS is a growing field, yet there is very little on how students understand the Bloch sphere, with Hu et al. (2024) as a notable exception. They mentioned the understandable temptation for students to graph $|0\rangle$ on the positive z-axis and $|1\rangle$ on the positive x-axis because they are called orthogonal (their inner product yields 0); however, in the geometry of the Bloch sphere they are antipodal. It is this potential curiosity that prompted us to look deeper into the Bloch sphere as an example of the intersection of linear algebra, complex numbers, and quantum computing, with a particular interest in inseparable aspects of mathematics and QIS.



(a) **(b)**

Figure 1: (a) The Bloch sphere (Ercolessi et al., 2024); (b) Uhden et al.'s (2012) model.

We draw on Uhden et al. (2012) for our theoretical framework and methodology to analyse the intersection of mathematics and QIS in the Bloch sphere. They proposed that models which analyse mathematical reasoning in physics need to acknowledge and make space for the conceptual entanglement of mathematics and physics. The authors do so by documenting degrees of mathematization and interpretation of structural skills, as well as technical skills with mathematical manipulations (see Figure 1b). Our research question is: What mathematization, interpretation, and technical skills may be involved in a didactic approach of deriving the Bloch sphere?

In this poster, we will use Uhden et al.'s (2012) model to analyse a didactical approach of the derivation of the Bloch sphere given by Glendinning (2005), with support from our own conceptual analyses and sources such as Ercolessi et al. (2024) and Watrous (2025). Findings include the importance of basis and dimension, algebraic and graphical representations of complex number (Cartesian, polar, exponential), spherical coordinates, the normalization condition, measurable quantities, and global phase and relative phase of a quantum state. This poster is part of a larger study investigating how students reason about and symbolize linear algebra concepts in coordination with QIS concepts across disciplines (U.S. National Science Foundation, Award No. #2500569).

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TWG2: The teaching and learning of university mathematics across different disciplines (including mathematical modelling)

From Mathematics to Economics: The Student Mentor Perspective on Relevance

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This study examines how student mentors perceive and communicate relevance of mathematics within economics education through video resources. Eight second- and third-year economics students participated as student mentors, addressing the challenge of perceived relevance during the transition from school mathematics to mathematics in economics education. The mentors emphasised the use value of mathematics, primarily by contextualising mathematical concepts for later courses, and reflected on their role as 'bridging people' drawing on recent experience to support first-year students. While prioritising use value, some examples intersected with exchange value, particularly through references to examinations, indicating that relevance may combine both dimensions.

Keywords: Mathematics for economists, relevance, use- and exchange value, qualitative, video resources.

INTRODUCTION

The SPVME (Student Produced Videos in Mathematics Education) project seeks to enhance first-year economics students' engagement with mathematics by integrating student mentors (second- or third-year economics students) as co-creators of digital learning resources. The student mentors, having previously engaged with the mathematical content through their own economics studies, are well positioned to act as "bridging" people between the disciplines. Our intention was that these mentors could support first-year students in recognising the relevance of mathematics to economics, by in their own way, contextualising and presenting mathematical concepts within economics. In this way, the project not only fosters interdisciplinary understanding but also contributes to the development of pedagogical structures that support and sustain such bridging roles within higher education and by that, strengthens coherence within the study programme.

POSITIONING THE STUDY WITHIN EXISTING RESEACH

The challenge of students' limited perception of the relevance of mathematical knowledge is well documented in mathematics education research. In teacher education, this issue is often framed through the concept of the double discontinuity (Winsløw & Grønbæk, 2014). In service mathematics contexts (such as engineering and economics), Florensa Ferrando et al. (2022) and Landgärds-Tarvoll (2025) extend this discussion to what can be described as a triple discontinuity. This refers to the successive transitions students must navigate: from school mathematics to university (service) mathematics, from university (service) mathematics to its use within their

study program, and finally to its use in professional practice. Even when mathematical content overlaps across these contexts, students frequently struggle to recognise its applicability (Christensen, 2008; Feudel & Biehler, 2021). Addressing such issues requires coherence, which involves creating clear connections across learning experiences so that students perceive mathematics as relevant and meaningful within their discipline (Canrinus et al., 2019).

Recent discussions in mathematics education emphasize the importance of interdisciplinary collaboration to overcome such relevance problems. TWG3: Mathematics and other disciplines at the INDRUM 2024 conference (Bašić & Landgärds-Tarvoll, 2024) highlighted the role of educators as "bridging people" who help students navigate interdisciplinary learning environments, while noting the lack of structures that support these roles in practice. TWG3 also called for research on technologies that connect mathematics to disciplinary applications.

The SPVME project responds to these challenges by engaging second- and third-year economics students as mentors and co-creators of digital learning resources for first-year students. This approach draws on Loch and Lamborn's (2016) work in engineering education, which showed that involving students as co-creators of learning materials can significantly improve the perceived relevance and foster engagement. Building on these insights, SPVME adapts the model to economics education, using student mentors to contextualise mathematical concepts within economics and strengthen coherence across learning experiences. In doing so, the project not only address the issue of students' perceived relevance of mathematics in economics but also contributes to exploring pedagogical structures that supports bridging roles in higher education.

THEORETICAL FRAMEWORK

While the overarching aim of the SPVME project is to address coherence, as defined by Canrinus et al. (2019), with respect to mathematics within economics education, the present study focuses on the issue of students' percieved lack of relevance in the study programme. Therefore we specifically focus on the second discontinuity, which Landgärds-Tarvoll (2025) defines to be between the mathemetics *for* economists to mathematics *in* economics eduction. In particular, we examine how student mentors percieve that they can strengthen coherence within the study programme through their appropirated *bridging role*.

The theoretical framing of this study is grounded in the work of Hernandez-Martinez and Vos (2018), who define relevance as a connection among an object (such as a mathematical task or topic), its usefulness, and the person engaging with it. Their framework draws inspiration from, and expands upon, earlier contributions to the theme and attends to four guiding dimensions of relevance: "*Relevance of what? Relevance to whom? Relevance according to whom? Relevance to what end?"* (Hernandez-Martinez & Vos, 2018 p. 247). In our study, these dimensions are operationalised as follows. The *relevance of what* refers to eight pre-determined

mathematical concepts or themes selected for video production. The *relevance to whom* concerns the intended audience of first-year economics students. The *relevance according to whom* is interpreted through the lens of the student mentors, who act as mediators of relevance through their own disciplinary experiences. In our study, these three dimensions are embedded within the design of the project and remain largely predetermined by its structure. Our analytical focus is therefore placed on the fourth dimension: *relevance to what end*. This dimension emphasises the goal-directed nature of relevance, clarifying that relevance must have a purpose (Hernandez-Martinez & Vos, 2018). Drawing on the conceptualisation by Hernandez-Martinez and Vos (2018) this dimension is analysed in relation to motives and motivations, both within the learning environment and in envisioned professional contexts and connects to notions of usefulness and value. Therefore, in this study, we examine whether and how student mentors express the relevance of mathematics in terms of use- or exchange value. Use value refers to the practical applicability of mathematical concepts within the economics context. It reflects how mathematics can be employed to solve problems, interpret models, or support reasoning in real-world or subject-specific scenarios. Exchange value, relates to the function of mathematics as a selection mechanism to obtain grades, pass exams, or enter further education. While use and exchange value are defined as distinct categories, Hernandez-Martinez and Vos (2018) as well as Pais (2013) both mention that these two forms of value may operate simultaneously and even complement one another within educational activity systems. Because the pursuit of disciplinary competence is inextricably linked to acquiring credentials within the institutional credit system, examining this interplay provides a holistic perspective for understanding how students actually perceive relevance.

Drawing on Loch and Lamborn (2016), we adopt a case study approach with a descriptive and explorative focus to investigate how student mentors perceive and communicate the relevance of mathematics in economics. This approach enables us to explore not only what student mentors say about relevance, but also how they mediate the relevance through their appropriated bridging role.

CONTEXT

The project is part of the University of Agder's initiative to enhance teaching quality through responsible use of new technologies and innovative pedagogical approaches. Therefore, the SPVME-project was developed to explore new ways of supporting student engagement in interdisciplinary learning contexts. The setting is a large first-year economics cohort (approximately 240 students) enrolled in a mandatory 7.5-credit mathematics-for-economics course. This course covers a broad curriculum ranging from algebra and single-variable calculus to multivariable calculus, the Lagrange method, and financial mathematics. The course is traditionally delivered through large lectures taught by a single mathematics instructor, supplemented by group-work sessions facilitated by student mentors.

The eight participating mentors were second- or third-year economics students formally employed as student mentors by the department. In their regular roles, they

assist with grading, provide feedback on mandatory assignments, and support first-year students during practice sessions following lectures. Having completed the mathematics course themselves and progressed within the economics curriculum, they are familiar both with the challenges of the first-year transition and with the later applications of mathematics that first-year students have yet to encounter. Positioned as 'student experts' on how mathematics becomes meaningful later in the programme, they are well placed to act as bridging people between the disciplines of mathematics and economics. To operationalise this potential bridging role, the mentors were invited to produce short videos on mathematical concepts pre-selected by the lecturer.

The task was intentionally designed with minimal constraints. Mentors were instructed to create two separate videos for each concept: one explaining the mathematics in their own words, and a second illustrating examples of the concept's relevance. Apart from this two-part structure, no further guidance was provided, allowing mentors to interpret "relevance" independently and to produce resources they believed would genuinely support first-year students. This design was informed by Guo et al. (2014) and Loch and Lamborn (2016), who highlight the benefits of concise, single-concept videos for first-year learners.

The resulting student produced videos (SPVs) were distributed to the first-year cohort in spring 2025 and were widely used. Usage analytics recorded 1,319 plays across the 16 videos, totalling over 61 hours of viewing time, with an average completion rate of 78%. Engagement peaked during the two weeks before the final exam, indicating that students primarily used the videos for revision. Questionnaire responses from 98 students confirmed this pattern: most valued the videos as concise, flexible resources for clarifying concepts and preparing for exams, though some noted limitations such as brevity and lack of interaction (Landgärds-Tarvoll & Skarpeteig Fidje, 2025).

Building on these initial insights, the present study investigates how the student mentors' perceived the relevance of mathematics within economics and how they understood their own bridging role. The research questions guiding the study are:

1. What characterises the student mentors' perceptions of the relevance of mathematics within the economics programme?
2. What do the participants perceive to be the benefits and limitation of student mentors as bridging people within the economics programme?

METHODS

This case study draws on data from eight student mentors and employs two analytical lenses: the established notions of relevance (use value and exchange value) and the new, underexplored concept of their appropriated 'bridging role' within the triple discontinuity. Because the first lens draws on well-defined theoretical constructs, a deductive categorisation approach was applied. In contrast, the bridging role required inductive, data-driven categorisation, as it represents a less explored phenomenon. The study is qualitative in design, focusing on how participants articulated their perceptions in their own words. The primary data source is the set of transcribed interviews, while

the analysis of the SPVs is used to support, refine, or nuance the interview-based findings as the SPV is viewed as the result of the mentor student's perceptions about relevance.

Interview data were collected via an asynchronous, self-produced structured interview routine. All mentors received an identical list of six prompts, asking them to reflect on: (1) their understanding of relevance; (2) how they identified relevance for each concept; (3) how they planned what to say; (4) how they planned what to show; (5) what they might do differently in future videos; and (6) perceived advantages and challenges of including student-produced videos in the first-year mathematics-for-economics course. Mentors were asked to record their reflections in a single, unedited video or audio file and to respond sequentially to the prompts. This method ensured standardisation across participants while allowing them to articulate their thoughts without interviewer influence.

RESULTS

When asked to explain their views on relevance, the student mentors consistently discuss the use value rather than exchange value of the concepts introduced in the mathematics course. All eight mentors provide descriptions in line with this view, and no statements or examples were provided in which exchange value is regarded as the only central relevance. However, some student mentors make references to what might be regarded as indications of exchange value. These references were to exams, particularly when students recalled exam tasks from subsequent economics courses to identify relevant contexts for the video tasks. These instances indicate that the concepts hold use value for subsequent courses while also carrying exchange value within those later courses. In line with Pais (2013) and Hernandez-Martinez and Vos (2018), such examples suggest that the two forms of value can operate simultaneously, with examination contexts implicitly reinforcing the perceived disciplinary usefulness of the concepts.

In terms of use value, two themes emerged. The student mentors predominantly discuss how to express that the mathematics course has use value in the second and the third discontinuity, being for (1) the later courses in the study programme (mentioned by all eight student mentors) and (2) later professional practice in economics and finance (mentioned by four). In the following, we provide brief summaries of four of the SPVs' content and present the students' explanations of their strategies for demonstrating the relevance within the economics study programme:

Ingrid was assigned the mathematical concept of the derivative. In her relevance video, she explains how firms use marginal cost (MC) and marginal revenue (MR), which are derivatives of the cost- and the revenue function respectively, to determine the profit-maximising output level. She clarifies that, in economic terms, MC and MR represent approximations of the additional cost or revenue generated by producing one extra unit. In the video, she discusses the implications for a firm when MR exceeds MC and when MC exceeds MR, concluding that the optimal production point is where

MC equals MR, that is, where the derivatives coincide. In her interview, Ingrid expands on her choice of example:

> Ingrid: Derivation is very relevant in very many subjects we have in the economics program. [...] Personally, I am very fond of macroeconomics. And derivation was used a lot there. So, it came very naturally that that was what I should use.

Henrik created a video about elasticity. He briefly mentions income elasticity and cross-price elasticity, but contrary to exemplifying these concepts drawing on content from later courses, he turns to price elasticity and points to an exam task from the mathematics course. When reflecting on his strategy for explaining relevance, he says:

> Henrik: I took an example from the math course that the lecturer delivered to us. And then I also remember that we will be using elasticity later, especially in microeconomics. So, I brought that up in the video too, that this will be relevant later when you are going to work with microeconomics, for example on cross-price elasticity.

Elena was assigned the Lagrange multiplier method. In her relevance video, she chose to present an exam question from the second-year microeconomics course, explaining that it is advantageous for students to master the Lagrange method early, as it reappears in later coursework and examinations. The example she selected involves a production function with labour and capital as inputs, where the task is to minimise total costs for a given output level. Through this example, she illustrates how constrained optimisation, first encountered in the mathematics course, underpins core economic analyses such as optimal producer choices. In her interview, Elena describes how she approached the task:

> Elena: And the way I did it was really just to think back to when I discovered the mathematical concept in other subjects and used that to explain that this will be relevant for the economics programme later. Because you will need this topic later in the program.

Both Henrik and Elena reveal an overlap in how relevance is understood. While their intention is to demonstrate the use value for subsequent economics courses, they do this by connecting it to the examination. In these instances, examination requirements (exchange value) are exploited to explain future utility (use value). For these mentors, the fact that a concept appears on a future microeconomics exam appears to be validation for it being disciplinary useful, and therefore relevant. This highlights that the students do not perceive use and exchange values as mutually exclusive. Rather, exchange value is leveraged as a proof of use value.

Eirik's contribution centred on the concept of the integral. His relevance video presents consumer surplus as an application of integration, showing how the area between the demand curve and the market price is found using definite integrals. He explains in the interview:

Eirik: When I was to calculate an example, I chose the definite integral. Because that is something you can use later. In later subjects, typically, microeconomics.

Although this was not included in their relevance videos, both Elena and Eirik also emphasise in the interviews that mathematics carries use value not only for their later courses but also for future professional practice in economics: Eirik says:

Eirik: I would think, what is it we are learning in the math subject? And how can it be connected to, first and foremost, what one does in the business sector? But perhaps most relevant for us right now would be: how is what we learn in the math subject relevant for what we are going to learn later in the course [study programme]?

And Elena mentions:

Elena: And for me, things that are relevant are mainly things I can use later, for example in working life or later in other subjects, when it comes to for example studies, because then I know that I will have use for certain topics or other things that I can actually use later in other circumstances, such as for example a job or other subjects and so on.

The mentors' emphasis on use value in the programme over professional practice reflects their current positioning as economics students. Consequently, they understand their 'bridging role' primarily through the second discontinuity: supporting first-year students transitioning from the mathematics-for-economics course to later economics courses. Six mentors spontaneously reflected on their own experiences with this transition, indicating they actively and meaningfully engage with this role. Furthermore, all participants highlighted empathy and relatability as vital components of their 'student voice'. Having recently faced the same challenges, they can translate mathematical relevance more accessibly, thereby adding substantial value to the programme's instructional structure.

The following extracts illustrate how Eirik and Sofie reflect on their role as bridging people and the second discontinuity in general:

Eirik: What I learned in the maths subject, I wasn't necessarily aware of what was relevant. It has come in hindsight. So, it is more that I - what I remember from - oh, we learned that in maths. I can use that here in this subject.

Eirik: We can relate a bit more to what they are going through. Because it is not so long ago that we learned the same thing ourselves. So, then it is perhaps a bit simpler to convey what it is they get stuck on, or what perhaps is - Yes, that is, what they get stuck on, and then how we ourselves, we who made the videos, how we got past what they get stuck on, and then take it from there.

Sofie: Because if you don't see [the] connection, you often feel that you see it as a bit meaningless, and you get much less motivation to learn anything. But if

you see that something is relevant for later, it often becomes more fun, and you get much more motivated to actually learn something about it.

Sofie: Student voices perhaps have a simpler language and often know a bit about what people find difficult. Because we have just been through it ourselves.

While the student mentors are positive about their role and confident in the value they contribute, several also point out their limitations within resource creation. In terms of these limitations, the participants mainly reflect on their lack of pedagogical training and the risk of being imprecise in their presentations. These themes are exemplified in the extracts from Henrik and Sofie below:

Henrik: Disadvantages are, at least for my part, that we are not knowledge communicators. Many of us are not experienced knowledge communicators. I have never had any role as a teacher or lecturer. The art of teaching something is something I have felt [challenged by] generally as a student mentor. It was something I had to learn along the way.

Sofie: But what can be a bit of a challenge, then, is that one must, in a way, be aware that what we student mentors say might not be as precise. Maybe someone unconsciously manages to omit some important details in their explanation. Or that we simply are not that precise, so maybe people interpret it in a slightly wrong direction, then, regarding what we are actually trying to explain. So, I think the videos can be very good, but it is very important that one shows that it is a supplement to lectures and the syllabus.

DISCUSSION AND CONCLUSION

This study explored how student mentors perceive and communicate the relevance of mathematics within economics education through their created video resources. A key finding is the student mentors' strong emphasis on use value of the mathematics for later courses in their explanations and reflections. This emphasis reflects their position as economics students who have recently navigated the first and are still themselves navigating the second discontinuity within the triple discontinuity structure typical for students in service mathematics courses (Florensa Ferrando et al., 2022; Landgärds-Tarvoll, 2025). Their strategies often involve contextualising concepts with economics terminology used in later courses and using examples drawn from their own experiences, reinforcing the idea of relevance as practical and discipline-specific.

While student mentors tend to articulate relevance primarily in terms of use value, the interview data and SPV content reveal that their accounts do not conform to the strict separation between use and exchange value proposed by Hernandez-Martinez and Vos (2018). Instead, a more complex pattern emerges. Several mentors draw on examples from examination tasks in later economics courses, framing these not as exam practice but as illustrations of concepts that "will be important later". Here, examination requirements (exchange value) are actively used to motivate and justify future disciplinary utility (use value). Because examples are tied to assessment and

progression within the programme, they simultaneously carry exchange-value significance. In line with Pais (2013), this suggests that the logics of use value and exchange value may function in complementary ways, with references to disciplinary usefulness also reinforcing mathematics' role in accreditation and selection. Similarly, Hernandez-Martinez and Vos (2018) highlight that mathematics often holds both values simultaneously, as learners pursue dual motives related to competence and credentials within the same activity system. Our findings reflect this duality: even when mentors intend to foreground disciplinary utility, their examples frequently raises progression-related concerns, indicating that the two forms of value can become intertwined in how relevance is constructed. This blended construction of relevance also offers insight into the contrast between the student mentors' intentions and first-year students' actual engagement patterns. Usage data reported by Landgärds-Tarvoll and Skarpeteig Fidje (2025) show that viewing of the videos peaked during the exam period, suggesting that the first-year students valued the resources highly, but primarily at moments when exchange-value pressures were most critical. Although the reasons for this selective engagement are not entirely clear, it points to a need for future iterations of the project to support both the conceptual, discipline-focused aspects of use value and the progression-related concerns that shape students' study behaviours. Designing resources that address these dual motives may help ensure that relevance is communicated in ways that resonate across different stages of learning.

The co-creation process highlights both the potential and limitations of mentors as bridging people. While they embrace the role, their lack of pedagogical expertise can constrain clarity, making collaboration essential. Lecturers must provide pedagogical coaching, but this support must be carefully balanced to preserve rather than override the authentic 'student voice'. Achieving this requires structured dialogue that recognises mutual expertise: mentors as experts in the student experience, and lecturers in pedagogy and mathematics. Ultimately, such collaborative practices strengthen the bridging role and enhance coherence across the economics study programme.

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Students' Difficulties when Sketching Trajectories of Differential Equation Models

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Keywords: modeling, calculus for life sciences, differential equations, covariational reasoning, phase plane trajectories.

INTRODUCTION

For more than two decades there have been calls to transform the mathematics education of biology and life science students in order to have a greater emphasis on mathematical modeling and dynamical systems (e.g., National Research Council, 2003). When analyzing and determining the long-term behavior of models arising in the life sciences, the notions of time series and phase plane trajectories play a central role. These two different representations of solutions to systems of differential equations provide complementary information about the evolution of a system. Understanding how to interpret and shift between these representations is a crucial skill to make sense of and analyze such models.

Previous research has explored students' conceptions of solutions to differential equations and the difficulties they encounter when interpreting and representing those solutions (e.g., Rasmussen, 2001). Trigueros (2000) noted that understanding the relationship between phase plane trajectories and time series "prove[s] to be a very difficult task for many students" (p. 96). More recently, Bennoun (2024) examined specific strategies used by students when sketching a time series associated with a trajectory and vice versa. Here, we extend this line of work by investigating how students reason about these shifts in representation.

THEORETICAL FRAMEWORK

When sketching a phase plane trajectory or time series of a two-dimensional model, students need to attend to how the two variables change in relation to each other. They thus need to engage in covariational reasoning, which refers to one "envision[ing] two quantities' values varying and envision[ing] them varying simultaneously" (Thompson & Carlson, 2017, p. 425). To analyze students' reasoning when sketching phase plane trajectories and time series, we use the covariational reasoning framework developed by Thompson and Carlson (2017). This framework is composed of six levels: 1) No coordination, 2) Pre-coordination of values, 3) Gross coordination of values, 4) Coordination of values, 5) Chunky continuous covariation, and 6) Smooth continuous covariation. Using this framework, our research question is: *What levels of covariational reasoning do students use when sketching the phase plane trajectory associated with the time series of a two-dimensional model?*

METHODS AND PRELIMINARY RESULTS

We conducted 10 task-based think-aloud interviews with students who had taken a calculus for life sciences course focusing on modeling. Students were given the time series of a model and asked to sketch the corresponding trajectory. To analyze the data, we conducted a thematic analysis (Braun & Clarke, 2006) using the levels of covariational reasoning as a priori codes.

By contrast to previous results that only described students reasoning at the smooth continuous and chunky continuous covariational levels (Bennoun, 2024), here we observe students at all levels of covariational reasoning. The majority of the participants reasoned between the *No coordination* and *Coordination of values* levels, which underlines that this task may be even more difficult for the students than previously thought. On the poster we will illustrate these different levels with sketches done by the students. We will also discuss the strategies used by students to build their trajectories as well as the types of difficulties they exhibited when completing this task.

This is work is supported by the National Science Foundation (NSF) grant DUE-2225258.

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Differences between integrals in thermodynamics and mathematics

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Students often struggle to connect their mathematical knowledge to other subjects, highlighting a discrepancy between how mathematics is taught and how it is utilised. This study investigates the single-variable integral within the institutions of university mathematics and thermodynamics. Using the ATD framework, we conducted a praxeological analysis of textbooks and interviews with two physics instructors and six students. The results revealed significant institutional discrepancies in techniques and technologies. Specifically, the institution of physics adopts a praxeology where dx represents an infinitesimal quantity and integration signifies summing them. This institutional relation was consistently manifested in the knowledge to be taught (textbooks), the taught knowledge (instructors) and the learnt knowledge (students).

Keywords: calculus, institutional relation, ATD, praxeology, qualitative analysis.

INTRODUCTION

In the context of university mathematics education for non-mathematics majors, it is often noted that while mathematical concepts are essential in other disciplines, students struggle to perceive the connection between their mathematics courses and their majors. One perspective suggests that this disconnection arises because mathematical concepts are taught and utilised differently in other disciplines (e.g., González-Martín, 2021). This raises the question of whether specific mathematical concepts transform when taught or applied in other disciplines. Consequently, a growing body of research has emerged in recent years analysing the discrepancies between knowledge taught in mathematics courses and knowledge taught in other disciplines. For instance, Feudel and Biehler (2022) focused on the economic interpretation of the derivative and investigated students' difficulties in bridging the formal mathematical concept with its economic application. Hochmuth and Peters (2021) analysed student solutions in electrical engineering signal theory, demonstrating that the operations and justifications for mathematical transformations differ from those in pure mathematics. González-Martín (2021) revealed distinct differences in the usage of integrals between engineering and mathematics courses through textbook analysis and instructor interviews. Notably, the latter two studies employed the notion of praxeology from the Anthropological Theory of the Didactic (ATD) (Chevallard, 1999) that has been adopted in numerous other studies (e.g., Hitier & González-Martín, 2022).

Sharing the concerns of prior studies, this study focused on the integral of single-variable functions in the context of mathematics and thermodynamics. Integral is a vital tool in physics; however, because integrals in physics are deeply contextualised with physical meaning, the interpretation and manipulation of mathematical expressions may differ significantly from mathematical contexts. Drawing on the ATD framework,

this study addresses the following research questions to clarify the institutional differences regarding the single-variable integral: 1) What are the discrepancies between the institutional praxeologies associated with the integral in the institutions of university mathematics and university thermodynamics?; 2) To what extent are these institutional discrepancies reflected in the personal relations of instructors and students towards the integral, and how do they manifest as differences in technological justifications? Thermodynamics was selected as the focus because it primarily employs single-variable integrals, a fundamental concept in first-year undergraduate calculus. While this study aligns with González-Martín (2021) in its focus on integrals, its scope extends beyond their use in textbooks and by instructors to include an examination of the individual knowledge of both instructors and students.

THEORETICAL FRAMEWORK

This study employed the ATD as its theoretical framework, following the precedent work. In ATD, any human activity can be described through the notion of praxeology, represented by the four-tuple $[T/\tau/\theta/\Theta]$, consisting of a *type of task* T, a *technique* τ to solve the task, a *technology* θ justifying the technique, and a *theory* Θ lying behind the technology (Chevallard, 2006). The pairs $[T/\tau]$ and $[\theta/\Theta]$ are referred to as the *praxis* block and the *logos* block, respectively. A praxeology is developed and situated within institution *I*. This study focuses on two distinct institutions: the institution of university mathematics (I_{math}) and the institution of thermodynamics (I_{phys}). Within an institution, the institutional relation $R(I, o)$ defines how an object o (in this case, the integral) should be known and handled. An individual's knowledge is described as a personal relation $R(p, o)$ of an individual p to an object o. These relations are shaped by the process of *didactic transposition* (Bosch & Gascón, 2006), which describes how knowledge is adapted and transformed as it moves from scholarly institutions to educational settings (e.g., from *knowledge to be taught* in textbooks to *taught knowledge* by instructors and finally to *learnt knowledge* by students).

METHODOLOGY

This study adopts a qualitative approach based on the levels of didactic transposition (Bosch & Gascón, 2006) to investigate how the integral is transformed between mathematics and thermodynamics. Prior to data collection, we established an initial Reference Epistemological Model (REM) of the integral, which contrasts the formal mathematical justifications of the integral (e.g., Riemann sums) in I_{math} with a hypothesised physics-specific praxeology consisting of the central hypothesis: **K0** (In I_{phys}, the integral $\int_a^b f(x)dx$ is used for summing up all infinitesimal changes represented as $f(x)dx$ from $x = a$ to $x = b$). The research was then conducted in two phases. In Phase 1, we performed a praxeological analysis of the textbooks to test and refine this initial REM as an adequate description of the *knowledge to be taught* in both I_{math} and I_{phys}. In Phase 2, the refined REM served as a framework for semi-structured interviews to investigate the *taught knowledge* of instructors and the *learnt knowledge* of students.

Textbook analysis.

To characterise the institutional relation $R(I_{math}, o)$ and $R(I_{phys}, o)$ to the integral, we conducted a praxeological analysis of 11 textbooks: (B_1) one university physics (Koide, 2023), (B_2) one calculus (Kazumi et al., 2016), (B_3) eight Japanese high school mathematics (e.g., Kawanaka et al., 2023; Tsuboi et al., 2022), and (B_4) a reference-style physics mathematics booklet for physics class (Asano, 2008). The first two are textbooks used in thermodynamics and calculus courses at the second author's university. High school textbooks were included as they represent the prerequisite institutional knowledge for university calculus in the Japanese context. Following the methodology of Bosch, Fonseca, and Gascón (2004), we identified representative task types (T) involving the integration of single-variable functions and reconstructed their associated techniques (τ) and technologies (θ) to identify institutional discrepancies.

Semi-structured interviews.

To examine the *taught knowledge* and the *learnt knowledge*, semi-structured interviews were conducted with two physics instructors (P_1 and P_2) and six students in non-mathematics majors (S_1–S_6). The interviews used a task-based format to explore their personal relations $R(p, o)$ to the integral and to test the hypothesis verified in the textbook analysis. All interviews were transcribed and then qualitatively analysed through the lens of praxeology. The first author served as the interviewer.

RESULTS1: TEXTBOOK ANALYSIS

Mathematics textbooks (B_2 and B_3) at both the university and high school level included the following type of problems as standard tasks, sharing a common praxis block.

T_M: Find the area between the given two curves $y = f(x)$ and $y = g(x)$ over a given interval $[a, b]$ on the xy-plane.

τ_M: Divide the interval $[a, b]$ into $D_i = [t_i, t_{i+1}]$ (a=$t_0 < t_1 < \cdots < t_n = b$) such that $f(x) \geq g(x)$ or $f(x) \leq g(x)$ over each D_i. Formulate the integral $\int_{t_i}^{t_{i+1}}\{f(x) - g(x)\}dx$ if $f(x) \geq g(x)$ over D_i, and otherwise, $\int_{t_i}^{t_{i+1}}\{g(x) - f(x)\}dx$. Then, calculate each integral by calculating the difference in an antiderivative and sum them up.

The logos block varied depending on the high school grade level or university level. The technology part consisted of a combination of some of the following:

θ_{M1}: A definite integral is a difference in the value of an antiderivative in the given interval.

θ_{M2}: For a non-negative continuous function $f(x)$ over $[a, b]$ and the area $S(t)$ of the region between the curve $y = f(x)$ and the x-axis over $[a, t]$, $\frac{d}{dt}S(t) = f(t)$ holds.

θ_{M3}: For a non-negative continuous function $f(x)$ over $[a, b]$, $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{b-a}{n}f\left(a + (b-a)\frac{k}{n}\right) = \int_a^b f(x)dx$. The left-hand side can be viewed as the area or volume of a certain region or the length of a certain curve.

θ_{M4}: The definite integral is the limit of the Riemann sum, which can be viewed as the area or volume of a certain region or the length of a certain curve.

θ_{M1} corresponds to the second part of the Fundamental Theorem of Calculus (FTC). It was shared among all textbooks. In the textbook at the university level (B_2), the technology consisted of θ_{M1} and θ_{M4}. In almost all high school textbooks for grade three in B_3 (e.g., Kawanaka et al., 2023), the technology consisted of θ_{M1} and θ_{M3}. In the other textbooks in B_3 (e.g., Tsuboi et al., 2022), the technology consisted of θ_{M1} and θ_{M2}.

Regarding the textbooks for physics courses (B_1 and B_4), we identified the following technology, which was not found in mathematics textbooks.

θ_P: An integral is the accumulation of infinitesimal changes, by which the total change is obtained.

In B_1, the following task is a typical type of task using integrals in the thermodynamics sections.

T_P: Find the total change in a certain physical quantity related to thermodynamics during a state change.

τ_P: Derive the equation regarding the relation between two infinitesimal changes of the form $d\blacksquare$. Formulate integrals of both sides, and then, after some transforming if needed, find the integral by calculating the difference in an antiderivative.

The technique τ_P for the task T_P is divided into two steps: first, the integral expression is formulated, and then it is calculated. The first step is based on the technology θ_P, whereas the second step is based on θ_{M1}. The technology θ_P was also found in the following sample problem and its sample solution in B_4.

T_C: Find the area of a circle with a radius a using integration.

τ_C: Express the area of a part of the circle over the interval $[t, t+dt]$ as in the form $h(t)dt$ by using an appropriate function $h(t)$, where dt is an infinitesimal change. Then, formulate the integral $\int_a^b h(t)dt$ and calculate it by calculating the difference in the value of an antiderivative.

The task T_C is a mathematical task. However, the steps in the technique τ_C and the technology $(\theta_P; \theta_{M1})$ were the same as the task T_P, and differ from those in mathematics textbooks.

The first half of τ_P and τ_C, as well as the technology θ_P, are considered knowledge specific to physics course textbooks. Textbooks for physics courses (B_1, B_4) first establish the relationship between infinitesimal changes and then use the integral symbol for summing up the infinitesimal changes. In contrast, mathematics textbooks (B_2 and B_3) directly find the integrand and set up the integral expression. Our analysis suggests that while mathematics textbooks (I_{math}) rely on the Fundamental Theorem of Calculus as a technology, the physics institution (I_{phys}) prioritises K0 in our REM to

provide an intuitive and physical justification for the integral process. Moreover, K0 is divided into two steps: first, identify the infinitesimal changes in question; second, sum them up. Based on this observation, we refined and reformulated the hypothesis K0 in our REM as a combination of the following two hypotheses:

K1: The symbol of the form $d\blacksquare$ is a quantity representing an infinitesimal change in ■.
K2: The symbol of integral $\int_{\blacksquare}^{\blacksquare}$ is the symbol for summing up all infinitesimal changes $d\blacksquare$.

RESULTS2: SEMI-STRUCTURED INTERVIEWS

Semi-structured interviews were conducted to examine whether K1 and K2 are found in the personal relation of physics instructors and students.

Interview setting.

The interview was designed so that participants would solve the following thermodynamics problem and then answer a series of questions.

Problem. When the volume of an ideal gas with a mass of m grams is changed while maintaining constant pressure, its temperature changes from T_1 (state A) to T_2 (state B). Given that c_p is the gas's specific heat at constant pressure, find the increase in the entropy S_{AB} between $A \rightarrow B$.

The anticipated solution to this problem is as follows.

Solution. The infinitesimal change in entropy, dS, is expressed as $dS = \frac{d'Q}{T}$ using the system's temperature T and the heat added, $d'Q$. Since S_{AB} is obtained by integrating both sides of this equation from A to B and using the formula $d'Q = mc_p dT$, we have

$$S_{AB} = \int_A^B dS = \int_A^B \frac{d'Q}{T} = mc_p \int_{T_1}^{T_2} \frac{dT}{T} = mc_p \log \frac{T_2}{T_1}.$$

Note that $d'Q$ appeared in the solution is a special notation in physics. Since the heat added is not a change in heat, it is denoted by using d' instead of d.

Question items consisted of 11 questions for instructors and 9 questions for students, including the following initial question Q0:

Q0 (for instructors): Explain this problem as if you were explaining it to students. Assume that the students are engineering students in their first or second year.

Q0 (for students): Solve this problem and explain your solution.

The questions from Q1 to Q7 were common for instructors and students. In this article, we focus on the following four questions (Q1-Q4).

Q1: For the entropy, how did you derive the expression $S_{AB} = \int_A^B \frac{d'Q}{T}$ representing the finite changes from the equation $dS = \frac{d'Q}{T}$ representing the infinitesimal change?

Q2: Did you use $\int_A^B$ as the symbol for summing up all infinitesimal changes dS in $A \rightarrow B$?

Q3: Regarding Q2, does it differ from the general way used in mathematics, that is, attaching $\int$ and dt to $f(t)$ formally, by viewing it as the symbol for finding the change in an antiderivative of a function f determined by a variable t?

Q4: Do you perceive symbols representing infinitesimal changes as if they were a single number or quantity?

The questions Q1-Q4 were designed to investigate the interviewees' personal relations $R(p, o)$ to the integral, with a specific focus on whether their relations align with the hypotheses (K1 and K2) established in our REM, which were refined through the textbook analysis. These questions aimed to examine the following four aspects of their technological justifications (θ).

R1: To what extent does the interviewee justify the integral as a process of accumulating infinitesimal changes to obtain a total quantity?

R2: Does the interviewee's personal relation to the integral symbol align with K2, where the symbol functions as an operational tool for the accumulation of infinitesimal changes?

R3: Does the interviewee explicitly distinguish between the institutional relations of I_{math} and I_{phys} regarding the justification of the integral?

R4: Does the interviewee's personal relation to the expressions such as dx or dS align with K1, treating them as infinitesimal quantities rather than formal integrators?

The instructors were given the problem prior to the interview. The students were generally given the problem during the interview; however, only one student (S_1) requested to receive it beforehand due to a lack of confidence in solving it. Additionally, students were provided with copies of textbook pages containing necessary knowledge and were allowed to use these as a reference while solving the problem.

Results of the interviews.

The results regarding the four aspects R1-R4 are summarised in Table 1.

	P_1	P_2	S_1	S_2	S_3	S_4	S_5	S_6
R1	yes	yes	yes	yes	–	yes	yes	yes
R2	yes	yes	yes	–	yes	yes	yes	yes
R3	yes	yes	no	–	no	yes	yes	–
R4	yes	yes	yes	–	yes	yes	yes	yes

Table 1: The responses regarding R1-R4 (the symbol '–' means 'indeterminate').

The following details the interviews regarding the personal relations associated with K2 (R2/Q2), K1 (R4/Q4), and the perceived differences between institutional relations (R3/Q3). In the following transcripts, the interviewer is denoted as 'I'. The transcripts were translated from Japanese by the authors.

Regarding Q2, all interviewees except for S_2 expressed a personal relation that aligns with the technology of K2. The response of P_2 most explicitly demonstrated the justification of the integral as a summation tool, consistent with the hypothesised institutional relation in I_{phys}.

P_2: [⋯] That's right. I think the essence of integral is as addition.

I: So, you use the integral symbol as a sign to sum up all those infinitesimal changes from A to B?

P_2: Yes.

P_2 drew the image of formulating the integral expression for solving the problem (Figure 1). Figure 1 also indicates that P_2's justification of the integral is based on K2.

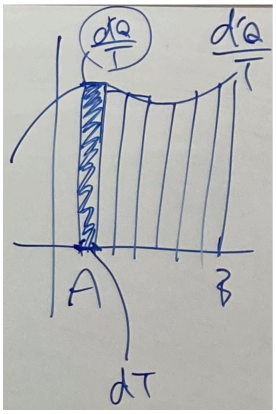

Figure 1: The image drawn by P_2 in the solution.

P_1 stated regarding the integral expression that "[...] by summing up such infinitesimal quantities [...]". The students, except for S_2, simply answered "yes" to Q2. These results indicate that most interviewees share a personal relation to the integral that aligns with the technology of K2, where integration is justified as an operational tool for summing up infinitesimal changes.

Regarding Q4, all interviewees except for S_2 expressed a personal relation that aligns with the technology of K1. For example, P_2 stated as follows in response to Q4.

P_2: I explain dS in that way during class too. […]

P_1 stressed the physical meaning of each symbol in response to Q4.

P_1: Well, yes. [...] Of course, since it's physics, each symbol must inevitably carry a physical meaning, […].

The students, except for S_2, simply answered "yes" or "well, ahh…yes" to Q4. These results indicate that most interviewees share a personal relation to the symbol of the form $d\blacksquare$ that aligns with the technology of K1, where $d\blacksquare$ is justified as a quantity representing infinitesimal changes.

Responses to Q3 regarding the differences between institutional relations were divided. Two instructors and two students (S_4, S_5) stated that their personal relations to integrals

in physics differ from those in mathematics. S_5, who recognised the discrepancy between the two institutional relations, described it as follows:

S_5: No, it's … completely different. A different thing.

I: By what kind of image do you use it in maths?

S_5: In maths, […] I just use it as an operator.

S_4 also answered that they "feel like it's more [...] formal way to use" in mathematics.

On the other hand, two students (S_1, S_3) stated that they maintain a consistent personal relation across both institutions. Below is an excerpt from the interview with S_1 regarding Q3.

S_1: In high school physics, I learnt it this way too. I learnt this before finding the definition of calculus in maths, so I might have been solving maths problems more with the natural sense of it being the sum of changes.

I: In mathematics too, you saw $\int f(x)dx$ as the sum of $f(x)dx$?

S_1: I saw it as the sum.

S_3 also showed the same response.

S_3: I add up the infinitesimal quantity $f(x)dx$ from A to B. In maths, I might use it that way too. [...] In maths, I think it as the sum of infinitesimals.

These results indicate that S_1 and S_3 also interpret the integral symbol in mathematics as the sum of $f(x)dx$, which aligns with K1 and K2.

DISCUSSION

This study utilised the ATD framework to investigate the praxeological discrepancies in the knowledge of the single-variable integral between the institutions of university mathematics (I_{math}) and thermodynamics in university physics (I_{phys}).

Discrepancies in institutional praxeologies.

The textbook analysis revealed that the *techniques* used to formulate integrals differ significantly between the two disciplines. In I_{math}, the praxeology is centred on the operation of finding an appropriate integrand function $f(x)$ and placing it between the symbols $\int$ and dx. In this context, the *technology* (Riemann sums) theoretically justifies the technique, yet the *praxis* (identifying the integrand and applying the Fundamental Theorem of Calculus) can often be performed formally without explicitly engaging with the concept of 'summation'. Consequently, in I_{math}, the 'summation of infinitesimals' perspective tends to remain implicit within the logos block. In contrast, I_{phys} explicitly adopts the praxeology where $d\blacksquare$ represents a 'infinitesimal change of a physical quantity' (K1) and $\int$ as the 'symbol denoting the summation of the infinitesimal changes' (K2). Here the technique is inseparable from its technological justification; formulation and calculation are driven by the physical meaning.

Diversity of personal relations.

The discrepancy in institutional praxeologies identified in the textbook analysis implies that students must navigate a gap between I_{math} and I_{phys} in their study of thermodynamics. The interview results (Q2, Q4) confirmed that the institutional relation $R(I_{phys}, o)$ to the integral—interpreting the integral through K1 and K2—is deeply internalised as a personal relation in the context of thermodynamics by physics instructors (P_1, P_2) and most students (S_1, S_3, S_4, S_5, and S_6). However, the divergent responses to Q3 highlight how students struggle to manage the conflicting institutional relations between I_{math} and I_{phys}. Students S_4 and S_5, along with the instructors, recognised the discrepancy. They manage it by switching their personal relations depending on the context. While this appears to be a successful strategy at first glance, it may also imply that their personal relations split into two disconnected ones. Conversely, students S_1 and S_3 applied the institutional relation of I_{phys} even within mathematical contexts. It may imply that the personal relation of S_4 and S_5 might come from the integration of $R(I_{phys}, o)$ and $R(I_{math}, o)$. Further details of these two types of personal relations should be investigated in future research.

Limitations of this study and future research.

This study has not addressed which type of students, students who apply the institutional relation of I_{phys} even within mathematical contexts and students who switch the personal relation depending on context, is successful in physics class. Elucidating how the difference in learning outcomes between these two types of students is and what impact (impediments or facilitation) this difference has on learning mathematics, physics, or other applied fields is the next crucial task.

This study has not addressed how the personal relation of students like S_1, S_3 arises. S_1 attributed this to having learnt calculus (integration as the sum of infinitesimal changes) through physics in high school prior to formal mathematics instruction. It may imply that the personal relation formed in one institution can strongly influence the requirement of another, but it should be examined in future research.

During the interviews, some students exhibited confusion in handling integrals such as $\int_A^B dS$ where neither A nor B are values of the variable S in dS. It remains to be clarified to what extent such technical difficulties are rooted in the institutional discrepancies identified in this study.

NOTES

1. This study was conducted with the approval of the Ethics Review Committee of the Graduate School of Science, Graduate School of Agriculture, and Graduate School of Veterinary Science, Osaka Metropolitan University. This study was supported by JSPS KAKENHI Grant Number JP23K22337.

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Reconstructing the Mathematical Organizations of *Modeling Life*

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This paper presents a reconstruction of the mathematical organizations (MOs) embedded in Modeling Life*, a curriculum that centres modelling as introductory mathematics for life science majors. We propose five MOs and compare them to those of standard Calculus. We observed a shift in the kinds of questions posed, the techniques used, and the basis for justifying mathematical work. The reconstruction provides a model for examining how such MOs are taken up in institutional contexts.*

Keywords: dynamical systems, modelling, ATD, case study

INTRODUCTION

Undergraduate mathematics programmes regulate, for each population of students, the foundational knowledge needed for entry to a discipline. For life science majors, this function has long been fulfilled by Calculus (or its lighter version, Biocalculus). Implementations of these courses reflect an epistemological order grounded in Analysis. As Stillman et al. (2020) argue, the prevailing tendency to equate "doing mathematics" with the practices of pure mathematicians constrains both how mathematics is taught and what mathematics is taught. At the current moment, the mathematical practices, and therefore needs, of the life sciences have diverged considerably from what is offered by Calculus. According to the Howard Hughes Medical Institute (HHMI) and the Association of American Medical Colleges (AAMC) the key mathematical practices desired of life scientists are to: quantify and interpret changes in dynamical systems, explain homeostasis in terms of positive and negative feedback, explain how feedback mechanisms lead to damped oscillations in glucose levels, and to use the principles of feedback control to explain how homeostatic and reproductive systems maintain an internal environment, and how disease arises from perturbations to homeostasis (AAMC-HHMI Committee, 2009).

In many mathematics programmes, the study of dynamical systems would follow multiple courses of Calculus, precluding its function as a *foundation* for life sciences. In response to this disparity, Garfinkel et al. (2017) authored *Modeling Life*. The authors selected and adapted their disciplinary know-how of modelling and applied mathematics, as used in quantitative biology, to be suitable as an introductory course for life scientists. What is at stake here is that the knowledge being transformed is not drawn from an existing domain within a university mathematics programme, but from active scholarly practice in quantitative biology, where biological phenomena and dynamical systems organize the questions posed and the techniques used to address them. The knowledge is not added to an implementation of Calculus; it supplants the standard Calculus (or Biocalculus) courses typically offered to life science majors.

THEORETICAL FRAMEWORK

Within the Anthropological Theory of the Didactic (ATD) knowledge is conceived as a system of practices, rather than a body of discrete facts. Because bodies of knowledge are constituted through their use in the disciplines that create them (Chevallard, 2015), a primary objective of ATD is to elaborate how scholarly knowledge (as practiced by disciplinarians) is transformed into knowledge-to-be-taught (in a classroom). This transformation takes place through *external didactic transposition*, whereby scholarly knowledge is "selected, adapted, and declared as 'matter to be taught', resulting in documents such as syllabi, textbooks, and other resources and regulations for teaching" (Bosch & Winsløw, 2020). Unlike prior ATD analyses that examined transpositions of historically remote mathematical knowledge, here the source field and its instructional form are developing concurrently.

The core unit in a system of knowledge is a *praxeology*, which models mathematical practices by connecting what is done (task), to how it is down (technique), to justifications for why doing it that way resolves the problem posed (technology, theory). A theoretical nucleus identifies the common mathematical idea that brings multiple praxeologies together within a *mathematical organization* (MO), giving rise to mathematical themes or domains (Bosch & Gascón, 2006). Multiple MOs operating together within a curriculum constitute a *reference epistemological model*. This model functions as a reconstruction of the mathematical knowledge-to-be-taught for the researcher to consider (Barbé et al., 2005, p. 56; Bosch & Gascón, 2006) rather than a replica of the scholarly knowledge.

In this paper, we treat the textbook and accompanying computational lab materials as artifacts of external didactic transposition. We address the question: *What mathematical organizations emerge from this transposition and how do they compare to those of standard Calculus*? We approach the question through synthesizing task families and discourses uniting them, as they are presented by these artifacts.

MATHEMATICAL ORGANIZATIONS OF STANDARD CALCULUS

The standard university Calculus, as represented in mainstream international textbooks has been characterized as a *stable mathematical corpus* whose content reflects an epistemological orientation that has been remarkably resistant to change (Barbé et al., 2005; Bosch, 2014; Bosch & Gascón, 2006; Winsløw et al., 2014). Its theoretical nucleus comprises limits and the fundamental theorem of calculus (FTC), consequently privileging task families organized around applying symbolic procedures justified by classical Real Analysis. Typical task families include the algebra of limits, computing derivatives, evaluating definite and indefinite integrals, optimizing quantities, and calculating areas or volumes. The associated techniques are the algorithms that carry out these tasks, such as rules for differentiation, substitution and partial-fraction methods for integration, and rules for computing finite limits and limits at infinity. The associated discourse justifies these procedures through geometric and rate-accumulation metaphors (derivative as the slope of a tangent line; integral as area under

a curve; the FTC as a link between the two) and appeals to Real Analysis as the broader theoretical undergirding (Winsløw, 2015). This framework uses notions of limit, continuity, and differentiability to formalize the geometric intuitions, although texts rarely provide proofs for these intuitions. Together, the task families, techniques, and their associated discourses define stable MOs in which understanding is expressed through an ability to apply procedures within this theoretical nucleus to a limited corpus of functions of one variable. This organization exemplifies the *visiting monuments* paradigm (Chevallard, 2012), with applications appearing only after the formal and procedural core are established for each task type. When real-world problems are formulated from physics, biology, or economics, they are simplified to fit existing task types such as finding rates, areas, or extrema. In this configuration, modelling has a demonstrative function: it shows where the established praxeologies of Calculus readily apply, rather than generating insights. This arrangement defines the epistemic boundary of Calculus and sets the point of departure for curricula where modelling supplies both the tasks and their justification.

MATHEMATICAL ORGANIZATIONS OF *MODELING LIFE*

The authors of *Modeling Life* describe it as a study of nonlinear dynamical systems featuring feedback loops as the mathematical explanation for biological phenomena. The text includes narrative exposition, worked biological and mathematical examples, and exercises. Some exercises focus on conceptual development while others are computationally oriented, requiring computational toolkits to simulate dynamical systems. The writing is expository, alternating between authorial modelling episodes, in which mathematics is introduced and developed to meet biological needs. The student tasks focus on constructing, simulating, and interpreting models of biological systems. We attended to the kinds of tasks the textbook proposes, the techniques it endorses, and the discourses that justify them (Wijayanti & Winsløw, 2017).

We operated at a meso-scale level of reconstruction, identifying MOs through recurrent task–technique–discourse configurations, rather than enumerating point praxeologies (Bosch & Gascón, 2014). We constructed two corpora through an AI-assisted extraction process using Claude.ai. The student corpus was extracted through retrieval of passages beginning with “Exercise” or “Further Exercises.” The authorial corpus was constructed by retrieving passages that either enacted a task-technique relation or provided a justification. Through a combination of prompting techniques (few-shot, role-based with ATD constructs, chaining, and meta) and iterative manual refinement, we tagged the mathematical object in focus for each passage (e.g., system behaviour, state spaces, plots), and the purpose, rationale, or logic of the passage (e.g. since $X \rightarrow$ property of the system; formulate modelling question). Next, we iteratively identified and refined task families and their associated discourses. Candidate groupings were generated through prompting and then examined manually, comparing our emerging candidate groupings to the textbook. This generated increasingly coherent task families (e.g., constructing models, stability analysis) together with the discourses that justify them. We retained groupings as candidate MOs when they met three conditions: (i) the

same task family appeared across both corpora (ii) the associated discourse justified the techniques in a consistent way across those contexts, and (iii) multiple task families could be interpreted through a shared theoretical idea that explained why the techniques were appropriate and what qualified as adequate. Ohterwise, groupings were merged or treated as dependent. The result was 9 candidate MOs with distinct but uneven praxeological configurations. Three appeared subordinate because we could not identify independent theoretical nuclei and one was externally motivated by noospheric conditions. In this way, we consolidated five MOs (Table 1).

MOs	*Praxis* Blocks	*Logos* Blocks
Modelling Change in Systems (MO-A)	Formulating first order ordinary differential equations (ODEs) as models from biological narratives; identifying variables, parameters, and interaction types; specifying parameter roles and initial conditions; creating stock-flow diagrams to support symbolic representations.	Theoretical nucleus situates modelling within logic of dynamical systems: a model is justified when its formulation expresses mechanisms of regulation, accumulation, or depletion observed in biological processes. Explains why rates provide sufficient representation of biological change. Links signs and magnitudes of rates to positive and negative feedback. Interprets derivative as an quantitative description of how one quantity changes with another, over time.
Stability Analysis & Equilibria (MO-B)	Determining equilibrium points; characterizing behaviour near these points; setting derivatives to zero; creating Jacobian matrices; using eigenvalues to classify stability.	Theoretical nucleus is *local linearization*: near an equilibrium, nonlinear behaviour can be approximated by a linear system whose eigen structure predicts return, divergence, or oscillation. Connects computations to qualitative reasoning about system response. Interprets stability through signs and magnitudes of derivatives or eigenvalues, which describe how perturbations to a system evolve.
Global & Qualitative Dynamics (MO-C)	Characterizing long term tendencies (oscillations, switches, sensitivity to parameter changes); numerical integration, phase-plane and nullcline	Theoretical nucleus draws on nonlinear dynamical systems and bifurcation theory: qualitative transitions (e.g., onset or disappearance of oscillations, sources, sinks) are governed by parameter values.

	analysis; parameter sweeps; comparing trajectories	Explains system behaviour in terms of patterns shown in trajectories and phase portraits. Parameter variation is a means for eliciting qualitative transformations in system behaviour.
Computational Modelling (MO-D)	Implementing and modifying simulations; designing computational experiments to examine model assumptions; use embedded python code; systematically vary parameters; produce time-series and phase-plane plots.	Theoretical nucleus presents *numerical approximation* as an alternative to solving differential equations. Technological discourse positions computations as *legitimate*: for carrying out mathematical analyses and as extension to by-hand analysis when examining dependence on parameters and functional forms. Euler's method produces trajectories in state space. Simulation permits inspection of model behaviour where closed-form solutions are unavailable.
Interpreting & Validating (MO-E)	Comparing model predictions with expectations from biology or data; revising models when outcomes disagree; articulating implications of outcomes; checking sensitivity and plausibility; adjusting parameters or functional forms to increase adequacy of models; summarizing outcomes verbally or graphically	Theoretical nucleus sets expectation that model coherence is an epistemic criterion. Links comparison, revision, and explanation as interdependent forms of mathematical reasoning. Explains interpretation as evaluating correspondence between simulated trajectories and expected quantitative relations among variables. Model adequacy is justified through coherence between quantitative results and known mechanisms. Model revisions improve coherence.

Table 1 Summary of reconstructed MOs for *Modeling Life*

MO-A: Modelling Change in Systems comprises candidate groupings centred on *modelling feedback and homeostasis* and *modelling population growth*. The former appeared across many biological narratives in the authorial and student corpora, while the latter seemed to be a special contextualized set of biological narratives that served as entry points to reasoning about feedback and the idea that models can grow in

complexity. Since they shared a common discourse relating rates, feedback, and system behaviour, we grouped them together. In the case of MO-B: Stability Analysis & Equilibria, an interconnectedness was observed between *equilibrium analysis* and *local linearization*. The discourse surrounding the latter provides the theoretical nucleus for classifying equilibrium points and interpreting stability; it further supplies a rationale for the task families constituting equilibrium analysis (e.g., setting derivatives to zero, creating Jacobian matrices, interpreting eigenvalues). MO-C: Qualitative & Global Dynamics combines candidate groupings about *qualitative dynamics* and *bifurcations*. The former focuses on characterizing long-term behaviour of systems, including oscillations, through examining trajectories and phase-planes. Across both corpora, the task families in the latter developed techniques to locate points where equilibria or periodic orbits appeared, disappeared, or changed character as parameters varied. The technological discourse treated parameter variation as a mechanism that alters the qualitative behaviour of the system. Together, the tasks and techniques were justified by nonlinear system dynamics and bifurcation theory.

MO-D: Computational Modelling combines candidate groupings about *simulation & sensitivity* and *computational implementation*. In the authorial corpus, the text introduced Euler's method and showed how to implement it in Python (or SageMath) to generate trajectories and plots. In the student corpus, the accompanying exercises and labs asked students to modify the provided code to run simulations, vary parameters, and interpret the resulting graphs. Although these activities initially appeared distinct (carrying out computations vs. interpreting the outputs), they enacted the same *praxis*: using numerical approximation to produce graphical representations that show how model behaviour depends on parameter values. The corresponding *logos* treated computation *as* mathematical analysis, and therefore a legitimate technique for producing evidence about system behaviour, rather than as an auxiliary tool or a means for obtaining equivalent representations. Since both candidate groupings drew on the same theoretical nucleus (numerical approximations of differential equations), we merged them into a single MO. Finally, MO-E: Interpretation and Validation comprises the candidate groupings *interpretation of biological meanings* and *model comparison & validation*. The former contains task families concerned with explaining model results in terms of their biological plausibility while the latter contains task families focused on assessing the adequacy and credibility of the models, typically through comparing alternatives or comparing models relative to how well they capture observed (expected) behaviours. Clearly, these activities are very closely linked in terms of their goals and both corpora position this co-occurrence as mathematical work, rather than commentary outside (or after) the mathematics. The two candidate groupings contrast in terms of the techniques we identified for carrying out the tasks such as mapping variables back to biological quantities and creating narratives for biological implications versus computing residuals or (qualitative) fits, evaluating competing formulations, and other tests of robustness. Ultimately, we were able to clarify shared *logos* blocks that explains these activities as part of the mathematical work testing the internal coherence of a model and its alignment with the phenomena

it is meant to represent, activities that are characteristic of modelling. We note that in MO-E, the theoretical nucleus differs in character from the other MOs as it concerns the criteria for the evaluation of models rather than their analysis.

We comment on distinguishing MO-A from MO-E, as both contain task families referring to *interpretation* and *validation* of mathematical models. Within the modelling competency literature (e.g., Blum & Leiß, 2007), these are separated as phases of a modelling cycle because they capture differing activities: *interpretation* gives real-world meaning to mathematical results whereas *validation* evaluates whether the model's behaviour or predictions are adequate. That distinction, however, is an artifact of framing modelling as a sequence of mental acts. From within the ATD, the activities that modelling cycles describe as successive phases emerge here as intertwined. In the *Modeling Life* corpora, interpretation and validation occur together in the authorial expositions and student exercises. The same task statement that might call for interpreting the results biologically also expects the agent to weigh whether those results make sense or fit the data. Such co-occurrences indicate that the tasks and techniques belong to a single or closely related organization, even when the techniques differ. Interpreting may involve mapping equations to biological mechanisms or describing trajectories in biological terms; validating may involve comparing outputs or carrying out revisions like adjusting parameters or reformulating equations. What unites these tasks is a common technological discourse that establishes *monitoring coherence between the mathematics and biology* as mathematical work. This discourse is embedded within, but does not exhaust, the broader *logos*. In MO-A, interpretation and validation concern *model construction*: ensuring that each mathematical term and relation adequately represent the biological mechanisms at stake. For example, adding aP to represent reproduction or subtracting bP to represent mortality. In MO-E, they shift focus to *model adequacy*: examining whether (or the extent to which) the completed model reproduces expected dynamics, such as stability, oscillation, or growth patterns, and whether parameter adjustments improve its fit to data or known mechanisms. This aligns with Czocher's (2018) findings that validation occurs throughout modelling as a form of sense-making about whether the mathematics continues to represent the system adequately.

Finally, the entirety of Chapter 2 is dedicated to MOs retained from Calculus: limits, differentiation, integration. These form a recognizable, partial, and self-contained set of task families and techniques. The corpora demonstrate computing rates of change and accumulated quantities, interpreting slopes, and approximating areas from rate information. The corresponding *logos* are underdeveloped. Explanations do appeal to geometric and biological intuitions of rate and accumulation; yet, the limit definition of derivative appears once without subsequent use or elaboration. The corresponding techniques (e.g., derivative rules) do not appear outside of this chapter. The remainder of the corpora rely on numerical and graphical techniques for estimating and interpreting rates of change and "integration" as numerical analysis and results in trajectories (rather than values or symbolic functions). The algebraic praxeologies of

differentiation and integration remain isolated from the modelling, simulation, and stability analysis that organize the course. We therefore identify Candidate MO-F: Differentiation and Integration as a distinct candidate grouping. Its presence is explained by noospheric constraints, specifically, the author's account of Springer's condition for publication: that elements of the standard Calculus persist in its publications aimed at introductory mathematics. The chapter maintains the institutional appearance of Calculus even as the course's epistemological organization shifted toward modelling and dynamical systems.

DISCUSSION AND IMPLICATIONS

We proposed a reconstruction of the MOs embedded in the *Modeling Life* textbook as a record of external didactic transposition of quantitative biology into introductory university mathematics. The reconstructed MOs differ from those of standard Calculus in the kinds of tasks they propose, the techniques they endorse, and the discourses that justify them. Tasks begin from questions about how a real-world system behaves ("When should we remove predators from a system to increase prey?"), how it sustains equilibrium or is altered when parameters vary ("Why does the same feedback mechanism that maintains homeostasis under normal conditions lead to oscillations when certain conditions change?"). These questions are not directed towards obtaining values or closed forms, but towards mathematically explaining biological phenomena. They generate task families that entwine formulation, computation, and interpretation because the theoretical justifications for the techniques arise from what those techniques allow one to observe or infer. As evidenced by MO-D, simulation and numerical approximation constitute the means through which the behaviour of a system is analysed and become the mathematical techniques for developing, testing, and informing revision of models. In this configuration, computation does more than overcome the limitations of paper-and-pencil techniques or provide optional alternatives; it is treated *as* mathematical analysis because generating and examining simulated scenarios are recognized as mathematical techniques for investigating the assumptions and consequences of models. In this sense, mathematical knowledge is produced through inquiry into systems rather than applied to predefined problems, aligning with Chevallard's (2012) paradigm of *questioning the world*.

Modeling Life's MOs displace the role of Real Analysis in justifying mathematical work. By focusing on *coherence* rather than *deduction* as the principal legitimizing criterion, *Modeling Life* reconfigures the content of introductory mathematical knowledge for life scientists. Prior ATD research argues that what *can* be taught or what *is* taught is constrained by higher levels of co-determinacy, such that the persistence of Calculus is tied to the stability of the praxeologies that sustain it (Barbé et al., 2005; Bosch, 2014; Bosch & Gascón, 2006; Chevallard, 2002). For these reasons, MOs rooted in modelling have tended to remain peripheral even when recognized as pedagogically valuable (e.g., Barquero et al., 2013; Barquero et al., 2015; Ruiz-Higueras & Garcia Garcia, 2011). Thus, implementing these MOs would place demands on the institutional ecology across multiple levels of codeterminacy: faculty

expertise to the didactic level where teachers' repertoires are formed; computational infrastructure to the material-institutional level that supports legitimate practices; and curricular structures to the top-institutional level that regulates which mathematical organizations may persist. At the same time, the inclusion of Calculus praxeologies in the textbook reflects noospheric efforts to preserve established curricular expectations by regulating their persistence within introductory mathematics, even at the moment of textualization of the didactic transposition. As a result, epistemologically incompatible organizations coexist (at least in the text) rather than replacing one another. Because the Calculus MOs are preserved (even in an atrophied form), we would expect institutions adopting the curriculum to use their presence to justify their inclusion in an implementation.

The present case differs from much prior work in ATD in that the transposed knowledge is not introduced as a supplement to existing Calculus but constitutes the basis of an introductory course that supplants Calculus altogether. It is also a case where the scholarly knowledge was transposed prior to achieving stability as a widely recognized domain within university mathematics. Thus, the knowledge-to-be-taught remains subject to ongoing elaboration in the noosphere, creating conditions under which external and internal transpositions are closely coupled. Our reconstruction provides a basis for examining how these MOs are taken up, adapted, and stabilized in specific implementations, and for studying external didactic transposition in cases where the knowledge-to-be-taught is not yet institutionally stabilized.

ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation under Grant No. 2417149

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Linking deductive reasoning and concept images Rolle's Theorem with undergraduate computer science engineering

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Keywords: Mathematical focus calculus, concept image, reasoning, intuition, proof

INTRODUCTION AND THEORETICAL BACKGROUND

Research shows that first-year university students struggle with mathematical concepts and with developing formal reasoning (Biehler et al., 2024; Gueudet & Thomas, 2020). The link between intuition and deductive reasoning is central in mathematics education (Fischbein, 1987, Lajos, 2023). Intuition allows students to grasp patterns, structures, and approaches directly. In Calculus, difficulties arise from the disconnection between concept image and concept definition (Tall & Vinner, 1981), as well as between symbolic manipulation and reasoning (Bressoud et al., 2016). This study analyses how students' concept images relate to their deductive reasoning in Rolle's Theorem, defined as the set of mental representations associated with the concept. We focus on analysing how students´ concept images relate to their deductive reasoning, specifically regarding deductive arguments and the framework used by the students for drawing valid conclusions from given premises. Concept image is defined as "*total cognitive structures including all mental pictures, properties and processes associate of the concept*" (Tall & Vinner, 1981, p. 152). We chose the Rolle's theorem because it establishes a connection between continuity and differentiability and it is a content in which less research has been carried out (Parameswaran, 2009).

RESEARCH QUESTION, METHODS & DESIGN

The research questions are: What are the concept images created by the students when working in the Rolle's theorem? What is the impact of the concept image on his or her reasoning about the theorem?

The study involved 34 Computer Science Engineering students in Calculus I. Although they have strong algebraic and programming skills, they show limited experience in formal reasoning and rely heavily on intuition. A qualitative design with participatory intervention was used. We also conducted an observation with participatory intervention, involving the design, implementation, and analysis of a didactic device. We present the results of the student evaluation questionnaire. The student questionnaire was divided into three parts: (i) concepts of functions and derivatives explored through an inquiry process. (ii) Rolle's Theorem: mental images and intuitive understanding. Students were asked to explain in their own words Rolle's theorem and to identify in different plots of graphs whether it is possible to apply the theorem. (iii) Assessing students´ understanding of the relationship between continuity and

differentiability and, exploring how metacognitive and affective factors may inhibit the effective use of their knowledge (Gómez-Chacón et al., 2021). For questionnaire analysis we use systemic networks (Bliss et al., 1983) that allow us to simultaneously look at all answers of the students. This way of presenting the data offers a global view and it highlights the complex relations between the different elements represented: conceptual image, the concept properties, logical antecedents and inferences. The results concern: (1) students' ability to state the theorem and apply it to reasoning tasks, (2) the influence of their concept images on deductive reasoning about the theorem, (3) their capacity to perceive the relationship between Rolle's Theorem and other related mathematical concepts, and (4) how metacognitive and affective factors may hinder the proper use of their knowledge and intuitions. The results of the case study are displayed. A combination of text, schematics, and diagrams is used to present and visualize the study.

Acknowledgments: Research Project PID2022-138325OB-I00, financed by MICIU/AEI.

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Teaching the Laplace transform: a mathematician's and an engineer's perspective

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Keywords: Mathematics for engineers, Laplace transform, teacher's perspective, anthropological theory of the didactic (ATD).

INTRODUCTION

Research has shown that university teachers working in engineering programmes mobilize different mathematical and didactic organizations depending on their academic trajectories. González-Martín and Hernandes-Gomes (2024) found that even when teaching the same calculus content, teachers with different backgrounds select and justify applications in distinct ways, reflecting their institutional positions. As part of a broader doctoral research project on the teaching of the Laplace transform in engineering mathematics, two exploratory interviews were conducted with a mathematician and an engineer. In this study, our research question concerns how these teachers organize the teaching of the Laplace transform in their courses, with particular attention to their forms of validation and to the discursive practices that characterize their institutional approaches. Using the Anthropological Theory of the Didactic (ATD), the analysis focuses on the mathematical and didactic praxeologies each teacher mobilizes and on the institutional conditions that influence their decisions.

THEORETHICAL FRAMEWORK AND METHODOLOGY

In ATD, a mathematical organization is described through types of tasks, techniques, the technological discourse that justifies these techniques, and the theories supporting this discourse (Chevallard, 2019). Didactic organizations arise within specific institutional conditions, including curricular constraints and program expectations. The Laplace transform provides an opportunity to examine how the same mathematical object can give rise to distinct praxeologies depending on the institutional role it serves.

Two semi-structured interviews were conducted with teachers who include the Laplace transform in their courses. Selected for teaching the same mathematical object to the same student cohort, these two instructors differ in academic background (mathematician vs. engineer) and course context (mathematics vs. electrical networks). The analysis interpreted the teachers' descriptions of how they introduce, use and validate the Laplace transform by identifying the corresponding praxeological components. Course syllabi, exercises and examination tasks were also collected. According to the course syllabi, the

Laplace transform is addressed in the second-year engineering mathematics course within a unit on differential equations, while in the electrical networks course it is introduced as a tool for analyzing electrical circuits. These documents were analyzed to identify the tasks and techniques proposed to students in each course and to provide a broader picture of how institutional conditions and constraints within the same institution shape the teaching of the Laplace transform.

RESULTS AND DISCUSSION

The two teachers construct distinct mathematical organizations of the Laplace transform. The mathematician develops the topic through definitions, existence conditions and general properties such as linearity and shifting, with proofs that form part of the technological discourse. The engineer introduces the transform through circuit and control problems and treats it as an analytical tool; statements of properties are included only when needed for computation, and proofs are omitted. These different forms of validation reflect the institutional roles of their courses. Similar contrasts are described by Castela and Romo-Vázquez (2011), who show how mathematical knowledge undergoes different forms of transposition depending on whether it is organized within a mathematics course or an engineering course. The interviews also show how each teacher adapts the topic within the conditions and constraints of their course. The mathematician excludes the explicit inversion formula because it relies on analytic tools that are not part of the students' preparation. The engineer limits theoretical detail, uses notation common in engineering practice and draws on professional experience when framing the role of the transform. In this poster we will present the outcome of this comparison, with emphasis on how the teachers' decisions result in two institutional "dialects" attached to the same object, marked by different symbolic conventions, discursive practices and forms of exposition.

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Didactic strategies for engineering mathematics: bridging the theory-application gap

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Keywords: mathematical modeling, blended learning, engineering mathematics education, integrative conceptual framework, qualitative method

INTRODUCTION

The teaching of mathematics in engineering education requires continuous updating to ensure its effectiveness in solving contemporary problems. While the scope of mathematical application varies by engineering disciplines, achieving professional competence demands a level of logical-mathematical reasoning specified by each program.

While the relationship between mathematics and engineering is well established through academic curricula, the mathematical praxeologies of their professional utility are often underexplored. This gap commonly leads students and even practicing engineers to question themselves on how mathematics will be useful in their jobs. Consequently, the research question is: What current didactic strategies are most effective for integrating educational mathematics into real-world engineering contexts?

The results presented are derived from a preliminary state-of-the-art analysis concerning applied mathematics in the professional life of engineers. This study forms part of a cotutelle doctoral investigation between France and Mexico.

METHODS

The methodology followed the phases of a systematic review of PRISMA (Moher et al, 2010). We proceeded in three phases: 1) Identification of pertinent studies, 2) removal of duplicate or ineligible records, and 3) evaluation and selection of the final sources based on predefined inclusion criteria. The search was done in scientific databases: Web of Science and Google Scholar, filtering sources from 2015 to 2025. To enhance the keyword selection, we utilized the generative AI tool, Copilot, and use a string to make the search using boolean operators

Inclusion criteria:

- IC1: Publications about pedagogical practices about mathematics implemented in engineering programs.
- IC2: Works that show innovative practices that link mathematical theory in engineering applications.
- IC3: Publications with a relevant number of citations.

RESULTS

The review affirms that mathematical modeling is one of the most critical didactical strategies that could bridge the gap between mathematical theory and real-world engineering applications. The effectiveness of this approach is maximized when it is implemented through blended methodologies, specifically Project-Based Learning (PBL), and Interdisciplinary Projects. (Chen et al, 2021; Kuo et al., 2019; MacLeod & Van der Veen, 2020).

Literature establishes a clear consensus that mathematical modeling, if well taught and implemented, provides the necessary framework to connect the theory to real-world problems. A valuable approach is realistic modeling, describe by Abassian et al. (2020), where the objective is to solve authentic, non-simplified and messy professional problems.

Modeling is also recognized as indispensable for innovation and the development of advanced algorithms within fields such computer science affirmed by the SIAM (Rüde et al., 2018). Research confirms that problem-solving methods, learned in engineering mayors, are effective for cultivating skills such as critical thinking, creativity, and peer collaboration (Szabo et al., 2020).

These skills are not only fundamental to successful engineering practice, but also highly valued in the modern professional workplace (McGunagle & Zizka, 2020). Research has shown that modeling activities have a positive impact on students, motivating them to develop their talents (Maass et al., 2019).

Future work and limitations

Integrating mathematical modeling is often limited by curriculum constraints and the significant time required for instructional design (Magana & Silva Coutinho, 2017). This ongoing doctoral research will continue to investigate mathematical praxeologies in professional engineering activities. A further objective of this research is to enhance pedagogical processes; consequently, future work proposes a technological platform to host research-based project repositories.

The poster will show a visual synthesis of the systematic review methodology, mapping thematic similarities and clusters of related literature alongside the insights of the main results.

Acknowledgments. Secretaría de Ciencia, Humanidades, Tecnología e Innovación (SECIHTI) and the French Embassy in Mexico for their support in this research.

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A taxonomy of parameter-related task types for analysing university students' modelling praxeologies

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A taxonomy of seven parameter-related task types was developed on the basis of an institutional reference praxeology for a modelling problem involving a physical system. Written solutions from 19 groups of students in science-related programmes, who worked on this problem, were analysed against this taxonomy. The students demonstrate competent computational and symbolic work but substantial difficulties with task types such as interpreting composite parameters and treating parameter variation. These findings indicate praxeologies dominated by a functional praxis block and a marginal logos block. From the perspective of the reference praxeology, the study highlights the central role of parameters in the dialectic of systems and models and the need for institutional conditions that facilitate this dimension of modelling.

Keywords: mathematical modelling, parameters, dialectic of systems and models, praxeological analysis, the anthropological theory of the didactic.

INTRODUCTION

Parameters play a fundamental role in mathematical modelling. They make it possible to express conditions, regulate variation, and analyse how model behaviour depends on measurable properties of the system. Yet research indicates that students often struggle to determine the role of parameters in a model, to distinguish them from variables or unknowns, and to interpret them in terms of the relations expressed in the model. One strand of research has documented broader difficulties with algebraic symbolism. Bardini et al. (2005) show that students often treat letters as numerical placeholders rather than as algebraic objects, which makes it difficult to work with generality and structural relations. These difficulties become particularly visible when parameters are involved. Wadoń-Kasprzak (2009) shows that students may treat parameters in several different ways, such as placeholders, generalisers, changing quantities, or unknowns to be determined, but that these roles are rarely coordinated coherently in their activity. Such difficulties persist at university level. Hernández-Zavala et al. (2025) demonstrate both that work with parameters in equations and systems may support students' reasoning about variation and that students need to distinguish carefully between unknowns, variables, and parameters in order to analyse how changes in parameters affect algebraic and graphical objects. Taken together, these studies suggest that parameters are not merely technical constants in symbolic work, but central objects in mathematical activity that require interpretation, coordination, and justification.

The present study addresses the following research question: "What types of parameter-related tasks characterise the modelling of a given physical system, and how

do students in science-related programmes perform these tasks?" Anchored in the anthropological theory of the didactic (ATD), the study draws on written solutions from 19 student groups. An institutional reference praxeology of the modelling problem was developed, from which a taxonomy of parameter-related task types was derived and used as the analytical lens.

THEORETICAL FRAMEWORK

In the ATD, mathematical activity is described in terms of praxeologies, and mathematical objects are understood as products of such praxeologies (Chevallard, 2019). A praxeology consists of a *praxis* block, comprising the types of tasks (T) recognised as legitimate within an institution and the techniques (τ) used to perform them, and a *logos* block, comprising the associated technology (θ) and theory (Θ) that explain and justify these techniques. Within this framework, parameters can be analysed through modelling praxeologies, that is, praxeologies centred on constructing and using models to answer questions about a system. The praxeological lens enables examination of how parameters function in modelling and how their associated logos may be weakened through didactic transposition.

Building on the diachronic didactic transposition analysis of elementary algebra developed by Strømskag and Chevallard (2022), a mechanism can be identified that may underlie students' difficulties with parameters in modelling. A key element of this mechanism concerns the distinction, introduced by Bosch and Chevallard (1999), between the ostensive and non-ostensive dimensions of mathematical praxeologies. In the present study, this distinction is mobilised to examine algebraic activity involving parameters in modelling. The ostensive dimension of such activity comprises the visible and manipulable elements, such as formulas, symbolic expressions, and the notational means through which this work is carried out. The non-ostensive dimension includes the technological and theoretical discourse that organises and justifies this work, notably distinctions between given, unknown, and to-be-determined quantities, and the interpretation of parameters as expressing conditions of the system under study. Thus, parameters are present ostensively as symbols in formulas, while their modelling function belongs to the non-ostensive dimension of the praxeology.

As shown by Strømskag and Chevallard (2022), when algebraic knowledge is transformed from scholarly mathematics into school mathematics, the ostensive layer is largely retained, while the non-ostensive layer is progressively reduced or omitted. In short, objects survive, but their raisons d'être fade away. Parameters, for example, remain visible as symbols in formulas, yet their conceptual status as markers of undetermined quantities or conditions is scarcely articulated in school institutions. Whereas formulas in scholarly mathematics function as algebraic models, in school mathematics they have gradually been transposed into fixed numerical recipes used mainly for inserting given values and carrying out routine computations. This transposition is reflected in the shift to "definite-number" versions and the marginalisation of formula transformation, which reduce the visibility and use of parameters in algebraic work and turn formulas into rigid statements cut off from their

raison d'être. Although this erosion of the non-ostensive layer takes place within school mathematics, its effects extend into higher education, where students continue to rely on the praxeologies shaped by this earlier transposition.

A foundational theorisation of mathematical modelling in the ATD is articulated through a dialectic of systems and models constructed in response to questions about a system (Chevallard, 1989). Modelling thus involves defining the system in relation to these questions, identifying relevant quantities and conditions, and specifying them through variables and parameters. Here, variables represent quantities that vary within the model, whereas parameters represent quantities that characterise the system and are typically treated as given in the modelling situation. The model is constructed by establishing relations among these quantities, typically in the form of algebraic expressions. These relations then make it possible to analyse and transform the model, and to interpret the results in order to answer the original questions and generate further knowledge about the system. In this dialectic, parameters occupy a central position: they relate the studied system to its mathematical model; the parameters' interpretation, their role in shaping model behaviour, and their dependence on measurable properties of the system all constitute crucial elements of the modelling process.

THE TASK AND THE INSTITUTIONAL REFERENCE PRAXEOLOGY

The task involved modelling the speed of a sphere falling vertically through a cylinder filled with oil. Based on the forces acting on the sphere—gravitational force (mg), buoyant force ($\rho_o Vg$), and viscous drag ($bv(t)$)—and taking the downward direction as positive, an ordinary differential equation (ODE) describing the motion was given: $m\frac{dv}{dt} = mg - \rho_o Vg - bv$. In the first part of the task, the following quantities were specified: variables t (time, s) and $v(t)$ (speed, m/s); parameters ρ_o (oil density, kg/m^3), ρ_s (sphere density, kg/m^3), r (sphere radius, m), η (oil viscosity, Pa·s), g (gravitational acceleration, m/s^2), and b (viscous drag coefficient, kg/s); and derived quantities V (sphere volume, m^3) and m (sphere mass, kg). Numerical values (ρ_o=860 kg/m^3, ρ_s=4 490 kg/m^3, r=6.3 mm, g=9.81 m/s^2) and observational data for $v(t)$ were provided in tables. In the second part of the task, students were given the terminal-speed relation, $mg - \rho_o Vg = 6\pi\eta r v_t$, which follows from Stokes' law and the force balance, and were asked to determine the oil's viscosity η using the measured terminal speed v_t.

To construct a taxonomy of parameter-related task types, an *institutional reference praxeology* of the modelling problem was developed. It serves as a tool for identifying the parameter-related components of the modelling praxeology and for interpreting how these components are mobilised in students' written productions. The types of tasks (T) characterising this praxeology are to model the motion of a sphere falling through a viscous fluid and to determine the fluid's viscosity from empirical data, more specifically: constructing a model of the speed $v(t)$ from the ODE; analysing the model's behaviour, the emergence of a terminal speed; using an empirically determined value of this quantity to determine the viscosity; and finally establishing a system-specific speed model and comparing it with empirical data. The associated techniques

(τ) include solving the differential equation symbolically to obtain a general model of the form $v(t) = \frac{(m-\rho_o V)g}{b} + \left(v_0 - \frac{(m-\rho_o V)g}{b}\right) e^{-\frac{b}{m}t}$, expressing derived quantities, $V=\frac{4}{3}\pi r^3$ and $m=\rho_s V$, and using the condition $\frac{dv}{dt}=0$ to obtain the terminal speed $v_t = \frac{(m-\rho_o V)g}{b}$. Further techniques include combining v_t with Stokes' law, $F_{drag}=6\pi\eta r v$, to determine the viscosity $\eta = \frac{(m-\rho_o V)g}{6\pi r v_t}$, subsequently determining the drag coefficient $b=6\pi\eta r$, and comparing the resulting model with empirical data by plotting the function and the measured values in a common coordinate system and assessing their agreement. The technology (θ) consists of the discourse that explains and justifies these techniques, and specifies the role of the parameters. Parameters ρ_o, ρ_s, r, g, η, and b are interpreted as properties of the physical system, while composite expressions acquire physical interpretation: the term $\frac{(m-\rho_o V)g}{b}$ represents the terminal speed determined by the balance between the net downward force and the viscous drag, while the factor $\frac{b}{m}$ governs the rate at which this state is approached. The exponential term expresses the transient phase of the motion, while the constant term corresponds to the steady state. This technological layer also includes recognising assumptions such as laminar flow and constant temperature. The theory (Θ) underpinning this praxeology draws on Newton's second law, Archimedes' principle, Stokes' law, and the theory of first-order linear ODEs that accounts for the exponential form of the solution and its convergence to equilibrium.

The taxonomy developed in this study is derived from the institutional reference praxeology by identifying parameter-related task types within its praxis and logos blocks, with particular emphasis on how parameters structure the model, support its interpretation, and enable analysis of its behaviour. These task types are not merely the explicit worksheet questions, nor are they reducible to a fixed solution order. Rather, they represent parameter-related components of the modelling praxeology brought into play by the problem. Some were directly prompted in the assignment, such as symbolic solution of the ODE, preservation of the parametric structure, determination of viscosity, and interpretation of parameter relations made visible by retaining a symbolic solution. Others, such as recognition of modelling assumptions and treatment of parameter variation, belong to the associated technological discourse of the modelling problem and were included because they are necessary for a fuller account of how parameters function in the model.

METHODICAL APPROACH

The empirical material consists of written solutions from 19 groups of first- and second-year students working on a modelling assignment in a mathematics course of approximately 120 students, taught by the author. The course covers differential equations, linear algebra, and functions of several variables. Students were enrolled in programmes in biology, biotechnology, chemistry, geology, and teacher education (with biology and chemistry). The assignment (Assignment 9) was completed in self-selected groups of 2–5, typically within the same programme, as part of a sequence of

13 hand-in assignments, of which 7 had to be approved to gain access to the final examination. This was the only assignment in the course to focus on extended modelling problems. The task analysed here was one of 4 modelling problems included in the assignment, which, like the other assignments, was assessed on a pass/fail basis.

Each group's written solution was analysed in relation to the taxonomy presented in the Analytical Results section. The analysis focused on whether and how groups performed the task types, which techniques they mobilised, and whether elements of technological discourse were visible in their written work. The categories consistent, partially consistent, limited, inconsistent, and absent were used to characterise task-type performance. Because the data consist of written solutions only, the analysis concerns explicit written praxeologies and technologies made visible in writing; it does not claim to access aspects of students' logos beyond what their written work supports.

ANALYTICAL RESULTS

The analysis of the students' solutions is structured by the taxonomy of parameter-related task types established through the reference praxeology. This taxonomy comprises seven task types arranged hierarchically, where ascending levels correspond to increasingly demanding requirements within the modelling praxeology—from the execution of techniques to the mobilisation of technological discourse that explains, justifies, and connects these techniques to the role of parameters and relations in the model. Table 1 summarises the taxonomy and how the 19 groups performed tasks of each type. The taxonomy is presented below, with each task type characterised at the level of the reference praxeology, then illustrated by excerpts from students' solutions.

Task type	**Group performance**
Primary parameter computation	*Consistent*: G1–G4, G7–G11, G13, G15–G19. *Inconsistent*: G5, G6, G12.
Symbolic manipulation of the differential equation	*Consistent*: G1–G4, G6, G7, G9–G12, G18. *Partially consistent*: G5, G8, G13–G17. *Limited*: G19.
Preservation of parametric structure	*Consistent:* G1–G4, G6–G8, G10. *Partially consistent*: G5, G9, G12, G17, G18. *Limited*: G11, G13–G16, G19.
Validation of model and assessment of parameter plausibility	*Partially consistent*: G1–G4, G6–G10. *Limited*: G11, G14, G15, G19. *Inconsistent*: G12, G13, G16–G18. *Absent*: G5
Physical interpretation of composite parameters	*Partially consistent*: G1–G4, G6, G7, G11–G13, G16, G18. *Limited*: G8–G10, G19. *Absent*: G5, G14, G15, G17
Recognition of modelling assumptions	*Absent*: All groups.
Treatment of parameter variation	*Limited*: G2, G4, G8, G10, G13, G18. *Absent*: G1, G3, G5–G7, G9, G11, G12, G14–G17, G19.

Table 1: Parameter-related task types and group performance

Primary parameter computation

This task type occupies the lowest level in the hierarchy because it involves executing straightforward techniques on given quantities, without requiring the technological discourse that explains or connects these techniques to the model.

All except three groups computed the sphere's volume and mass correctly from the given radius and density. Incorrect values typically lead to distorted or implausible results in subsequent steps. G6 provide a clear example of a unit-conversion error:

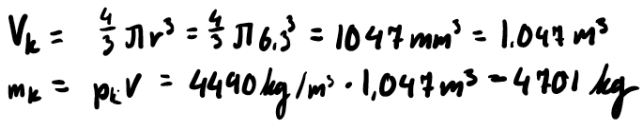

Figure 1: G6's calculation of volume and mass of the sphere

They used 1 $mm^3 = 10^{-3}\ m^3$ rather than the correct 1 $mm^3 = 10^{-9}\ m^3$. This inflates the volume by a factor of 10^6, and the resulting mass (here calculated as 4 701 kg) becomes physically impossible for a sphere of millimetre size.

Symbolic manipulation of the ODE

This task type is positioned above basic computation because it involves applying algebraic techniques to work with the model's core expression, yet remains largely technical and only minimally dependent on associated technology.

A large proportion of groups demonstrated fluency in solving the given ODE. It was generally approached as a familiar structural object, and the algebraic techniques required to obtain the general solution were largely handled competently. Figure 1 shows G2's consistent solution of the ODE.

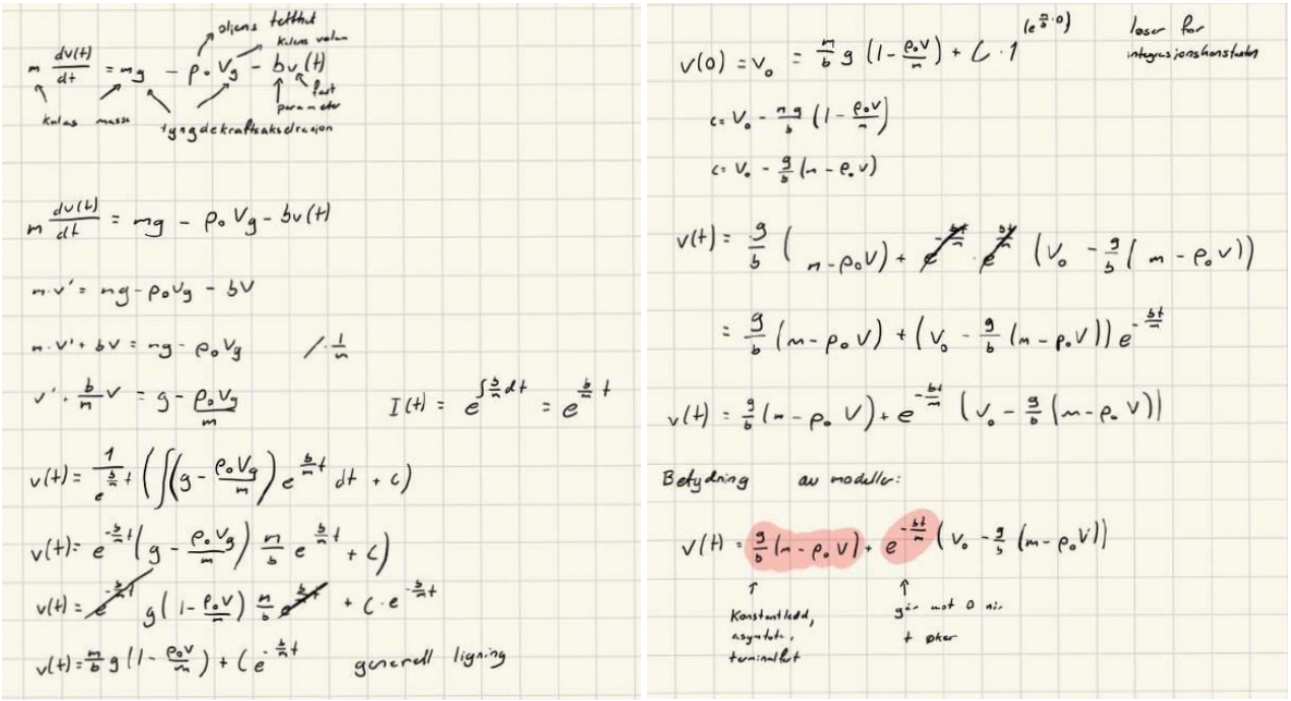

Figure 2: G2's solution of the ODE (pink markings are in the original)

However, several groups exhibited structural breakdowns in symbolic manipulation, including incorrect separation of variables or loss of the speed term. These errors undermined their ability to develop a coherent speed model.

Preservation of parametric structure

This task type is ranked higher because retaining parameter symbols throughout the derivation entails a more global organisation of techniques and begins to mobilise technology that clarifies the relationships between quantities as they are expressed symbolically in the model.

Groups varied markedly in their ability to retain parameter symbols throughout their symbolic work. Maintaining generality in the model is essential for identifying relationships between physical quantities, interpreting composite parameters, and linking the solution to broader modelling aims. Some groups (e.g., G2 in Figure 2) handled parameters consistently and derived general expressions that preserved the structure of the differential equation. Others substituted numerical values prematurely or omitted parameters entirely, limiting the scope for subsequent interpretation. A subset of groups relied on pre-structured functional expressions—typically exponential templates or regression curves—to describe the relationship between time and speed. For example, G19 substituted $mg - \rho_o Vg = k$ and proceeded to solve the ODE by separating variables. They then inserted the numerical value $k = 0.037298$ and arrived at the model $v(t) = C_3 e^{\frac{-b}{m}t} + \frac{0.037298}{b}$. Their next step was to use regression, treating this expression—as inferred from their solution—as a template function of the form $v(t) = ae^{-ct} + d$, and then estimating the parameters a, c, and d directly from the measured data given in the task. As a result, the parametric structure of the solution was no longer visible, and the relationships between the system parameters could not be examined. The reliance on a template function rather than a symbolic solution indicates a shift from modelling to curve-fitting. Regression adjusts interdependent parameters to preserve fit, at the cost of their physical interpretation.

Validation of model and assessment of parameter plausibility

This task type is placed after symbolic and structural work because it draws on local elements of the technology, requiring students to judge whether the results of their techniques remain coherent with empirical data and the model's expected behaviour.

Nearly all groups produced some form of comparison between the model and the empirical data, typically through plotting. However, the depth of this validation varied substantially. Although the task specified a linear drag force in the formulation of the model, none of the groups discussed how this assumption might influence their results or the model's behaviour. Likewise, the explicit assumptions of constant temperature and pressure—which underpin the treatment of viscosity and density as fixed—were not taken up in any of the written solutions.

Limited attention was given to diagnosing mismatches, assessing the consequences of earlier parameter choices, or articulating why a particular parameter value might be reasonable for the investigated physical system. A few groups addressed the empirical adequacy of the model as part of their technological work, considering whether parameter values (e.g., the drag coefficient b or the viscosity η) were plausible. For

instance, G6 remarked on irregularities they encountered in the modelling process, which led them to suspect earlier mistakes in their parameter calculations: "I think our mass is wrong because the value obtained for b is implausible and the graph looks strange" (see Figure 1). However, their solution contains no indication that this prompted a recalculation.

A further indication of the variation in how groups assessed parameter plausibility appears in G15's treatment of the viscosity. This group used the correct expression for η, inserted the physical quantities with appropriate units, and obtained a numerical value of the correct magnitude. Their final unit assignment, however, does not match the dimension of dynamic viscosity: instead of $N \cdot s/m^2$ (equivalent to $Pa \cdot s$), they wrote an incorrect combination of units. Given the otherwise consistent setup and intermediate unit work, this is plausibly interpreted as a transcription error.

Physical interpretation of composite parameters

This task type is located at a higher level in the hierarchy because it demands the mobilisation of technology to connect composite expressions to their physical interpretation, linking mathematical organisation to the behaviour of the system.

A few groups articulated the physical meaning of composite expressions such as $\frac{(m-\rho_o V)g}{b}$ (terminal speed) or $\frac{b}{m}$ (decay rate of the speed). Most interpretation remained descriptive, focusing on the observable flattening of the speed curve rather than on how this behaviour follows from parameter relationships or causal mechanisms. G8, for instance, described the increasing oil resistance that halts acceleration but did not link this to the symbolic expressions, nor did they account for the role of buoyancy—a pattern shared by several groups. This contrasts with G11, who took a more structurally oriented approach. They rewrote the ODE by substituting $m = \rho_s V$ and dividing through by ρ_s (density of the sphere), which exposed the driving term $\frac{g(\rho_s-\rho_o)}{\rho_s}$ as a buoyancy-corrected acceleration, labelled "modified gravitational acceleration." They also identified $\frac{b}{m}$ as a damping factor, recognising that this composite parameter governs the exponential decay rate. A similar structural focus appears in G2, who identified the constant term $\frac{g}{b}(m - \rho_o V)$ as the model's asymptote and note that the exponential term "tends to zero as t increases" (see Figure 2).

Recognition of modelling assumptions

This task type is positioned near the top because identifying the assumptions that justify the model requires a more reflexive form of technology, involving consideration of the conditions and principles under which the model and its assumptions are valid.

This task type was not performed by any of the groups. The assumptions relevant to this task type lie in the reference praxeology: the problem presupposes, for example, that the drag coefficient b determines the flow regime and governs the exponential decay rate in the model. In the first part, the proportional relation $F_{drag} = -bv(t)$ is given

as part of the force model from which the ODE is already derived, and students are not asked to examine or justify it. In the second part, where the terminal-speed equation derived from Stokes' law is used to determine the viscosity, the physical conditions for its validity (such as laminar flow and low Reynolds number) remain implicit, and students apply the equation directly without engaging with the underlying assumptions.

Treatment of parameter variation

This task type appears at the highest level because analysing how parameter variation shapes model behaviour requires integrating techniques and technology, approaching the theoretical dimension of how parameters structure the dynamics of the system.

Such technological discourse was largely absent. Parameters were typically treated as fixed numerical inputs rather than as quantities that structure the behaviour of the system. For example, G10 demonstrated some performance of this task type by computing different values of the drag coefficient b from several data points and comparing the resulting model behaviour, a restricted form of parameter variation. However, their work remained confined to addressing discrepancies between the model and the data, and did not extend to a principled analysis of how variation in b impacts the model's dynamics.

DISCUSSION

The analysis of students' solutions shows that they generally manage the computational and symbolic work at the lower levels, but encounter increasing difficulty in task types involving composite parameters, model–data correspondence, model assumptions, or parameter variation. These difficulties indicate that students' modelling praxeologies are characterised by a dominant praxis block alongside a marginal logos block. This pattern is consistent with research showing that students across educational levels struggle to treat parameters as elements with distinct roles in mathematical expressions rather than as numerical placeholders (e.g., Hernández-Zavala et al., 2025; Wadoń-Kasprzak, 2009). In modelling contexts, this difficulty extends to interpreting parameters in relation to the structure of the model and its assumptions, through which the properties of the studied system are represented. The present study shows how the weakness of the logos block in students' modelling praxeologies is manifested. The parameter-related task types developed here provide an analytical means of identifying this: students rarely engage with the technological discourse needed to explain, justify, or connect their techniques to the structure of the model and its parameters.

These findings point to the need for institutional conditions that support the mobilisation of technological discourse involving parameters in modelling work. When opportunities to express how parameters shape the model and mediate its relation to the physical system are limited, the dialectic between system and model can only be partially realised. The taxonomy developed in the present study makes visible the parameter-related task types through which this dialectic is institutionalised and highlights where students' modelling praxeologies fall short: when higher-level task types are not engaged with, modelling fails to yield insight into the studied system.

One possible implication of these findings is to design modelling problems that bring these task types to the fore, creating opportunities for students to engage with the technological discourse that gives coherence to their techniques and supports interpretation of parameters as representing properties of the system. This aligns with work emphasising the importance of explicit attention to parameters in instruction, particularly in calculus courses (Belding & Quinn, 2025). For such work to take hold, it must be supported by institutional conditions that make this mobilisation possible. This includes a didactic contract that recognises modelling as inquiry oriented towards studying systems and addressing questions that draw on knowledge from other disciplines, in line with the paradigm of questioning the world (Chevallard, 2019), where knowledge is brought to bear on questions, and modelling is understood as a means of investigating systems rather than merely applying prior knowledge.

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Comparative study of modelling tasks in post-reform mathematics courses in engineering education in Spain and Sweden

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This study compares how modelling is introduced in first-year mathematics courses for engineering programs in Spain and Sweden. Using the anthropological theory of the didactic, we analyse the introduction of modelling tasks in curriculum reform processes in both countries. Despite reforms aimed at fostering modelling and interdisciplinarity, tasks remain intramathematical, focusing on algorithmic techniques and theoretical explanations rather than genuine modelling processes involving engineering systems. Both cases show strong technical–theoretical coherence but limited openness, reflecting the persistence of applicationism. We discuss the conditions and constraints that shape these practices and highlight the potential of those tasks as a didactic tool for praxeological reconstruction.

Keywords: Modelling, curriculum reform, engineering education, applicationism.

INTRODUCION

Numerous studies have shown that mathematics courses in engineering degrees often struggle to bridge abstract concepts with practical applications, highlighting the need for more integrated forms of instruction (Florensa et al., 2022; Pepin et al., 2021). Barquero et al. (2014) had already identified the tendency to present mathematical models as ready-made tools to be applied in engineering or scientific problems– a phenomenon they termed ‘applicationism’.

In response to this persistent issue, several institutions involved in engineering education have called for curricular reforms that better align mathematics teaching with professional practice, emphasising modelling, problem-solving, and interdisciplinarity. The SEFI working group on mathematics education has published two influential reports (e.g., Alpers, 2021) advocating for mathematics curricula that more accurately reflect the demands of engineering practice. More recently, UNESCO (2021) has emphasised the importance of interdisciplinarity in engineering education, aiming to prepare future engineers who can effectively connect mathematics with data analysis and emerging technologies.

Nevertheless, research examining these reforms at the undergraduate level reveals a high degree of stability in the syllabus, with limited changes to the knowledge taught. For instance, Bosch et al. (2021) analysed the criteria used in the design of new undergraduate mathematics programmes, showing that institutional constraints often hinder innovation. Similarly, Pleština et al. (2024) reported the limited impact of the Bologna process on mathematics courses within engineering degrees in both Spain and Croatia.

At the ICTMA conference, Asami-Johansson et al. (2025) presented a comparative study of two reforms of mathematics courses in engineering degrees. Their analysis combined an analysis of the syllabus changes—using adapted indicators of *applicationism*—with interviews of eleven experienced lecturers involved in these reforms. The authors highlighted persistent institutional and epistemological constraints that continue to limit the integration of modelling into mathematics courses for engineering education. This study builds upon this work, extending the analysis of the specific modelling tasks newly introduced in the reformed mathematics courses for the 1st-year of engineering programs in Spain and Sweden. We examine the content and structure of these tasks and analyse the extent to which they challenge the *applicationist* conception of mathematics in engineering education. Accordingly, this study addresses the following research questions (RQs):

1. How do modelling tasks, introduced in the reformed mathematics programmes, transform the mathematical knowledge to be taught?
2. What conditions and constraints shape the implementation of new teaching devices supporting modelling practices in mathematics courses?

MODELLING TASKS IN POST-REFORM MATHEMATICS COURSES

At the University of Gävle (UoG), the reform launched in autumn 2025 introduced a common mathematics curriculum for first-year bachelor's engineering students. The previous structure (22.5 ECTS: *Algebra and Geometry* 7.5, *Linear Algebra* 7.5, *Calculus in One Variable* 7.5) was streamlined to 20 ECTS, with an equal distribution between *Linear Algebra* and *Calculus in One Variable*. Although the mathematical content remained essentially unchanged, the reform introduced new forms of student work.

The objective "model and solve application problems in Linear Algebra using mathematical software" appeared in both pre- and post-reform. However, its implementation was substantially expanded. Lecturers designed nine *computer-based laboratory* tasks applying linear algebra concepts to contexts from different engineering fields (Electronics, Building, Computer Science, Energy Systems, Industrial Economics, and Mechanical Engineering). This group assignment accounts for 1.5 ECTS (15% of the course credits) and involves groups of three to four students selecting one of the nine tasks. Each group consists of three to four students and selects one of the nine tasks. For each task, five groups are formed, resulting in a total of 45 groups (approximately 160 students) working over a period of five weeks. Students submit a report containing the code and solutions, followed by an oral examination in which each group explains its coding and problem-solving processes.

At the Escola Universitària Salesiana de Sarrià (EUSS), an engineering school in Barcelona (Spain), the reform increased the number of credits for mathematics courses from 21 ECTS to 24 ECTS, in line with Engineers Europe's recommendations. The courses were restructured from three (*Mathematics, Calculus, Statistics*) to four (*Mathematical Analysis, Calculus, Applied Mathematics, Statistics*), mainly involving

a redistribution of content and the inclusion of topics on numerical methods (Asami-Johansson et al., 2025). The most significant change, implemented during the autumn semester of 2025, was the implementation of some applied activities related to the use of numerical methods. These were solved in two-hour sessions with students working in pairs (15 groups) on computers using Excel to implement and present their solutions.

THEORETICAL FRAMEWORK AND METHODOLOGY

This study adopts the anthropological theory of the didactic (ATD) (Chevallard, 1999 and 2019) as the main theoretical and methodological framework to analyse how mathematical modelling is conceived and organised within reformed engineering mathematics courses. The ATD provides tools to describe both the internal organisation of mathematical knowledge—through the notion of *praxeology*—and the set of conditions and constraints, internal or external to the university and engineering education, that shape the possibilities for modelling.

Mathematical modelling and *applicationism* in the ATD

Within the ATD framework, most mathematical activity can be described as a *modelling activity*, where questions arising within a system lead to the construction of mathematical models aimed at providing answers. Following García et al. (2006), modelling can be proposed as a recursive process of constructing and articulating mathematical praxeologies of increasing complexity, each emerging from the questioning of previous ones. This dynamic view contrasts with the more static conception of modelling as a final "application" of pre-taught mathematics.

A praxeology $\wp = [T/\tau/\theta/\Theta]$ connects the practical and theoretical dimensions of human activity (Chevallard, 2019). The *praxis* comprises a *type of tasks* (T) and *techniques* (τ) used to solve it. At the same time, the *logos* includes the *technological* and *theoretical* discourses (θ, Θ) that explain and justify the praxis. Any human activity—whether solving an equation, driving a car, or engaging in mathematical modelling—can be described in terms of praxeologies with the development of a praxis and logos.

This praxeological interpretation of modelling has important consequences for analysing how modelling is taught and learnt. When the modelling process is reduced to a mere application of predetermined techniques, the logos component tends to remain implicit or absent. Several studies (e.g. Wozniak, 2012) have shown that in university settings, modelling is often reduced to what Wozniak calls *mute* or *weak* praxeologies—activities in which the practical component is visible. However, the justificatory discourse often remains implicit. These situations result in "fake inquiries" (Bosch, 2019), where students perform apparent modelling processes without engaging in genuine questioning, validation, or theoretical developments of systems and models.

This tendency aligns with what Barquero et al. (2013) characterise as *applicationism*—an institutional epistemology that conceives modelling as a unidirectional application of already-established mathematical knowledge. Under this approach, (1) mathematics is treated as autonomous and independent from other scientific disciplines; (2) its

organisation follows the internal logic of contents, rather than the systems to be modelled; and (3) applications appear only after basic mathematical instruction. In this study, these indicators of *applicationism* have been adapted to the specific case of engineering education (Asami-Johansson et al., 2025). They are used to examine modelling tasks and lecturers' accounts of reform processes.

Ecological analysis and levels of didactic codeterminacy

The praxeological structures within modelling tasks in both countries are shaped by multiple conditions and constraints that extend beyond dominant epistemologies. These factors operate across various levels of didactic codeterminacy (Chevallard, 2019)—from *disciplinary* levels, where mathematics for engineering is organised as teachable knowledge, to broader *institutional* and *societal* levels, which determine how universities structure engineering education and define pedagogical norms across the teaching of university disciplines. Using this scale (see Fig. 1) allows us to place and interpret the constraints affecting the implementation of modelling in university mathematics courses.

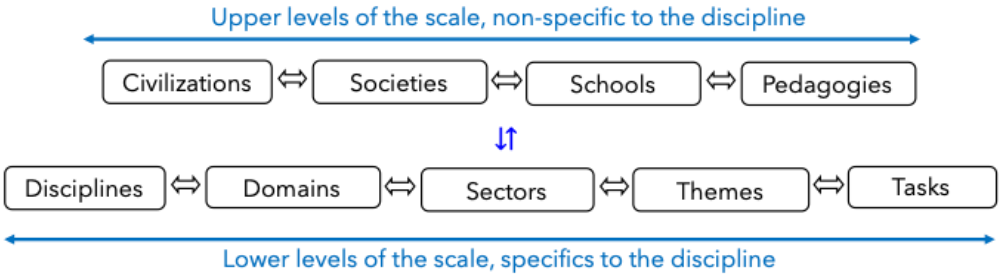

Figure 1. Scale of levels of didactic codeterminacy (adapted from Chevallard, 2002)

RESULTS AND ANALYSIS

At the UoG in Sweden, the laboratory tasks within the course Linear Algebra, developed by the mathematics faculty, were organised under the following themes: 1. Transport Networks and Graph Analysis–intended for students in Industrial Economics; 2. Image Processing with Linear Transformations–for Computer Science; 3. Modelling of Electrical Circuits–for Electronics and Automation; 4. Inventory Control and Linear Optimisations–for Industrial Economics; 5. Structural Mechanics and Force Analysis–for Building and Mechanical Engineering; 6. Triangulation and Positioning – for all programs; 7. Singular Value Decomposition for Signal Analysis–for Electronics and Automation, Computer Science; 8. Linear Regression for Energy Forecasting–for Energy Systems; 9. Mixing Systems and Linear Equation Systems–for Energy Systems and Industrial Economics. However, students can choose any assignment regardless of which program they belong to.

Students are required to use MATHEMATICA or Python to implement their solutions. As outlined earlier, the work is conducted in groups over a period of five weeks. Each group submits a written report and participates in an oral examination. While students are allowed to use tools such as ChatGPT for coding assistance, the oral examination will assess their ability to adapt and modify the code in response to changes in the task

description. The mathematics teacher delivers guidance lectures, during which he walks students through similar tasks. In the following section, we provide a detailed description of one of these tasks.

At EUSS, the newly designed modelling activities in the Calculus course were devoted to introducing numerical methods. The activities were incorporated at various points throughout the semester, with two of them introduced in the themes on Differential Calculus and Integration of Single-Variable Functions. Students were required to use Excel spreadsheets to implement and propose solutions. In the activities carried out in the computer lab, students were not allowed to use any GenAI. Instead, similar examples were provided as a starting point for students to implement their solutions. The students worked in groups of two or three and were assessed accordingly.

Praxeological analysis of the tasks

Modelling praxeologies at a glance: the case of UoG in Sweden

The task "*Transport Networks and Graph Analysis*"[1] was proposed as the first computer-based laboratory activity in the reformed Linear Algebra course. It aims to connect matrix algebra with applications in engineering through a contextualised example of transport networks. Students, working in small groups, are asked to analyse a directed graph with six nodes that includes both unidirectional and bidirectional arcs, some of which form cycles, while node 6 acts as a potential sink.

The main objectives of the task are:

1. Accessibility: Construct the adjacency matrix A (6×6) of the network and compute $R = A + A^2 + A^3$ to determine which nodes are reachable from node 1 within three steps.
2. Route analysis: Count the number of distinct paths of lengths 2 and 3 from nodes 1 to 6 and identify two different routes, not sharing the same first node.
3. Flow balance: Calculate $A^\top 1$ (inflow), $A1$ (outflow), and $b = A^\top 1 - A1$ to classify nodes as sources ($b_i < 0$), sinks ($b_i > 0$), or balanced nodes ($b_i = 0$). Verify that $1^\top b = 0$ as a consistency check.
4. Redundancy: Investigate whether an arc can be removed without affecting the reachability from node 1 to node 6 within three steps, justifying the answer using powers of A.

Before working on the main problem, the teacher introduces a simplified example involving a 4-node graph. This example guides students through the process of building an adjacency matrix, computing accessibility using matrix powers, and interpreting the results in terms of reachability and flow.

Tasks (T)	*Techniques* (τ)	*Technology -Theory* (θ-Θ)

[1] Supplementary material providing a more complete description of the task 'Transport Networks and Graph Analysis' is available at: [link]."

T_1: Build the adjacency matrix for a simple directed graph and infer basic accessibility. T_2: Construct the 6×6 adjacency matrix A for the given directed transport network. T_3: Compute $R = A + A^2 + A^3$ and determine which nodes are reachable from node 1 within three steps. T_4: Count the number of distinct paths of exact length 2 and 3 from node 1 to node 6; identify two different routes 1→6 that do not share the same first node. T_5: Compute $A^\top 1$ (inflow), $A1$ (outflow) and $b = A^\top 1 - A1$; classify nodes as sources ($b_i < 0$), sinks ($b_i > 0$) and balanced ($b_i = 0$); verify that $1^\top b = 0$. Determine whether a single arc can be removed without affecting reachability 1→6 within three steps, and justify using powers of A.	τ_1: Encode the graph as an adjacency matrix ($A_{ij} = 1$ if $i \to j$). τ_2: Compute matrix powers (A^2, A^3) and their sum R (manually or using software). τ_3: Read accessibility: $(R)_{1j} > 0 \Rightarrow j$ is reachable from 1 in ≤ 3 steps. τ_4: Count paths using entries of A^2, A^3; identify distinct first-step nodes. τ_5: Compute inflow/outflow vectors and classify nodes via b. τ_6: Check redundancy by hypothetically removing an arc and re-evaluating reachability in A^2, A^3.	Definition of the adjacency matrix for directed graphs. Concepts of directed graphs, connectivity, sources/sinks, and cycles. Interpretation of $(A^k)_{ij}$ as the number of walks of length k from i to j—transport systems as discrete directed graphs; simplified assumptions (no capacities or costs). A product with a vector of ones yields in/out degrees; $1^\top b = 0$ expresses flow conservation.

The task provides explicit θ-level discourse (definitions, meaning of A^k) and references to Θ (graph theory, linear algebra). However, students may complete the activity without articulating why the operations work or under which assumptions. The "transport" context is superficial. The modelling remains intramathematical—translation from a picture to matrices—without honest feedback from the original system being investigated. Validation is purely algebraic, not contextual.

Modelling praxeologies at a glance: the case of EUSS in Spain

The activity introduces students to two classical numerical methods for finding the roots of nonlinear equations: the bisection method and the Newton–Raphson method[2]. The context is purely mathematical, focused on algorithmic reasoning rather than real-world modelling. Students are asked to:

[2] Supplementary material providing a more complete description of this task is available at: [link]."

1. Use the bisection method $e^{-x} = x$ in the interval [0, 1], identifying the existence of a root via Bolzano's theorem, iterating until the interval is smaller than $\varepsilon < 0{,}005$.
2. Apply the Newton–Raphson method to the same equation and implement the iterative formula $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$ for $n \geq 1$. Compare the convergence speed with the bisection approach.
3. Discuss potential limitations (divergence, dependence on the initial guess) and perform additional exercises on new equations using both methods.

Tasks (T)	*Techniques* (τ)	*Technology -Theory* (θ-Θ)
T_1: Verify the existence of a root using Bolzano's theorem. T_2: Implement the bisection algorithm until conditions are satisfied. T_3: Apply the Newton–Raphson iteration to approximate the roots. T_4: Compare the efficiency and convergence of both methods. T_5: Identify limitations and sources of error through further exercises.	τ_1: Define $f(x) = e^{-x} - x$, choose interval [0,1] and compute successive midpoints. τ_2: Calculate iterates x_k, monitor error $\lvert x_{k+1} - x_k \rvert$, or $\lvert f(x_{k+1}) \rvert$, and apply stopping criteria. τ_3: For Newton–Raphson, compute successive iterates using $f'(x) = -e^{-x} - 1$. τ_4: Tabulate results, track number of iterations, and interpret numerical convergence.	Justification of the existence of roots by Bolzano's theorem. The iterative methods and error analysis. Interpretation of forward and backward error as measures of numerical accuracy. Comparison of convergence rates.

The task seems to organise a praxeology mostly focused on numeral methods rather than a genuine modelling praxeology. The students' activity centres on mastering techniques for solving equations numerically, supported by a strong logos (theoretical discourse on convergence, error, and existence). The didactic contract prioritises procedural fluency and conceptual correctness over problem formulation or validation against an external system. Thus, the modelling dimension is intra-mathematical: the "problem" is not derived from a real engineering situation but serves as an example to illustrate algorithmic methods. Validation is internal (numerical precision, convergence proof), not external (empirical adequacy).

CONCLUSIONS: CONDITIONS AND CONSTRAINTS FOR MODELLING

The analysis of the two tasks—*Transport Networks and Graph Analysis* (UoG) and *Numerical Calculation: Bisection and Newton–Raphson Methods* (EUSS)— reveals strong structural similarities in how modelling is conceived and enacted in reformed engineering mathematics courses. By interpreting these observations through the scale

of didactic codeterminacy, we identify the conditions and constraints that shape the implementation of modelling-oriented teaching devices.

At the level of *sectors* and *themes*, both tasks can be described as intra-mathematical modelling praxeologies. The modelling process unfolds entirely within the mathematical universe, without reference to empirical systems or engineering practices. The *praxis* in both cases is defined by a sequence of computational procedures for constructing or manipulating algebraic objects (matrices, iterates, computational routines), often supported by software. The *logos* is explicit and theoretically coherent (graph theory, linear algebra, numerical analysis), serving mainly to illustrate the 'theory' given in advance. This produces tasks with strong technical-theoretical coherence but a limited openness: the modelling is directed toward illustrating established mathematical principles rather than generating a modelling process and validating the new knowledge produced. Both tasks exemplify what Wozniak (2012) characterises as *mute* or *weak* modelling praxeologies: explanations exist, but the questions and systems are predetermined and offer little space for a rich system-model dialectic. Consequently, the context becomes irrelevant to the solution process, reinforcing an *applicationist* conception of modelling. Mathematics is conceived as a reservoir of ready-made tools to be applied to pre-formulated problems rather than as an adaptive modelling process capable of approaching the study of relevant phenomena. The activities aim to contextualise abstract mathematics through illustrative 'applications', yet these applications remain internally organised by the conceptual organisation of mathematics itself.

At the *pedagogical* level, the tasks designed, and teaching conditions contribute to significant constraints for modelling. At the University of Gävle, software use is formally required by the syllabus, as part of the expected learning outcomes: students must "model and solve application problems using appropriate tools." One of the teachers responsible for the laboratory assessment emphasised that "the most important aspect is for students to understand the problem and be able to explain it using paper and pen." The use of software—mainly MATHEMATICA or Python—is therefore treated as a supportive but secondary tool for visualisation and structuring reasoning rather than as a genuine instrument of modelling. Interestingly, students reported that software use improved their understanding of problem-solving, as it encouraged them to think in a more visual and structured way. As one of the teachers observed, "solving the problem through coding can help students view the problem from a different perspective." This reflection highlights a potential didactic gain: programming may foster a form of instrumental and visual reasoning that aligns with the concept of praxeological reconstruction in ATD, even if this is not explicitly stated in institutional discourse. The same teacher, after attending a seminar on modelling within the ATD framework, explicitly recognised that "the modelling tasks we provide to our students are not modelling, according to the ATD, which the speaker applied during the seminar" Such awareness signals a reflective stance and possible evolution in institutional epistemology. However, curricular and evaluative norms still constrain it.

At the EUSS, tasks are performed in pairs and delivered as a joint report, and often completed during class time. Assessment emphasises accurate execution and reporting rather than exploration or validation. Students tend to perceive these tasks as routine exercises, with little awareness of any modelling dimension. Teachers encourage precision and clarity in algorithmic implementation, but do not explicitly frame these activities as modelling processes. The observed student engagement is functional rather than critical—no strong reactions emerged regarding the task itself. However, oral assessments occasionally prompt deeper reflection on how and why the methods work, although this remains peripheral.

At the *university* level, curricular requirements and assessment cultures create structural constraints on the space available for modelling. Both institutions operate within rigid programme-level expectations that prioritise procedural fluency and the coverage of foundational mathematical content. As a result, tasks introduced within reformed programmes, though intended to support modelling practices, often end up reinforcing rather than transforming existing epistemologies. Modelling remains framed primarily as an illustration of mathematical theory instead of an adaptive inquiry capable of producing new knowledge about systems.

Future work should explore how modelling praxeologies can be reconstructed to foster richer dialectics between *praxis* and *logos*, enabling students to engage in authentic modelling and contributing to broader transformations in university mathematics programmes and assessment practices, so as to create more favourable conditions for mathematical modelling as a serving tool for initial engineering training and for students' future professional practice.

ACKNOWLEDGMENTS

The research reported in this paper was funded by the grants PID2021-126717NB-C31 and PID2021-126717NB-C33, funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe"

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Promoting graph understanding through a short intervention: an eye tracking study

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Keywords: motion graph understanding, dual process theory, eye tracking, short intervention.

INTRODUCTION

Understanding function graphs is fundamental in mathematics and in the analysis of everyday and scientific phenomena. Research in mathematics and physics education has extensively documented students' difficulties with interpreting graphs (e.g., Janvier, 1981; Leinhardt et al., 1990) also at the university level (e.g., Planinic et al., 2013). Dual process theory (Kahneman, 2013) provides a useful framework for interpreting such findings in mathematics and science education (e.g., Kryjevskaia et al., 2020), highlighting the contrast between fast, intuitive thinking, which often leads to erroneous responses even among university mathematics students, and slow, reflective thinking, which supports analytical reasoning and self-monitoring. Eye tracking methods have increasingly been employed to investigate these processes by revealing patterns of students' visual attention during graph-based tasks (e.g., Sušac et al., 2018). Within this perspective, fast thinking is typically associated with rapid initial fixations and intuitive gaze patterns, whereas reflective thinking is indicated by longer fixation durations and more systematic visual exploration.

RESEARCH METHODOLOGY AND RESULTS

The aim of our study is to examine these cognitive processes and to explore how analytical and critical thinking can be fostered to help students overcome difficulties in understanding motion graphs. Our research question is: How does a brief intervention designed to promote slow analytical thinking, as described by dual process theory, affect university students' comprehension of motion graphs?

Thirty participants - university students in mathematics and pre-service mathematics teachers - took part in the study. The procedure comprised three phases: a pretest, a set of guiding questions, and a posttest. Both the tests required students to translate real-life motion scenarios into mathematical models as speed-time graphs. After each pretest task, students answered short supporting questions that served as heuristic prompts for analysing the situation; no feedback on correctness was provided. In the posttest, students solved the same tasks as in the pretest.

Eye movements were recorded using a Tobii Pro X3-120 eye tracker (120 Hz), and the data were analysed with Tobii Studio software. Students frequently relied on intuitive "graph-as-picture" interpretations, leading to incorrect responses driven by fast, automatic thinking. In the pretest (Fig. 1, left), students mainly focused on completely incorrect options (1–3), whereas in the posttest (Fig. 1, right), attention shifted to the correct option (5) and the partially correct option (6), which were also the most frequently chosen. Selected tasks and results will be present in the poster.

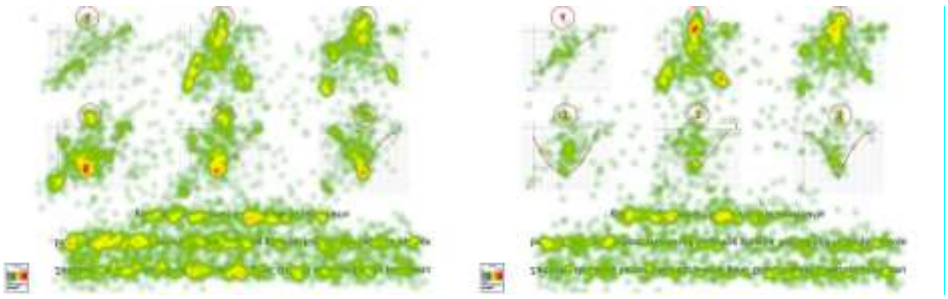

Fig 1: Heatmaps showing visual attention during the pretest (left) and posttest (right).

After the brief intervention, correct answers increased. The guiding questions effectively directed students' attention to relevant graph features, supporting deeper engagement and more deliberate reasoning. The findings illustrate how fast, intuitive thinking can mislead even students who have been trained in mathematics. They also demonstrate that brief, well-designed interventions, as well as the insights provided by eye tracking, offer valuable tools for improving and investigating graph understanding.

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Mathematics in Two Engineering Industrial Contexts: What Didactic Implications for Mathematics Teaching

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This paper presents a methodology for identifying mathematical praxeologies in engineering workplaces developed through an empirical case study conducted with an engineer in Mexico and further tested in a French case. Using the Anthropological Theory of the Didactic (ATD), we show that workplace mathematical activity is shaped by institutional conditions, including client demands, resource availability, and time constraints. Interviews with engineers enabled the reconstruction of praxeologies, while subsequent interviews with an academic expert revealed the mathematical models underlying workplace "black box" practices. We argue that praxeological reference models informed by both institutions can support the design of instructional proposals for engineering mathematics education.

Keywords: engineers' workplace, praxeologies, differential equations, engineering education

INTRODUCTION

Research in mathematics education has shown the persistent tension between the mathematics taught in school and its application in workplaces, particularly for engineering students who often struggle to recognise how university mathematics is used in practice (Pollak, 1988). Establishing connections between these forms of mathematics requires understanding how they are used in working environments (Sabra & Tabchi, 2024). However, elucidating the role of mathematics in industry remains challenging. Studies have shown that mathematical practices go beyond calculations, especially as sophisticated software automates many computational procedures (FitzSimons, 2013). As a result, mathematics often functions as a shared communicative medium that supports reasoning and decision-making (FitzSimons, 2013; Mustoe & Lawson, 2002). But it does not stop there; the industrial sector is not static. Technological change continually reshapes companies' internal dynamics, altering the mathematical demands placed on engineers. Additionally, mathematical practices also depend not only on engineers' knowledge and abilities but on external factors that influence their choices and constrain their applications, such as 'codes of practice' (guides made by engineers in industry to solve problems) that determine what counts as an acceptable solution within companies (Castela & Romo-Vázquez, 2022). For these reasons, studying engineers' mathematical practices in workplaces requires methodological tools capable of capturing workplace dynamics through the experiences of engineers in practice: the projects they engage with, the ways they approach them, and the processes in which they participate.

In this paper, two industrial cases involving engineers in practice are analysed, leading to the development of an initial two-stage methodological approach that enables an examination of how mathematical practices arise in engineering work and the factors that shape them. This approach is grounded in an interview grid based on elements of the Anthropological Theory of the Didactic (ATD).

CONTEXT AND RESEARCH QUESTIONS

As a part of the development of our methodological approach, efforts were made to establish collaborations with industrial companies, both of which specialise in designing and developing projects for external clients. One company is based in Mexico and the other in France, hereafter referred to as Company W and Company X, for reasons of confidentiality. Across these companies, different types of projects related to process automation were identified. In particular, the Company W case shown a project centred on the implementation of controllers. Understanding this project requires engaging with several concepts from the control theory, the field that provides the foundations for the design and validation of such controllers.

To illustrate the basic ideas of control theory, consider temperature regulation in a room: the user sets a reference temperature, the thermostat measures the output, and the controller adjusts heating to reduce the error between them. Disturbances, such as an open window, require the system to compensate automatically. This illustrates the basic principle of feedback control. In industrial practice, one of the most common strategies is the PID controller (Hägglund & Guzmán, 2024), which adjusts the system based on mathematical manipulation of the error signal using three parameters: proportional, integral, and derivative. Designing such controllers usually requires obtaining a transfer function, that is, an expression of the system's dynamics derived by applying the Laplace transform to its governing differential equations (Ogata, 2010). The study of projects involving control strategies remains essential as many processes in industry are run autonomously and are expected to remain stable despite external factors. Therefore, mastering control theory is a required ability for engineers in practice. However, the development and implementation of controllers depend on possessing knowledge regarding different fields such as physics and mathematics, and on knowing how to combine and apply this knowledge appropriately, which has been shown to be challenging for students (Zenger, 2007).

These considerations have led us to formulate the following questions: How can the activity of engineers be reconstructed to identify and analyse the mathematical knowledge used in their professional practice in industry? And how can the outcomes of this reconstruction be used to improve connections between workplace mathematical practices and the mathematical knowledge taught in engineering education?

THEORETICAL FRAMEWORK

For the Anthropological Theory of the Didactics (ATD) (Chevallard, 1999), human activity is shaped by the institutional settings they have participated throughout their lives. Institutions are socially stabilised organisations that legitimise shared ways of

thinking and acting (Chevallard, 1999). Industrial companies, for example, are institutions, and engineers act as their subjects. Within institutions, expectations and rules function as constraints, while available resources are conditions; these frame what subjects can do and how they may do it. For instance, companies make available specific tools and software to engineers, but their work is bound by deadlines. To study the human activity within institutions, ATD proposes an epistemological model known as praxeology $[T, \tau, \theta, \Theta]$ composed of four elements: the type of task (T) to be accomplished, the techniques (τ) used to develop it, the discourse that justifies such technique(s) called technology (θ) and the theories (Θ) supporting the technology.

Over time, institutions develop and legitimise praxeologies that reflect their way of solving recurring problems. For example, companies may develop procedures or design internal norms (Castela & Romo-Vázquez, 2022). The praxeologies performed by engineers may be shaped by their expertise, their current role in the company, and their professional background. An electronic engineer and a mechanical engineer engage in different types of work and therefore of tasks, while one may deal with circuit design, the other carries out the study of materials, and thus they develop techniques respectively, validating them according to what is accepted in their own company. Although most visible praxeologies reflect institutional ways of doing, validated and shared within that institution, there also exist personal praxeologies. These refer to the repertoire of tasks and associated ways of performing them that an individual member develops, often accompanied by specific technologies. Personal praxeologies may remain private or be shared within small groups (e.g., a working team), but they are not standardised and therefore are not accessible to the broader institution. However, they may offer valuable information into how subjects adapt institutional praxeologies.

METHODOLOGY

Research has shown the challenges involved in analysing the mathematical practices of engineers within industrial dynamics (e.g., Kent et al., 2007; Gainsburg, 2006). To address these challenges, the methodological approach presented in this study was progressively developed through an empirical process based on case studies. The first implementation of this approach took place in the context of a study with a Mexican Engineer working in Industrial Engineering. The initial aim was to identify and reconstruct a mathematical modelling praxeology emerging from the engineer's professional activity. This was documented through semi-structured interviews with an engineer in practice. During the analysis of this first case, the need emerged to confront the engineer's approach to the task with mathematical techniques recognised within academic institutions. This led to the introduction of an additional analytical step consisting of interviews with an academic expert in the relevant field. The purpose of this stage was to re-examine the task identified in the workplace and to articulate the mathematical processes involved in a way that may be recognised within the Engineering University institution. Following the first empirical implementation, for a methodological concerns, we characterise more clearly the two complementary stages. The methodology was therefore structured as a two-stage process: (1) reconstructing

engineers' workplace praxeologies through interviews, and (2) re-examining the same task from an academic perspective with an expert in the field.

First Stage: The Engineer's Interviews

To gather information from engineers in practice, previous studies addressing similar aims offered useful insights (e.g., Walkington et al., 2025). In this sense, a pre-interview questionnaire and a follow-up interview were designed. Questions cover the four components of a praxeology as well as the institutional constraints and conditions that shape the techniques in industry. We focused on three analytical categories: engineers' profiles, institutional context, and task and technique(s). The pre-interview questionnaire, sent to the participant before the interview, focuses on the obtention of the engineer's profile. This profile reflects the experiences obtained by the institutions they have belonged in life and is examined by exploring their formal education (e.g., bachelor's, master's degree, certifications), and their roles within companies. Therefore, their profile shapes the mathematical tasks they may develop in practice. As mentioned before, mechanical engineers may engage with machine design and therefore they're involve with mathematical concepts such as kinematics and dynamics, while electronic engineers may apply mathematics in the design of signal filters, where the frequency domain analysis becomes central.

The follow-up interview, designed to be executed in one hour, has two main goals: 1) to characterise the institutional context of the company and 2) elucidate a mathematical modelling praxeology. In industry, factors such as the availability of technologies (conditions) or deadlines (constraints) could determine the mathematical tools to be used, and the way they will be applied. To understand how the industry uses, produces, and validates mathematical knowledge, we focused the questions on understanding its internal dynamics (e.g., hierarchy, project stages, type of clients, etc.). For the identification of the task and techniques, questions explore three approaches: the types of tasks; the collection, analysis, and data representation; and the decision-making. The first approach identifies activities in engineers' routine to determine the relevance and frequency of each task, to distinguish which are institutionalised practices and which may be unique, context dependent. Collecting, analysing, and representing data relies on methodological decisions (e.g., type of data and its management), which may be linked to mathematical tools (e.g., trend analysis, graph representations). The decision-making involves techniques, feasibility, and internal validation (e.g., criteria to choose techniques leading to technologies). Consequently, a full understanding of the mathematical modelling praxeology may require following interviews that detail techniques, technologies, and theories.

The Expert's Interviews

The consultation with an academic expert has served to situate engineers' practices within the academic context. The expert interviews aim to reconstruct the same task identified in the engineers' workplace. In this way, we obtain a complementary perspective on the mathematical activity, free from the practical constraints of industry.

This allows the elucidation of mathematical content that may be only partially visible in practice. Moreover, comparing both approaches makes it possible to identify points of transition between these two institutions and to highlight tensions between the expected academic resolution of problems and how they are addressed in workplaces. In addition, this re-examination provides insights into how these mathematical processes in practice may be adapted to be recognised and accepted in mathematical coursed from engineering education.

DATA COLLECTED

Two studies were conducted following this methodological process: one with a Mexican engineer and another with a French engineer, the latter still ongoing. The first case (Mexico) served as the empirical basis for the development and refinement of the methodological approach described above. The second case (France) constitutes an additional implementation intended to test the scope of the methodology in a different industrial context. In both cases, interviews were video-recorded and transcribed for further analysis. The Mexican participant is a mechatronics engineer with three years of experience in industry working at Company W, dedicated to specialised projects for the automotive sector. Three sessions were conducted with him: the follow-up interview and two subsequent interviews examining the techniques from the task identified initially.

The French participant specialises in real-time systems, holds a PhD in automation, and has seven years of industrial experience. He is currently a technical manager at Company X, which provides computer-science services, particularly in automated systems programming and software development. The follow-up interview has been completed, but further interviews are still required to detail a type of task and its techniques.

The expert consulted in the second stage of this methodology, specifically for the re-examination of the Company W case, is a researcher in control strategies who holds an electrical engineering degree and a PhD in automatic control. His trajectory includes three years working in a company dedicated to industrial automation of processes. Three one-hour sessions were held with him to solve the task performed by the Mexican engineer. All sessions were audio-recorded, and the expert provided additional material, including research papers and program simulations to support his approach.

DATA ANALYSIS

Based on the information obtained through the interview process, a type of task related to mathematical modelling emerged from the analysis of the engineer's discourse. The associated resolution process was examined through the identification of potential strategies, the appraisal of alternative approaches, the selection of a resulting technique, and the validation of these choices made by the engineer. This analysis was inspired by the notion of epistemic tasks (Castela & Romo Vázquez, 2022), particularly those related to appraisal, motivation, and validation. These categories helped to examine

how engineer evaluated possible approaches, justified the selection of particular techniques, and validated the solutions adopted in practice. Particular attention was given to whether these moments reflected institutional conditions and constraints, or whether they appeared as explanations for the effectiveness or relevance of a technique. On this basis, the corresponding praxeology was reconstructed.

The Engineer's Practices in a Mexican Industry

As described by the Mexican engineer, the company W operates through an interdisciplinary dynamic where projects are assigned to teams composed of five engineers with different profiles. He noted its hierarchical organisation, emphasising that managers intervene in major decisions, imposing strategies considered effective, which shows a restriction in the range of techniques available to face tasks. The role of the clients also emerged as a significant constraint, as project requirements are defined by them. As he explained, "The client provides us with precise parameters... and the objective is well documented". He added "If we choose a power source... perhaps the client has an agreement with a brand, then the client chooses... [In a specific case] certain instruments that we had already chosen were changed once the purchase order was placed, so we had to adapt what we already had to what the client requested". The interviews showed other factors influencing the choice of techniques, such as implementation time, functionality, scalability, and budget. As the engineer explained: "In this type of company, the main objective is speed of development… If PID is the fastest to implement, we choose that. If, due to the complexity, fuzzy control is faster, then fuzzy control is used". He added "If the strategy is not scalable, it is not optimal, because each new implementation requires development, which generates costs. The higher the costs, the less profitable it is." These elements are institutional constraints. The type of task identified – namely, the implementation of a PID controller for torque regulation in a magnetic particle brake –, consisted of a test where the objective is to maintain the same back-and-forth movement on a motor regardless of load changes applied by a brake. Interviews allowed us to reconstruct techniques as practices shaped by their contextual factors, showing the adaptation of mathematical knowledge in practice. For instance, in this case, the engineers initially attempted to manually set the three parameters P, I, D, following a common practice. As he explained: "If it is very simple [the system], you do it manually, and in an hour or an hour and a half you get the values for all three". In this case, the manual set was discarded due to the system's complexity.

A second approach involved developing a differential equation to represent the motor-brake dynamics to derive a transfer function. The model's complexity and the limited time available led them to change the technique and treat the system as a black box. The final technique consisted of developing a program that automatically tested different combinations with the system functioning in real time. Parameter ranges and order of variation were guided by the engineer's previous practical experience and their interpretations of control theory. As the engineer stated, "If P is too large, you reach the target very fast but with high oscillation". He added "I helps to stay within the curve

[desired behaviour] and avoid rising too much" and "*D* works with future errors." The program executed iterative small changes, saving results in spreadsheets. Parameter selection was determined by performance rules established by considering acceptable error values, the system's graphical behaviour and clients' final validation. We observed that engineers mobilised knowledge related to control theory but reinterpreted it empirically. Their explanations about the PID parameters reveal mixed conceptualisations: while the effect of *P* on overshoot is correctly identified, *I* is associated with reducing oscillations (a function typically attributed to *D*), and *D* is linked to "future errors", a description closer to the integral behaviour as *D* works with the error rate. These reinterpretations can also be understood as elements of a personal praxeology that engineers progressively construct with their experience of their workplaces, developing their own techniques to address this type of task and elaborating justificative discourses (technologies) that are coherent within the institutional constraints of Company W, where practical validation is accepted.

The Engineer's Practices in a French Industry

The French case provided an initial understanding of Company X. Its internal organisation follows a hierarchical structure led by a manager in charge of administrative responsibilities, followed by an operations manager who oversees the commercial and technical dimensions of the projects, and finally the engineering teams who develop the projects. The interviews helped to clarify the engineer's current role in Company X as a technical manager, where he focuses less on direct technical development and more on technical team leading, as stated by him: "[I do] day-to-day monitoring of teams, listening to them... Anticipating their problems and working with them". This structure suggests the presence of institutional constraints influencing engineers' autonomy regarding decision-making in techniques' selection. However, a more detailed description of their roles may be necessary to conclude it. The engineer's discourse also showed the client's main role in project development and validation, which imposes constraints on the design process. As he explained: "Customers come to us to test their project in simulation mode, potentially several times, until they say we're ready to go live at the customer's site". He noted that "Clients set a certain standard for us […] but all they want is for it to work". This illustrates how functionality operates as a minimal requirement defined by the client, constituting a criterion of validation. Other institutional factors, particularly time and budget, also emerged as major constraints. As he stated: "The deadline is the main criterion... For us, the cost is closely linked to the deadline: once we have purchased the materials, our only cost is the human time devoted to the project. So, if we manage to meet the deadlines, the cost will be under control. Right now, the deadline is the priority". The interview provided an overview of two projects: the modification of a toll barrier system, where the client did the physical component replacement while the company X adapted the control logic; and the migration of a loading system in a refinery, involving the programming of five automatons for managing the tanker train loading process. The two projects may be explored in depth during subsequent interviews to

clarify the tasks and techniques in detailed, and therefore, elucidate the mathematics. The analysis of Mexican and French cases reveals similarities and differences. Both engineers revealed key institutional constraints such as the client, who defines objectives, sets operating criteria, and validates processes and outcomes; time and budget as factors that restrict the range of feasible techniques to those offering fast implementation and practical viability; and how functionality is prioritised over formal justification. These elements illustrate that the choice of techniques depends not only on technical or mathematical considerations but also on organisational, economic, and functional demands. The cases differ in the engineers' roles. The Mexican case is directly involved in technical development, whereas the French engineer owns a managerial and supervisory position, which results in different types of tasks assigned and therefore different praxeologies developed in each case.

The Expert's Perspective on Mexican Case

The Mexican case provided a mathematical modelling praxeology; however, its empirical approach could lead to difficulties in being brought into engineering education. For this reason, during the second stage of our methodology, we consulted an expert in the field of automatic control. This expert provided us with access to an academic praxeology where mathematical tools are explicitly mobilized to obtain the model engineers needed to design their controller. This process also revealed the process of abstraction of the physical phenomena to be converted into a model from the control theory perspective, as well as the expert's intention to control the complexity of the model to make it reproducible in educational settings but at the same time keeping its efficient in practice. During these sessions, the expert proposed a solution based on the modelling the brake, while recognising that an accurate representation would be highly complex due to its electromagnetic and fluid-dynamic characteristics. Therefore, he suggested a simplified approximation based on an analogy with a well-documented system: the magnetic levitator. The analogy was based on comparing the operating principles of both systems: in the brake, an energised coil generates a magnetic field that alters particle viscosity to produce torque, whereas in the levitator, a magnetic field produced by an energized coil suspends an object. As the expert explained: "This model [the levitator] is already known... Normally, it represents a distance, but you can adapt it; it's similar".

Starting from this analogy, he adapted the levitator equation (Balko & Rosinová, 2017), by redefining variables, neglecting terms like gravity, and introducing a motor-speed-dependent term. With these modifications, a simplified differential equation could be obtained as $\frac{d^2x}{dt^2} = K_i u(t) - k_m|\omega(t)|$, where $x(t)$ is a magnitude related to the braking torque; $\omega(t)$ is the angular velocity of the motor; $u(t)$ is the input current; and K_i and k_m are constant proposed to encompass non-linear phenomena that are not explicitly modelled. From this, diverse control design methods could be applied. We observed that although the engineers in the Mexican case initially attempted to follow a more academic modelling path, the institutional constraints of industrial practice, and the

complexity of the modelling made them manage the task from an empirical approach, while the expert could abstract, simplify, and select models without the restrictions that shape workplace activity. Moreover, he possesses a level of academic training and professional experience that enables the use of theoretical tools not necessarily accessible through university education alone.

DISCUSSION AND CONCLUSION

The analysis of the Mexican case allowed us to develop and progressively structure a methodological approach for identifying mathematical modelling praxeologies created and used in the engineering industry. The French case, currently ongoing, provides an additional context for examining the scope of this approach. Although one member of the research team has an engineering background, understanding how engineers mobilise mathematics in industrial tasks proved difficult, highlighting the opacity of workplace mathematical activity for external observers. The interview design, grounded in praxeological components and in the institutional conditions and constraints associated with an engineer's work, proved essential for identifying praxeologies and for uncovering several factors that shape how tasks are carried out. Client's expectations, available resources, and time-sensitive decision-making emerged as key elements influencing the choice and development of mathematical modelling techniques. The Mexican case revealed the considerable difficulty for external observers in distinguishing between personal and institutional praxeologies in industry. This challenge led us to consult a second institution, the academic institution, to clarify the goals underlying the engineer's actions, understand why 'the personal mathematical-modelling praxeology' functioned effectively, and examine how a mathematical technique treated as a black box in industry could be unfolded into a white box from an academic perspective. This analytical step subsequently led to the explicit formulation of a two-stage methodological approach.

From this analysis, we argue that a praxeological reference model, informed by both institutions and centred on the same type of task but articulated through different techniques, can serve as a foundation for designing instructional proposals in engineering education, whether in mathematics courses or in modelling-focused courses. Such a model would make visible the transitions between workplace and academic mathematical practices, thereby strengthening the connections that engineering students must learn to navigate.

ACKNOWLEDGEMENTS

This work forms part of the research carried out within the DeMETeRE project, and the financial support provided by this project is gratefully acknowledged.

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Mathematical modelling for future teachers: Case of sea tides

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Keywords: mathematical modelling, general sine function, teacher education, Anthropological Theory of Didactic (ATD).

INTRODUCTION

This study forms a part of a broader project examining the current state of mathematical modelling in Croatian secondary schools and university teacher education. While many curriculum documents worldwide explicitly emphasize the central role of mathematical modelling in school mathematics, the Croatian National Curriculum for Mathematics adopts a less pronounced perspective. Modelling is listed as one of five general competencies, yet specific objectives and related sub-competencies are not further elaborated. Likewise, mathematical modelling is not yet systematically addressed in Croatian teacher education or professional development programmes. This aligns with the observation that "a widespread difficulty in mathematics teaching is the students' (and sometimes teachers') lack of preparation to link the knowledge acquired in the classroom to real-life problems" (Cabero & Winsløw, 2023).

THEORETICAL FRAMEWORK AND CONTEXT

Our analysis is framed within the Anthropological Theory of the Didactic (ATD) (Chevallard, 2019), which describes all mathematical activity as modelling characterised through praxeologies – at the level of *praxis* (how an activity is carried out) and *logos* (why it can be done in that way). ATD also supports an integration-oriented approach to modelling, centred on the process of constructing a model and emphasising the interdisciplinary, recursive and reversible relation between a system and its model (Puchaczewski et al., 2024). Constructing a model activates a range of student praxeologies, but also makes visible, for instance, "missing praxeological development about quantities, units, and their measurement – the 'no man's land' in the mathematisation of quantities" (Bosch & Chevallard, 2024). Our research question is: *What modelling praxeologies can be identified while constructing a mathematical model of a periodic phenomenon, and how are they realised or missing in the praxeological equipment of future teachers?*

METHODS AND RESULTS

The study is based on 79 written exam papers produced by two generations of pre-service teachers in the course *Didactics of Mathematics 3*, in the 5th year of mathematics teacher education programme in Zagreb, Croatia. The data corpus

consists of students' written solutions to a modelling task on periodic phenomenon of sea tides, which is partially accessible to high school students.

The poster will present theoretical considerations, a qualitative praxeological analysis, and selected results from students' work discussing their praxeological equipment. The analysis of students' work is coded using the results of the praxeological analysis. To ensure consistency, interpretations were discussed within the research team. The results focus on students' (1) observations of the phenomenon (e.g., periodicity, possibilities for data collection), (2) introduction of variables and a coordinate system, (3) proposal of initial functional models (such as a trigonometric sine function or a piecewise linear function), (4) use of advanced models (e.g., using sums of trigonometric functions or a Fourier style sum to describe patterns with alternating high tides of different amplitudes), and (5) interpretation and analysis of parameters.

The results highlight the complexity that real-world contexts introduce into mathematics lessons when engaging in mathematical modelling, while also creating opportunities to identify and articulate elements of the practically inexhaustible range of modelling praxeologies – particularly those that are natural and accessible to (high-school) students. We also present preliminary results from an ongoing longer inquiry activity with the same group of students, which may provide insight into the recursive and reversible nature of the modelling process.

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The dialectic of systems and models at the interface between mathematics and advanced physics

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In Mathematics Education Research, modelling has often been modelled as a cycle. In this paper, we wish to question this perspective by considering a context where professional modellers work: the interface between mathematics and advanced physics. To shade light on modelling processes there, we interviewed three researchers in mathematics on their relation to physics. We infer the modelling cycle is at best a special case, insisting on the contrary on the reversibility and recursivity of the dynamics of systems and models.

Keywords: mathematical physics, modelling process, study process, Anthropological Theory of the Didactic.

INTRODUCTION

In Mathematics Education Research (MER in the following), modelling has often been modelled as a cycle. In this case, two entities are distinguished: the so-called "mathematical" and "real" worlds, the latter being sometimes called "extra-mathematical" (Greefrath & Vorhölter, 2016; Ye et al., 2024; Barquero, 2023). In this paper, we wish to question this perspective by considering a context where professional modellers work: the interface between mathematics and advanced physics. Our methods consists in considering teaching and learning situations outside school in order to better characterise how teaching and learning unfolds within the school system. Within the framework of didactics of mathematics seen as an experimental epistemology (Blum et al., 2019, p. 15), studies of this kind play a key role in the *a priori* analysis of didactical situations (Barquero & Bosch, 2015).

In the continuation of an investigation of transposition phenomena at the interface between mathematics and quantum mechanics (Lombard, 2023), this study may be seen as a report on the dynamics, through various modelling processes, of pieces of scholarly knowledge, that is, at an early stage of their didactic transposition (Chevallard & Bosch, 2020). To shade light on such processes, we interviewed three researchers in pure mathematics working at a French university and questioned them about their relation to physics. To make both our questioning and our analysis more precise, we adopted the epistemological framework provided by the Anthropological Theory of the Didactic (ATD; Chevallard, 1980; Bosch & Gascón, 2006; Chevallard et al., 2022). It comes with a collection of theoretical tools that we used in order to identify and observe patterns throughout our inquiry, the most significant of which for the present work being the dialectic of systems and models (Chevallard, 1989a; Garcia et al. 2006). The goal of the interviews was then to recognise effective dynamics of this dialectic, at the scholarly interface between mathematics and advanced physics, asking ourselves the following research questions:

What does the dialectic of systems and models actually look like at the interface between pure mathematics and advanced physics? In particular, what can be said of the recursivity and reversibility between systems and models in this context?

We consider researchers as people who study in order to learn, and we wish to better characterise the dynamics of the study process they live, as well as the ones their work permits. So, we investigated the source of their questions, the role of modelling in their response, and the destination of the answers consequently produced on both sides of the interface. In particular, the framework of the ATD offers a common ground to model both study processes (see for instance Bosch, 2018) and modelling processes, the nexus of which is rather a blind spot of MER.

THEORETICAL FRAMEWORK AND DATA COLLECTION

In an article recalling the history of the subspace of MER devoted to mathematical modelling, Barquero (2023) presents the most standard approaches following the workings of the CERME group on the matter. When she comes to presenting the anthropological approach, one may read:

> Since the initial works by Chevallard (1989a, p. 53), he described modelling processes in different steps: (1) delimitation of the system to be studied, specifying its relevant aspects, (2) construction of model(s) and work within the model; (3) work with the model to generate knowledge about the system. […] The interest and value of a model lies in its ability to produce knowledge about the system being modelled that another approach would not provide as easily. Work on/within the model may involve the construction of successive models, better adapted to the study, and which imply a redefinition of the systems to be modelled, so that the systems, models, and knowledge generated are part of the new systems to be considered. In the interaction or dialectic between system definition and model construction, the reversibility of the system-model relationship and the recursivity of the modelling process are two key properties for understanding the conceptualisation of modelling proposed in this theoretical framework.

In this spirit, our study aims at focusing on three aspects of the articulation of systems and models: its so-called cyclicity, its reversibility and its recursivity. As Marí et al. (2025) explain:

> The models used (if they are already known) or constructed (if they need to be created) are added to the system under study, transforming the model-system set into a new system, which might be modelled again. In this sense, the modelling process constantly evolves generating successively new system-model sets, each step calling for more complex models. This property can be identified as the *recursivity* of modelling. The *reversibility* property in the modelling process highlights the fact that, at certain steps, systems and models can reverse their functions: for instance, when a previous system is used as a model of a new system or of the used model.

This characterisation was first given by Chevallard (1989a) but scarcely scrutinised since.

In the context of a previous study of transposition phenomena at the interface between mathematics and quantum mechanics, this dialectic of systems and models had been used as an analytical tool to probe epistemological dynamics at the taught

level (Lombard, 2023, section 10.3.1). We had also interviewed researchers in theoretical and experimental physics in order to survey the scholarly practice (Lombard, 2023, section 7.1). However the emphasis was not on the dynamics of systems and models. In this paper we combine both focuses, by presenting interviews of scholars and analysing them thanks to the theory of models provided by the ATD.

Concretely, we interviewed three researchers in pure mathematics working at a French university. In the Department of Mathematics, their research falls within an axis devoted to the study of geometry, topology and algebra. However, they all engage with mathematical physicists and theoretical physicists, hence our choice to interview them. R_1 works in knot theory, while R_2 and R_3 work in algebraic geometry. They interact with physicist who study theoretical high-energy physics, more specifically (topological) quantum field theories. During the two-hour long conversation we had with each of them, we asked our interviewees about their past and current collaborations with physicists: the topic, the mathematics at stake, the origins and the background of the collaboration. The goal was, firstly, to collect first-hand information about the pieces of knowledge involved, and secondly, to record the various relations to those pieces of knowledge (Chevallard, 1988) which cohabitate at the interface: reading and writing methods, modelling and validation habits.

The resulting interviews were analysed using the theoretical tools just mentioned, using an overall approach quite typical of didactic transposition studies (see Bosch & Gascón, 2006, for an overview). The following excerpts are not included merely as anecdotes, but summarise the recurring themes identified in the analyses derived from these interviews

QUESTION-ANSWER DYNAMICS ON BOTH SIDES OF THE INTERFACE

An important feature of the interface between mathematics and physics is the presence of objects which are shared by the communities, even though this may not be apparent at first sight:

> We really tried to make an effort to understand each other, and so typically there was a moment when the PhD students and post-docs at the conference [who were mathematicians, mathematical physicists and theoretical physicists] got together to try to explain to each other what is the Jones polynomial [...], one of the basic invariants of knot theory [...]. And so we spent an hour or two saying, let's start with something very simple [...] everyone gives their definition and we try to see why it's the same thing. (R_1)

This allows for a specific work on concepts and their meaning on both sides, which R_1 called a “dictionary phase”. Such objects will then appear at some step in the researchers' studying process, either as starters or as intermediary models. Hence the following question regarding an object living at the interface, a question which has to do with modelling strategies: “Can I reinterpret it entirely within the framework I am familiar with, or does it require changes, and if so, what kind?” (R_1)

So, objects are shared by communities of practice, even though they may be approached qui differently by their members. This leads to discrepancies regarding their use and manipulation, the questions they may raise or which, on the contrary,

are stifled. Calculating Feynman integrals (a kind of path integrals) is an example of how such discrepancies may fuel study processes at the interface:

> Basically, since Feynman, we know that to understand the interactions between elementary particles, we need to calculate integrals [...] particle physics produces integrals that are very complicated to calculate, and they really come from physics, meaning they are not objects that a mathematician would have pulled out of a hat, saying, "I want to calculate this integral." (R_2)

Adopting the anthropological stance on modelling, the initial integral may be seen as a "system" which is recursively modelled, calculation step after calculation step. Such reformulations are deemed necessary in order to provide answers regarding this system, but the corresponding gestures may be quite different depending on the community:

> For some people, calculating means having a value with thirty digits after the decimal point for a certain quantity, and for some people, who are more like the people I talk to, calculating can mean much more abstract things, such as expressing certain classes of integrals in terms of already known functions or things like that, or trying to understand identities between integrals. (R_2)

Actually, the "system" (the integral) from which the calculation process departs may be constructed differently in function of the questioning at stake:

> And then there's also the type of integral that people look at, i.e. do they look at integrals that are very interesting from a physical point of view, do they look at physical models that are somewhat simplified compared to reality, so to speak, etc. (R_2)

So, we already witness the relativity of the epistemological status of being a system or a model. Then, the "reality" mentioned in the previous excerpt denotes written models, in terms of integrals, of sometimes hypothetical phenomena, like field excitations, which are described or alleged in theoretical particle physics. The very constructed, already quite modelled character of this "reality" is actually noticed by R_2 herself ("so to speak"). Rather, the "system" is what poses question:

> Physicists working on these objects tend to know how to do calculations without really knowing what the definitions of the objects are, so [...] sometimes they can say things that don't make sense, but overall they are people who know how to do calculations. When they say that two quantities are equal, it means that they are equal. (R_2)

As we will develop shortly, the notion of making sense is quite idiosyncratic. In any case, physicists produce equalities, which model the equations they started with. Those equalities question our interviewees, as they were obtained by people embodying rather different praxeologies (Bosch & Gascón, 2006). It appears clearly that those writings — which function both as models resulting from the physicists' work and as systems to be investigated by the mathematician — cannot be called "real" neither "extra-mathematical". At the same time, they are robust and mysterious enough to raise questions, hence the need to model them, to work on and with them.

Interestingly, this is an instance of an "informational leap" (Brousseau, 1997, p. 98), when students understand the assignment but do not know yet how to solve it. The objects dealt with by physicists should be familiar enough to mathematicians for

them to understand what is at stake, yet dissimilar enough for them to have work to do. Hence the generation of a study process, which is also a modelling process: "You may not necessarily understand why they say things, but normally you understand what they say." (R_1)

From this point onward, the question may evolve separately in mathematics, which means (crucially) that the dynamics of systems and models goes on, simply following the practice of manipulations and validations proper to this community. Models and systems from various sub-fields of mathematics will then play a homologous role as those from physics here. This is another way to question the existence of an alleged "reality" or "extra-mathematical world" that would go with any modelling process:

> [the field now known as quantum topology] has become independent of the questions that initially motivated it. It is actually related to the theory of quantum group representations, the theory of some moduli spaces in algebraic geometry. There is a whole interesting combinatorics of this galaxy of invariants, interesting algebraic relations between them, and so in fact it is now a field that has broken away from the questions that gave rise to it, and which has its own intrinsic logic. (R_3)

Hence a rich variety of relationships between mathematical and physical modelling processes, on one side of the interface or the other. In our context, it seems that modelling intra-mathematically amounts to moving away from physical instances. One witnesses a spectrum of modelling practices which resemble more linear paths in divergent directions fuelled dialectically than cycles fixed on the same ("real") point:

> Sometimes, with some things I do that were very motivated by and closely related to physics, I can clearly see that people who do physics are drifting away […] they feel that you can provide them with tools that can be useful to them, but when it goes too far [they] look at you and say: OK, I'm happy to talk with you about derived stacks informally, for the sake of intuition, but I'm not going to read the technical stuff you write because, in fact, I'm not interested in it. It's too far removed from my concerns as a physicist. [...] That doesn't mean you won't cross paths again later on, eh (R_3)

A modelling process which takes the form of a cycle then appears as a somewhat fortunate special case. This effect is strengthened by considering large scale processes, where cycles from physics to physics via pure mathematics famously caused the bewilderment of both actors and observers of the interface between pure mathematics and advanced physics (Wigner, 1960). In the following section, we will try to unveil smaller scale phenomena, which point to a more balanced relationship between mathematics and physics at this level of the practice.

PHYSICAL MODELS AND PHYSICAL INSTANCES

At such scales, the dynamics of systems and models at the professional level and at the interface appears as quite different from what is customary observed in schools. Instead of seeing mathematics as a powerful heuristic tool, if not a perfect analyser of physical situations, it appears that physics also provides models for mathematicians to guide their study.

> What surprises me most is that [physicists] have an understanding of the underlying system that is completely beyond me, and as a result they are able to make incredible predictions, which are usually correct. And that gives us things to work on. (R_1)

In the previous section, we described this kind of assertions as physical systems, which may be modelled mathematically in order to answer some of the questions they raise. However, such propositions could also be seen as physical models of cases which are particular from a mathematical point of view. Physicists would extend their domain of validity, thus constructing a more general model than what mathematicians would naturally do. Mathematicians may then try to do the same, according to their own norms of validation, thus searching for alternative models, grounded otherwise:

> When a physicist tells you, "Well, there must be something there, in that place, that has such and such properties [...] and we'll get it by counting such and such," to me, it just comes out of nowhere. And in fact there's a reason behind it, and even without being given the reason, once they've given you all the properties, then you can try to work on the maths side and formally define that thing. (R_1)

Physics raises questions about objects which, from the point of view of mathematics, are special cases. In this respect, their models, generated by physicists, may be modelled themselves, mathematically this time, to raise the level of generality:

> So clearly, yes, the questions that arise in physics lead us to develop techniques for which we would not have had any motivation without these things. Then [...] partly due to culture, habit, we really [...] try to implement techniques that will always be general. That is to say, we start from the implicit principle that Feynman integrals are not that specific, and so we will try to implement techniques that do not only refer to Feynman integrals but refer to integrals or certain types of integrals in general. [...] Because we know from experience that it will actually be simpler, because we won't be bothered by a whole bunch of peculiarities, situations that are actually of no interest. (R_2)

Such more general models may then play a role in physical explanations of new phenomena, compared to the physical model everyone started with. In this case, the modelling process may appear as a cycle: from physics to mathematics and back. Even then, the fact we come back to the initial system is questionable though, as the knowledge physicists have of the initial system will have necessarily improved. That is, having moved on in terms of questioning, one may still remember the point of departure as a measure of the path travelled, but the view on it has changed; new questions emerge at the same time as old ones find answers. Put another way, there are no pieces of knowledge, only relations to pieces of knowledge (Chevallard, 1988), and those relations have changed. In this respect, we never go back, but rather adapt to perpetually evolving ecologies.

So, as we hinted at in the previous section, the ubiquitous vision of modelling as a cycle (Greefrath & Vorhölter, 2016) which lives in the noosphere[1] is merely a special case of a more general model. However, this refined model becomes necessary when it comes to accounting for the intermingling of study and modelling processes at the

[1] "the people who devote time and energy to thinking about the teaching of mathematics" (Chevallard, 1989b)

interface between pure mathematics and theoretical physics, a fact which should draw the attention of mathematics (or physics) education researchers.

Coming back to our context, let us remember that the questions arising from physics live and develop in mathematics because they refer to mathematical objects. So, they could as well live and develop in mathematics:

> The mindset is that integrals from physics are not that specific. So, automatically, when we think about them, we put them into a larger class that has nothing to do with particles, but only with the concept of integrals and the concept of algebraic varieties (R_2).

Sometimes even, an answer may be given to physicists because it is not very costly, from a mathematical point of view, and may be of interest for them: "We realised that, with all the formalism we were putting in place, we could answer a question that people were asking themselves in string theory." (R_2)

Here, we touch upon the ease one has to work with a given kind of models given her praxeological equipment (Bosch & Gascón, 2006), as well as the strategies existing in every scientific field to promote one's own questions and the diffusion of specific answers to them (Bourdieu, 2004). However, and more generally, this shows how the circulation of models at the interface is actually both ways, mathematical systems being theorised by physical models, and the other way round. R_3 even mentioned a striking example involving Edward Witten, a theoretical physicist famously known for being awarded the Fields Medal in 1990. Even though Witten's practice cannot be considered as common, it reveals still uncovered facets of the epistemological relationship between pure mathematics and theoretical physics:

> [Witten] shows that topological field theories can be used to rediscover things that are fairly well known to mathematicians, so he has a whole approach to Morse theory, similarly, using path integrals. [...] Morse theory is the fact that you can calculate the homology of a manifold with certain Morse functions. It was something that took a long time to develop, and then this guy comes along and writes a path integral and finds, not all of Morse theory, but part of it, just like that! That really impressed people. (R_3)

All in all, these are illustrations of the fact that systems and models are epistemological functions rather than absolute properties, and that pieces of mathematics should not be confined to the status of models.

MATHEMATICAL WORK ON PHYSICAL MODELS

The dialectic of conjecture and proof is another significant terrain to appreciate the modelling dynamics between these two disciplines. Indeed, physical models of mathematical problems can provide deep insights to tackle conjectures:

> There is a kind of conjecture, that we have several topological invariants. There are a few that we know how to define well […] but we conjecture that we are missing a whole bunch of them. […] But we conjecture that there are about as many as there are pairs of integers. […] If I want to try to define these things, well, I have a few ideas, but I'm still starting from scratch, and, well, typically I've heard lectures [by a theoretical physicist] where he explains, precisely, between which pairs of such homologies there will be morphisms, which ones can dominate which ones, what kind of counting needs to be done to deduce at least their dimensions, things like that. And so there we can really take

ideas and say, OK, if we assume that what he says is true, here are a number of properties that my invariant must have, and that greatly restricts the choice. (R_1)

To better understand the dynamics at stake, one should consider this refined version of the conjecture, provided by physicists, as a model extending to a wider class of mathematical objects something that had been noticed only in some special cases:

if a mathematical phenomenon occurs every time we test it on an object, and we feel we have tested enough objects, then it's not crazy to think that it's a general phenomenon. But then, a physicist will say, well, it's a fact, and a mathematician will say it's a conjecture. That's the difference in approach. (R_2)

Such an extension, in the realm of physics, of a peculiarity from the mathematician's point of view, is grounded in proper modes of validation, which are not purely deductive but rather involve a whole semantics in terms of physical objects:

Physicists conduct tests on classes of Feynman integrals, observe phenomena, and conjecture or assert that these phenomena are entirely general, draw conclusions from them, and then, well, it is the mathematician's job to try to prove what has been conjectured or asserted by physicists, yeah. (R_2)

Given such physical models, mathematicians often seek an equivalent formulation. The first reason is, they simply wish to reformulate the same result in a manner that would be better grounded from their point of view (which is of course a relative notion). In this case, the mathematical model is often trivial. The second reason is, mathematicians will now look for an even more general result, which would then encompass the physical model as a special case. Then, this mathematical model would appears as a model of the model. We find our feet: maths is a model for physics. In both ways, the physical result functions like a model, which validity should be tested mathematically:

You have to look at what class of examples was actually tested, and whether, for the question at hand, I consider this class of examples to be broad enough. Or whether, in fact, when I look at the class of examples, it is very specific, which makes me a little doubtful. (R_2)

The dialectic between systems and models at the interface then enriches the practice on both sides. Then, some physicists will develop tools which only live in pure physics, while others will pursue a study path closer to mathematical physics:

very quickly we replace questions about integrals with questions about algebraic geometry, particularly cohomology groups of algebraic varieties, and as a result, in a certain community of physicists, well, they are also increasingly manipulating these concepts: cohomology of algebraic varieties, the theory of motives and a whole bunch of things like that, which are quite sophisticated mathematics, let's say, quite modern (R_2)

On the other side of the spectrum, mathematicians develop their own practice. Its relative obscurity, even for theoretical physicists, mirrors the mysteriousness of the physicist' insights we mentioned in the previous section. R_2 thus compares her practice with that of mathematical physicists:

So this is really what I would call a maths article, there is a statement and there is a proof. And they [theoretical physicists] demonstrate, let's say, a weak version of what we demonstrate, which does not answer the conjecture but it still proves something very

> interesting, which is, let's say, a weak version of the conjecture. […] Our article is unique in the sense that the methods used are quite advanced, […] we really get our hands into the cohomology of algebraic varieties, and the physicists in question are not really capable of doing that, not with the level of experience and technical expertise that we have in these areas. (R_2)

The relativity of expertise appears clearly, together with the various ways in which models from one discipline may fuel the other. Even at the rather confidential interface between pure mathematics and mathematical physics, we can see practices differ, transpose, intertwine. This last excerpt shows the vivacity of such exchanges, throughout the spectrum of modelling practices habiting this interface. In this last case, the function of mathematics as models of models stands out:

> There is a theory called Chern-Simons theory, which is a topological field theory. So, it has changed a bit recently, but topological theories are really, for physicists, theories that have no physical content. They were really just toy models. Well, actually, for mathematicians, it's already very complex, it says a lot about topology, it produces fantastic invariants. In fact, it's great, it's fun to think that it's a toy model for physicists, or something that has little physical interest, where in fact you've simplified a lot of things. […] The problem is that the mathematically rigorous definitions of Chern-Simons theory are not defined as physicists introduced them. [...] Physicists write formulas, they have a Lagrangian, they use the principle of least action, they write a Feynman integral. This produces things, and what it produces, the way it is organised, has a mathematical structure, and this is what mathematicians call topological field theory. So already, what physicists call topological field theory and what mathematicians call topological field theory are not the same thing. What physicists call this way is a set of initial data and a construction that is not completely mathematically rigorous. What mathematicians call topological field theory is the result of this construction, which we know how to produce in another way mathematically. (R_3)

Interestingly, the very notion of elementariness appears to be relative. It appears the objects of the interface are never really the same after all. Systems of various kinds, rival models of numerous levels of generality, grounded in as many practices of validation: once we have looked more closely, once we have modelled it carefully, the “modelling process” we started with looks inevitably different.

CONCLUSION

Thanks to interviews conducted with professional actors of the contemporary interface between pure mathematics and advanced physics, we could peek into the behaviour of the dialectic of systems and models before those pieces of knowledge are transposed didactically. Thanks to the flexibility of the general model provided by the Anthropological Theory of the Didactic, we could decrypt some of the phenomena taking place there. We questioned the “reality” of systems, we recognized the high-frequency recursivity, as well as the reversibility of the alternation of systems and models. The modelling cycle thus appeared as a fortunate coincidence, opening our mind on what modelling dynamics could actually look like in classrooms.

Indeed, this survey could be used as an epistemological reference model (Bosch & Gascón, 2006) to better investigate modelling phenomena as they actually unfold in academic contexts. Following the principles of didactic engineering as an observation apparatus (Brousseau, 2006; Barquero & Bosch, 2015), the dynamics of systems and models in the classroom could, in well chosen conditions (Bosch, 2018), exhibit similar aspects to those we unveiled here. Then, this preliminary model could work as a monitoring device for both the analyst and the study aide.

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Computational thinking and programming in first year STEM courses at bachelor level

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Keywords: computational thinking, programming, STEM-education.

INTRODUCTION

Following the strategy to focus on so-called "generic skills" (analytical and logical thinking, problem solving, interdisciplinary competence and communication skills) in all bachelor courses at the Faculty of Science and Technology at University of Bergen, we study how computational thinking (Wing, 2006) and programming (as part of generic skills) is being integrated in beginner courses for various bachelor's degree studies at the faculty. Every student at the Faculty of Science and Technology will in their bachelor´s degree take a course focusing on the technical programming aspects that are needed, so that they can include programming and computational thinking into their own bachelor´s program. In this project we will focus on how the curriculum and teaching material of various courses (first- and second-semester courses of calculus, linear algebra, chemistry, physics, biology) contain traces of computational thinking and programming, via document analysis. From this baseline, we will conduct further research into how the introduction of computational thinking and programming has an impact on the teaching and learning of these courses at the faculty. To get a deeper insight into the matter, we will in the extension of this short communication conduct (i) document analysis of the curriculum and teaching materials, (ii) interviews of teachers and (iii) interviews of students.

THEORETICAL FRAMEWORK

The project is grounded in the theoretical framework of Communities of Practice (Wenger, 2011). We connect the theoretical work to the current project in the same manner as described in a similar and still ongoing project, where a single mechanics course at another university was studied. We refer the reader to another paper for a detailed description of the domain, community and practice in the current project (Lie et. al. 2025).

METHODOLOGY

We focus on document analysis of curricula and teaching material for each course. The unit of analysis will be on how the curricula and associated teaching material contain traces of programming and computational thinking. Comparative analysis might be relevant, as we hypothesize large differences in computational focus, both between courses and across fields. The findings from the document analysis will be supplemented by content analysis on teacher and student interview transcripts. The methodological approach supports a well-founded and nuanced view of the state of the faculty's computational thinking integration at bachelor level in STEM courses.

PRELIMINARY RESULTS

Initial reviews of the curriculum for various beginner courses at the faculty reveal that references to programming and computational thinking are often quite vaguely written. "The student knows about standard techniques from computational science" and "The student can see simple, realistic problems from a mathematical perspective and solve them" are examples of vague references to computational thinking. More explicit examples include "The student knows how to use numerical methods for mathematical modelling in science", "The student can solve systems of differential equations numerically" and "The student can make simple computations within population dynamics and genetics". These are only a few of the traces we find in the curricula engaging computational thinking and programming. The communication will delve deeper into more examples and present how computational thinking and programming is made explicit in the teaching material of the courses.

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TWG3: Transitions to, across, and beyond university mathematics

Revisiting group axioms for prospective teacher training

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This article analyses the first session of a preliminary experiment conducted with third-year university students, as part of a wider research project dedicated to the design of a training module that revisits group theory from both mathematical and didactical perspectives. The analysis is grounded in Activity Theory, adapted to the didactics of mathematics. An a priori analysis, based on epistemological, curricular, and cognitive dimensions, anticipated students' possible activities in a task centred on the four group axioms. The a posteriori analysis compares these expectations with observed work. The session fostered reflection on the interdependencies between axioms, providing insights for designing future sessions supporting a "double institutionalisation" of group theory for prospective teachers.

Keywords: group theory, Klein's second discontinuity, prospective teacher training, Activity Theory adapted to the didactics of mathematics, case study.

INTRODUCTION

In France, prospective secondary school mathematics teachers usually study group theory, as part of a degree in mathematics, but often fail to see its relevance for their future professional practice (Zazkis & Leikin, 2010). This phenomenon is part of what is known in the literature as "Klein's second discontinuity" (Kilpatrick, 2019), which refers to the transition between prospective teachers' university studies and their future profession as secondary school teachers. However, several studies have identified numerous links between group theory and secondary school mathematics, be they explicit or implicit (Rolland, 2023, 2024; Wasserman, 2016).

This paper forms part of my PhD thesis [1], which focuses on developing a short training programme, or capstone course, for prospective teachers. The aim is to revisit notions of group theory covered in the bachelor's degree and to draw explicit connections with secondary education, both mathematically and didactically. Indeed, it does not seem enough for students to know mathematics well to be able to teach it well (Wasserman et al., 2019). They should also reflect upon *how* to teach mathematics. This endeavour has been coined as a "double institutionalisation" (Pézard, 1985, p. 4). The institutionalisation is the officialization of knowledge in class by the teacher. It is "double", because it applies both to the mathematical content and the didactical choices made by the teacher, made explicit to the students. In summary, my approach to addressing Klein's second discontinuity is to highlight and model disciplinary practices via explicit connections between university and secondary mathematics, and between mathematics and didactics (Grenier-Boley & Robert, 2024).

This paper analyses the first session of a preliminary experiment conducted with third-year university students, whose purpose was to test the relevance of certain choices for

the design of the module later implemented for prospective teachers at master's level. This session focuses on the mathematical content, specifically on the group axioms.

The research questions addressed in this paper are: What are the students' activities (in the sense defined later) during the session? How could they contribute to reducing Klein's second discontinuity? More specifically, what kind of sets do the students tend to work with, and which semiotic registers do they invoke?

THEORETICAL FRAMEWORK AND METHODOLOGY

This research falls within the theoretical framework of Activity Theory, adapted to the didactics of mathematics (ATDM) (Bridoux et al., 2016). The core hypothesis is that the analysis of the activities of students provides insight into their learning. An activity is defined as everything a subject does while performing a task, that is, actions, internal states, thoughts, etc. (Rogalski, 2013, p. 4). The students' activities can thus be only partially observed by the researcher, through what they say, write, or do.

Two main hypotheses underlie the acquisition of knowledge in mathematics, drawn from the work of Piaget and Vygotsky. On the one hand, the students should be working independently on the tasks proposed by the teacher (Piaget). On the other hand, the collective discussion during the session enhances the learning (Vygotsky) (Grenier-Boley, 2019, p. 76). According to these two hypotheses, it is thus relevant that our session allows the students to work independently, partly alone (in order to facilitate devolution [2] and tackle the proposed tasks), and partly in small groups. Working together should facilitate discussions when faced with an unusual task, that is when the direct application of knowledge is not sufficient (Robert, 2008, p. 38).

A semiotic register is a system of signs and rules that allows one to represent, express, and manipulate mathematical objects (Duval, 2006). An object always has multiple representations in different registers. Groups have, in particular, representations across many registers: natural language, algebraic, geometric, and tabular, to name a few.

To analyse what happened during the session, the researcher builds a reference named the "*relief on the concepts to be taught*" at the very moment of teaching studied (Grenier-Boley & Robert, 2024, p. 1462). The possible activities of students are then analysed by comparing what was expected in the relief and what we think a majority of students actually did (Bridoux et al., 2016, p. 191). This relief combines epistemological, curricular, and cognitive analyses of the notion to be taught. The epistemological aspect encapsulates its mathematical characteristics, including the contents' semiotic registers of representation; the curricular aspect corresponds to the programmes, or syllabi of courses taken in previous years or earlier in the year, while the cognitive aspect refers to the known difficulties of students. Given the lack of programme at university level, my curricular analysis is based on the comparison of three sets of lecture notes of an introductory course on group theory, while my cognitive analysis is backed by previous research (Lajoie, 2000; Rolland, 2023).

The *a priori* analysis aims to identify what could have been, but did not actually happen during the session. The lack of actions of the students becomes thus as informative as their actions. It is built thanks to the relief and gives access to the expected activities of the students (Grenier-Boley, 2019, p. 68). The *a posteriori* analysis consists of reconstructing the students' activities and comparing them with what was expected. However, since the actual activities are largely invisible because they are internal, we only have access to the possible activities. These are activities that we assume were performed by a majority of students, considering the analysis of the session, broken down into phases of work (collective, individual) (Grenier-Boley, 2019, ibid.).

RESEARCH CONTEXT

The preliminary experiment was conducted over four sessions, of one to two hours each. The class was led by the usual teacher, also a mathematics education researcher. The sessions formed a component of a course designed for students intending to pursue a master's degree in education the following year [3]. The course was attended by ten students, all of whom had previously completed an introductory course in group theory. During the first session, one student was absent, and the teacher divided the class into three groups. The students were not accustomed to working in small groups.

During each session, the class was recorded with multiple microphones, pictures of the board were taken regularly, and the written productions of the students were kept. After the fourth session, short group interviews were conducted with the students to collect their thoughts on the experiment.

A PRIORI ANALYSIS OF THE FIRST SESSION

This paper focuses only on the first session of the preliminary experiment, which lasted one hour. The session aimed to let the students work on the four group axioms [4]. Any axiomatically defined structure articulates underlying constraints with possibilities enabled by its combined axioms. The goal of the exercise is to show the role of each axiom, as well as their relationships. Indeed, they are all independent from each other, except for the neutral element, which is required for the invertibility.

Firstly, a discussion guided by the teacher should enable students to recall the axiomatic definition of a group. Then, they should collectively reflect upon "What is the purpose of groups?"

Research showed that students often struggle with the axiomatic definition (Lajoie, 2000; Rolland, 2023). It seems thus necessary to explicit it. Moreover, the question should uncover part of the conception the students have of groups. The value of studying groups is not always made clear to them; they often have to figure it out for themselves. Drawing on mathematician interviews conducted for the relief, I anticipate responses highlighting the unifying and classificatory role of group structures.

Secondly, the students should work together in small groups on the following exercise:

> Give examples of sets G endowed with operations $*$ such that

a. The operation $*$ is closed.
b. The operation $*$ is associative.
c. The operation $*$ has a neutral element.
d. All elements of G have inverses under the operation $*$.
e. The operation $*$ is not closed.
f. For each axiom, find a set G and an operation $*$ that satisfy that axiom and as few of the other axioms as possible.

I expect that students have never thought about axioms independently, since they are always introduced together in the lecture notes I analysed. I therefore assume that students will start by looking for a set with an operation that is closed, but will give an example of a group that they can use again for the first four questions. For questions five and six, I assume that most students will manipulate sets of numbers using the four arithmetic operations (addition, subtraction, multiplication, and division), as they will probably seek to restrict known groups. Indeed, the study in my master's thesis showed that the students were particularly familiar with these groups (Rolland, 2023, p. 37). They are also likely to manipulate simple sets composed of a few elements, in particular integers, such as $\{0; 1; 2; 3\}$. For the fifth question, I therefore expect students to give the example of $(\mathbb{N}\backslash\{0\}, -)$, which does not satisfy any axioms, or $(\mathbb{Z}\backslash\{-42; 42\}, +)$, which satisfies all axioms except closure.

The point of the exercise lies in question six. Once the students reach it, they will review their answers to the previous questions and will probably find that their examples they gave satisfy more properties than the one requested. The aim is to get them to think about the interdependencies between the axioms and the constraints they generate. I expect students to identify that for a set with a given operation, it is sufficient to remove the neutral element from the set to "lose" the identity axiom. Furthermore, I expect them to understand that it is not possible to find an example in which the elements are invertible without having defined a neutral element.

Regarding the lack of inversibility, I expect that some students will look for sets in which only one element does not have an inverse, while others will look for a set in which no element, except the neutral element, has an inverse. This is due to the misinterpretation of the negation of the statement "all elements are invertible". Indeed, Durand-Guerrier et al. (2015, p. 33) note that no student was able to make completely correct use of quantifiers in their definition of a group.

I expect that students will not attempt to define an operation using a table, as they have not worked with tables since the previous year. I therefore assume that knowledge related to the tabular semiotic register is not available. I also do not expect them to define operations by combining several arithmetic operations, as they are unfamiliar with this type of operation. In fact, of the sixteen students who completed the questionnaire I designed for my master's thesis, only eight attempted to determine

whether the operation defined on $\mathbb{R}$ by $a * b = \frac{1}{3}(a + b)$ was associative, and three of them had difficulty recognising the operation in question (Rolland, 2023, pp. 31-32).

The figure below (fig. 1) shows some expected answers for each combination of the axioms.

	closure	identity	inversibility	associativity
$(\mathbb{Z}, +)$	x	x	x	x
$(\mathbb{Z} \setminus \{-42, 42\}, +)$		x	x	x
$(\mathbb{N}, +)$	x	x		x
$([0, k], *)$, with $a * b = \lvert b - a \rvert$	x	x	x	
$(\mathbb{Z} \setminus \{-1, 1\}, \times)$, $(\mathbb{N} \setminus \{0\}, +)$	x			x
$(\mathbb{N}, *_1)$, with $*_1$ defined below	x	x		
({-1, 0, 1}, $*_2$), with $*_2$ defined below, or see the table of $*_3$ below		x	x	
$(\mathbb{Z} \setminus \{2\}, +)$		x		x
$(\mathbb{Z}, *)$, with $a * b = -a - b$	x			
$(\mathbb{N} \cup \{-42\}, *_1)$, with $*_1$ defined below		x		
$(\mathbb{Z} \setminus \{0\}, +)$				x
$(\mathbb{N} \setminus \{0\}, -)$, $(\mathbb{Z} \setminus \{0\}, -)$				

$$a *_1 b = \begin{cases} a + b & \text{if } a \neq b, \\ a & \text{if } a = b, \end{cases} \qquad a *_2 b = \begin{cases} a + b & \text{if } a + b \in \{-1, 0, 1\}, \\ 2 & \text{if } a + b \notin \{-1, 0, 1\}, \end{cases}$$

$*_3$	e	a	b
e	e	a	b
a	a	e	x
b	b	x	e

Figure 1: Examples of answers to the exercise.

A POSTERIORI ANALYSIS OF THE FIRST SESSION

Following a ten-minutes introduction to present the aim of the sessions on group theory, students spent eleven minutes individually recalling the axioms before the teacher formalised them. Some students questioned the necessity for the set to be non-empty. Guided by the teacher, they agreed that it is implied by the neutral element. Hence, as expected with the *a priori* analysis, some students tried to add a fifth axiom.

After this, the class collectively discussed the purpose of groups for ten minutes. One student interpreted the question from a pedagogical point of view. For him, groups are a good intermediate between university and the introduction of sets in high school, without any justification. Another student explained that groups allow us to gather similar objects according to given criteria. This answer highlights the classifying role of groups, which agrees with the *a priori* analysis.

Then, the students worked on the exercise in small groups for about ten minutes. Group 3 searched a set endowed with an operation which verifies none of the axioms, and came up with $(\{1; 2\}, *)$, with $x * y = x^y$. As foreseen in the *a priori* analysis, they worked on a small set of integers, but went a bit further than the four arithmetic

operations. It is to be noted that after about four minutes of independent work, the teacher invited the students to consider sets of numbers, with arithmetic operations. It is an indication I specifically asked him to provide, in case the students got stuck.

After the work in small groups, a collective discussion followed during the remaining twenty minutes. The teacher asked the class for an example of a set equipped with a closed operation. Group 3 gave $(\mathbb{R}, +)$, which verifies all axioms, and Group 2 gave $(\mathbb{N}, +)$, which verifies everything but the inversibility. Thus, as expected by the *a priori* analysis, these students started by giving a group before progressively moving to sets with more restrictive operations. Group 3 modified their first example so that the operation is only closed, and gave $(\{0; 1\},*)$, with $x * y = x^y$. Group 2 also modified their example and gave $(\mathbb{N}\backslash\{0\}, +)$, which is only closed and associative.

The class then tried to find as many combinations of the axioms as possible. One of the students of Group 2 noted that to remove the axiom of identity, it is sufficient to remove the neutral element from the set. The exercise therefore encourages reflection on the dependencies between axioms.

Group 1 gave a table, for which the operation is only closed (see fig. 2). One student of the group justified that there is no neutral element by saying that "there is no line that goes $x1$, $x2$, $x3$", which therefore means there are no inverses. He then computed $(x1 * x2) * x3$ and $x1 * (x2 * x3)$ to show that the operation is not associative.

	x1	**x2**	**x3**
x1	x1	x3	x2
x2	x3	x2	x1
x3	x2	x1	x3

Figure 2: Table created by the students of Group 1.

Contrary to what I had anticipated, the tabular framework was available to the students of Group 1. Moreover, they seemed to have embraced the idea mentioned previously that it is necessary to have a neutral element to define inverses.

The class then searched a set endowed with a solely associative operation. Group 2 gave $(\mathbb{Q},\times)$, which is associative, but also closed and has a neutral element. Group 2 then gave $(\mathbb{N}, -)$. The teacher checked the properties of the operation by finding a neutral element and computing $a - (b - c)$ and $(a - b) - c$, after which the class agreed the operation only verifies the axiom of identity (it is actually incorrect since 0 is only neutral to the right side).

Group 1 then asked for help finding a set with a solely associative operation. The teacher gave them $(\mathbb{Z}^*, +)$ (which stands here for $(\mathbb{Z}\backslash\{0\}, +)$) and asked the students to check its properties. Group 3 tried to come up with another example using matrices, but failed. Yet, this group worked with objects that are not numbers, contrary to what had been anticipated in the *a priori* analysis.

Group 1 realised there are fewer than 2^4 possible combinations of the axioms, and together with the teacher, one student of this group tried to compute the number of combinations, but they did not conclude.

To close the session, the teacher wanted to gather the class around the search for a last example. He asked for a set endowed with an operation that has a neutral element and inverses, but is neither closed nor associative. Several students gave $(\mathbb{Q}, -)$, but realised the operation is closed. A student of Group 1 suggested to "take something of $\mathbb{Z}$". The teacher took off 2, but then realised -2 is not invertible. The student suggested that they take -2 out too. The class finally agreed that $(\mathbb{Z}\backslash\{-42; 42\}, -)$ has the desired properties. Yet, 0 is once more only neutral to the right.

CONCLUSION

Discussion of the results

The following figure (fig. 3) shows the propositions of students. A red cross indicates the identity is only partially verified.

group		closure	identity	inversibility	associativity
G3	$(\{1,2\}, x^y)$				
G3	$(\mathbb{R}, +)$	x	x	x	x
G2	$(\mathbb{N}, +)$	x	x		x
G3	$(\{0,1\}, x^y)$	x			
G1	$(\mathbb{N}^*, +)$	x			x
G1	<table><tr><th></th><th>x1</th><th>x2</th><th>x3</th></tr><tr><th>x1</th><td>x1</td><td>x3</td><td>x2</td></tr><tr><th>x2</th><td>x3</td><td>x2</td><td>x1</td></tr><tr><th>x3</th><td>x2</td><td>x1</td><td>x3</td></tr></table>	x			
G2	$(\mathbb{Q}, \times)$	x	x		x
G2	$(\mathbb{N}, -)$		x		
Teacher	$(\mathbb{Z}^*, +)$				x
G1	$(\mathbb{Z}^*, -)$				
Teacher and G1	$(\mathbb{Z} \setminus \{-2, 2\}, -)$		x	x	

Figure 3: Answers given by students.

The proposals are varied and come from all three groups. Hence, the students were fully engaged with the task. Working independently, alone and then in groups, likely helped with the devolution. Sets of numbers and elementary operations are largely predominant, confirming that students are familiar with them. Yet, given that the teacher enticed them quickly to work with such sets, I must remain cautious. However, one of the groups used a table, without any reminders about this representation register having been given. Thus, the tabular semiotic register, covered in second year, was available to some students, contrary to what I had anticipated. In addition, Group 3 considered using matrices, but was unable to produce an example. The students did not therefore limit themselves to manipulating sets of numbers. This had been confirmed during the second experimentation with master's students, during which examples such as $(\mathbb{R}\backslash\mathbb{Q} \cup \{1\}, \times)$ and the symmetric group $(S_n, \circ)$ were given.

It seems necessary to recall the definition of a group at the beginning of the session.

Some of the students seem to have understood that the existence of inverses requires the prior definition of a neutral element. Furthermore, since they were all involved in finding examples that satisfied certain combinations of axioms, the objective of getting them to work on the role of each element and their interdependencies seems to have been achieved. The entire session emphasised the object nature of groups. In particular, the collective discussion at the beginning of the session showed that the tool nature was not widely understood. It therefore seems appropriate to approach the sessions about groups from the object nature perspective, as this is the one with which they are most familiar, and then to introduce the tool nature at a later stage.

The students had the opportunity to experience working in groups, which is unusual for them. This may explain why the working groups took a while to get started and why the teacher had to intervene more than expected. I suggest that keeping the same groups for all sessions, as far as possible, will facilitate the work later on, when the students are more familiar with the system. No institutionalisation has been made of this working method or the reasons for its implementation. In the context of this pre-experiment, I simply wanted the students to experience group work for themselves, to see what they would say about it during the post-session interview.

Elements for a double institutionalisation

The final version of the module aims to enhance students' mathematical and didactical understanding, a process coined as "double institutionalisation" (Pézard, 1985, p. 4). This pre-experimentation focused on the mathematical institutionalisations, but the second experimentation aimed at master's students included didactical institutionalisations. I shall explicit some of them as well of some of the changes made.

My analysis of the first session shows that some combinations of axioms are easier to exemplify than others. Choosing carefully some combinations, instead of letting the students search freely, seemed a good opportunity to discuss the concept of didactical variable with the students. During the second experimentation, the concept was defined, and the teacher explained why each variable had been chosen. The wording of the exercise has thus been rewritten in this regard. This also allowed the teacher to institutionalise that this type of work on a 'difficult definition' is a way of enabling students to construct meaning around this definition and to reveal the possibilities and constraints of the underlying concept. Moreover, given that some students were able to recall the tabular register, the first session has been lengthened and enriched to include work on groups of small cardinals, to give way to a discussion on what can be gained from using a diversity of registers, and to reveal some of the constraints inherent in the group structure. During the second experimentation, the students were furthermore asked to reflect upon the benefits of working in small groups.

Bridging Klein's second discontinuity regarding group theory

This pre-experimentation was designed as a test run for the second experimentation. The goal was to control the pertinence of the mathematics content chosen before adding a layer of didactical institutionalisation. This first session showed that the students

engaged fully with the proposed task and gained insights into the role and relationships between axioms. This is particularly valuable for secondary school teachers, since the group structure underlies many objects taught at this level. Indeed, understanding that manipulating these four axioms can be complex could for example allow the students to reflect on difficulties faced by pupils when solving equations such as $x + 5 = 12$, in which all axioms are at play. During the final session of the module, students had to solve this equation and justify each step with an axiom, hence uncovering layers of potential difficulties for pupils. Moreover, the task exposed in this article is a good starting point to discuss didactical concepts such as didactic variables, semiotic registers, and group work. Thus, after the second experimentation, students should better understand the benefits of studying advanced university mathematics, and I hope that their group theory course will not "remain only a more or less pleasant memory which has no influence upon [their] teaching" (Klein, 2016/1908, p. 1).

NOTES

1. My PhD is supervised by Frédéric Brechenmacher (LinX) and Nicolas Grenier-Boley (LDAR). I wish to thank Nicolas Grenier-Boley for his thorough proofreading of this paper.
2. That is, it should be easier for students to engage with the task.
3. This course was a pre-professional elective for final-year mathematics undergraduates preparing for the national competitive examination for secondary teacher recruitment.
4. A group is a set G endowed with an operation $*$, such that the latter is closed ($\forall a, b \in G, a * b \in G$), associative ($\forall a, b, c \in G, a * (b * c) = (a * b) * c$), has a neutral element ($\exists e \in G, \forall a \in G, e * a = a * e = a$), and all elements have inverses ($\forall a \in G, \exists a^{-1} \in G, a * a^{-1} = e$).

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Understanding students' affective experiences in the transition from high school to university mathematics

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Investigating the transition from high school to university mathematics, this exploratory qualitative study examines students' affective field during the transition from upper secondary to university mathematics. Using photo-elicitation and focus group interviews with a small cohort of student, this study examines the perspectives of first- and third-term mathematics students using Schindler and Bakker's affective field (2020). The focus is on how students navigate in this transition in connection to their affect-orientation related to mathematics. Qualitative methods, including visual methods and focus group interviews, are employed. The findings indicate a complex emotional journey, with initial excitement giving way to mixed feelings of lower self-efficacy and a change in their mathematics-related beliefs and values.

Keywords: Transition, university mathematics, visual methods, affect theory.

INTRODUCTION

STEM programmes at the University of Southern Denmark are characterized by high dropout rates, with a 17% dropout rate in the first year according to the University of Southern Denmark's education report for the academic year 2022/2023. This highlights the importance of taking a relational perspective in addressing dropout rates, emphasizing the relationship between students and study programmes (Ulriksen et al., 2010). To achieve a comprehensive understanding of students' experiences, this article argues for an expanded approach that considers students' affective orientations in relation to their studies. From this perspective, students' affective experiences become a key lens for understanding how transitions into university mathematics are lived, navigated, and sustained.

This study investigates how mathematics students at the University of Southern Denmark perceive the transition from high school to a university mathematics programme, with a focus on their mathematics-related affective fields (Schindler & Bakker, 2020). Current didactic research in university mathematics has highlighted the pertinence of student experience in transitions (Di Martino et al., 2023; Gueudet, 2020; Thomas et al., 2010). Thomas et al. (2010) pointed out that students are affected differently, but the motivation towards mathematics changes with time. Further research on transition has shifted from a purely cognitive focus to a holistic approach, with a focus on the affective dimension (Kirsten, 2022; Geisler og Rolka, 2021). This perspective provides valuable insights into students' active adaptation to new circumstances in higher education (Gale & Parker, 2014, p. 735). However, some studies tend to oversimplify these aspects, focusing on one or a few affective

factors, which can limit our understanding of students' complex realities (Grootenboer & Marshman, 2016; O'Keefe et al., 2017). While research on the transition from secondary to tertiary mathematics has increasingly acknowledged the importance of affect (Schindler & Bakker, 2020), much of this work has relied on observations, interview-based approaches or questionnaire data focusing on isolated affective constructs. Fewer studies have explored students' lived affective experiences of transition through holistic affective frameworks combined with visual methods.

RESEARCH QUESTION

This article contributes to the literature on transitions in university mathematics by offering an exploratory, qualitative account of students' affective orientations towards mathematics, analysed through the lens of the affective field and elicited via visual methods. By combining photo-elicitation and focus group interviews, the study provides insight into how shifts in affective orientations are expressed and negotiated during the transition from high school to university mathematics.

From this, this study explores the following research question: *How do students' affective orientations toward mathematics change during the transition from upper secondary school to the first year of university mathematics, as expressed through visual representations and interviews?*

To answer this question, the focus will be on exploring and understanding the emotional, attitudinal and motivational aspects of students' experiences with their study. This approach is based on the theory that students' perceptions and experiences in a subject are deeply influenced by their affective responses, which include their emotions, beliefs, attitudes and motivations (Schindler & Bakker, 2020). In the Danish upper secondary system, student autonomy is typically exercised within a clearly structured framework, where tasks, learning goals, and progression are largely teacher-defined, and where students receive regular feedback and guidance. In contrast, university mathematics places a substantially greater responsibility on students to structure their own study practices, interpret course expectations, and identify what counts as relevant mathematical activity.

THEORY

The examination of students' affective states is grounded in Schindler and Bakker's (2020) concept of the 'affective field', which outlines a collection of various affective factors influenced in learning situations. The notion of the affective field is adopted precisely to avoid an atomistic treatment of affective constructs and to capture how students' emotions, beliefs, self-efficacy, interest, values, and attitudes co-occur and mutually influence each other in concrete educational situations. Rather than treating these constructs as separable variables, the affective field provides a holistic and dynamic analytical lens. The theoretical framework encompasses a comprehensive range of psychological constructs, which have been pertinently addressed by research in university mathematics didactics (Suthar, 2010). A central quality of the

framework is that the affective field is analogised as a magnetic field where affective factors act as poles that mutually influence each other. This metaphor is particularly relevant for the present study, as it allows us to attend to moments of attraction and repulsion in students' accounts of mathematics, for example when enthusiasm coexists with insecurity, or when increased interest is accompanied by decreased self-efficacy.

The following are the affective factors encompassed within the affective field:

- **Emotions:** These encompass a range of emotional states, from joy to frustration, experienced in response to mathematical tasks, teaching methods or interactions with peers.
- **Self-Efficacy:** This refers to one's belief in their ability to successfully tackle mathematical tasks.
- **Beliefs:** These are deeper assumptions about mathematics, such as whether it is seen as challenging or beneficial.
- **Interest/Motivation:** This relates to the intrinsic drive and curiosity that prompt individuals to engage in mathematics studies.
- **Values:** Values refer to principles of mathematics including the importance of precision or real-world applications.
- **Attitude:** This is the overall disposition towards mathematics. (Listing based on Schindler & Bakker, 2020, p. 307).

While all of these affective factors are considered constitutive of the affective field, not all of them are equally visible or articulated in every empirical situation. In line with Schindler and Bakker (2020), the affective field is treated as a whole, while acknowledging that certain affective factors may become more salient than others depending on the activity, context, and mode of expression. By viewing these affective factors holistically, we can achieve a deeper understanding of students' experiences in their mathematics studies and their transition from high school to university. This theoretical perspective is particularly compatible with visual and dialogical methods, such as photo-elicitation, as these methods allow students to express affective orientations that may be difficult to articulate verbally and to capture the coexistence of multiple affective factors.

METHOD

This section outlines the methodological design of an exploratory qualitative case study aimed at investigating students' affective experiences during the transition from high school to university mathematics. The data were collected for the bachelor project titled 'Transition from High School to University: Students' Mathematics-Related Affective Fields'. This involved interviewing students enrolled in the mathematics programme to gain insights into their experiences.

The study's approach included two focus group interviews. The first group consisted of three first-term students (Adam, Ben and Cecilia), who were interviewed a week into their university journey. The second group comprised four third-term students

(Dennis, Eric, Frida and Ginna), who were a month into their term studying applied and pure mathematics. The inclusion of both first-term and third-term students was intentional, as it allowed for a comparative perspective on students' affective orientations towards mathematics at two different points in the transition process being immediately after entry into university mathematics and after initial experiences with university-level coursework. These participants were selected through announcements made in lectures and by engaging with students in faculty facilities to ensure a diverse representation of experiences. To maintain confidentiality and respect gender identities, pseudonyms were used for all participants. The interviews were conducted using a mobile phone for audio recording and were later transcribed in full for analysis, and lasted approximately 60 minutes and followed a semi-structured format.

A significant methodological tool used in this study was photo-elicitation, inspired by Rose's (2016) visual method. This approach is particularly useful in exploring the experiential and affective dimension of students' transitions. Photo-elicitation allows for a more comprehensive and nuanced understanding by combining visual representations with the perspectives of the informants. Photo-elicitation was chosen as a methodological tool to access affective experiences that may not be readily articulated through verbal language alone (Rose, 2016). In mathematics education research, students' affective orientations are often tacit or taken for granted, visual prompts can support reflection and enable participants to express nuanced and embodied experiences of mathematics (Svendsen et al. 2026). In this case, the students were asked to select photos from a set of 120 photos titled 'Visual Explorer', relating to the words 'mathematics', 'high school' and 'university'. The students were asked to select one photos for each prompt. They were instructed to choose photos that, in their view, captured their experiences and perceptions of the relationship between high school and university mathematics.

To enrich the data, this method was supplemented with semi-structured group interviews. The students choice of photos, formed the basis of a semi-structured interview. The interview questions delved into students' likes and dislikes about mathematics, their reasons for choosing to study it at university, self-assessment of their mathematical abilities and any changes in their perception of mathematics since starting university. The combination of photo-elicitation and focus group interviews allowed for a more holistic exploration of students' affective orientations towards mathematics.

The analysis followed an iterative process. First, the interview transcripts were read in full to identify episodes where students articulated affective experiences related to mathematics and the transition. These episodes were then interpreted through the lens of the affective field, attending to how beliefs, emotions, self-efficacy, interest, values, and attitudes co-occurred in students' accounts. The visual materials were analysed in conjunction with the interview data, with particular attention to how students explained and interpreted their chosen photos and drawings. The analysis

focused on identifying patterns and contrasts between the students. Two overarching themes related to these transitions emerged: "From tool to science" and "Among peers".

DATA AND ANALYSIS

The results illustrate how students articulate and negotiate their affective orientations towards mathematics during the transition from high school to university. The analysis highlights shifts in students' beliefs about mathematics, their sense of responsibility and autonomy, and dynamics in their affective field across different stages of the transition. In the analysis, particular analytical attention is given to moments where students' beliefs about mathematics and their self-efficacy are articulated or visually represented, as these emerged as especially prominent in the photo-elicitation material.

Photos of the word 'mathematics':

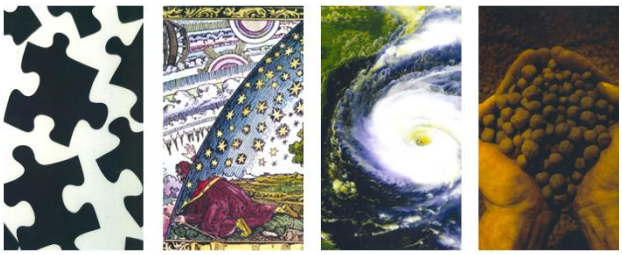

Figure 1. From left, first-term students Adam's and Cecilia's and third term students Dennis' and Fridas' photos on the word 'mathematics'

Across students, the selected photos for 'mathematics' reveal a shift from viewing mathematics primarily as a practical tool or puzzle towards conceptualising it as a broader and more differentiated scientific discipline. For first-term students, mathematics is often associated with solvable tasks and application, whereas third-term students more frequently emphasise structure, subdisciplines, and abstraction. Adam, a first-term student, said that, to him, mathematics is like a puzzle where the goal is to find a solution. He compared it to playing computer games, which he is very fond of. Frida, a third-semester student, described how her view of mathematics has become more nuanced after one year, with a greater focus on the various disciplines that mathematics encompasses. She did this based on an photo of what she described as two handfuls of stones gathered into a large heap. She used the photo to describe how she has now realised that mathematics is a collection of many sub-elements but also has an application perspective, which she calls 'everyday mathematics'.

From tool to science:

Overall, the comparison between first-term and third-term students indicates a transition in how mathematics is perceived: From a tool used across subjects to a discipline characterised by multiple interconnected domains and abstract reasoning.

This shift in beliefs is closely entangled with changes in students' self-efficacy and values, as increased complexity simultaneously evokes fascination and insecurity.

From the interview with Eric, it is evident that although his view of mathematics has evolved since high school, he strongly identifies with the subject. He firmly believes that his current understanding of mathematics is correct, differing from the general public's view. This indicates not only a deep connection to his field of study but also that such a study can lead to isolation in terms of other perspectives. Throughout the interview, the students often defended mathematics as a subject. For instance, Frida addressed the societal attitude towards mathematics, stating,

> It's okay not to be good at math. But I also think it's because people underestimate their own abilities. (Own translation, Frida)

This comment reflects a belief that mathematics is a fundamental truth in the world, and thus, holds great value. The students' orientation include notions of accessibility and repetition in high school, contrasting with the specialised and demanding nature of university education.

Photos of the words 'high school' and 'university':

Figure 2. From the left, Ben's photos of high school (traffic) and the university (nomad) and Adam's photos of high school (toast) and university (wood)

Across the photos depicting high school and university, students consistently describe a shift from structured, guided learning environments to increased responsibility and self-direction. High school is portrayed as collective, predictable, and instructional, whereas university is associated with autonomy, uncertainty, and individual accountability.

To the words 'high school' and to the word ' university' Ben chose a picture of a traffic jam for 'high school' and that of a nomad for 'university'. He justified this by explaining that, while in high school, one follows a set path along with others, at university, there is greater flexibility in pursuing a specific direction, but one tends to do so more on their own. Ben feels that high school and university represent different experiences in terms of personal freedom and individual responsability, where high school is understood as more structured and collective. Adam chose the picture of a toast for 'high school' and that of stacked wooden beams for 'university'. Adam first justified his choice of the toast picture by mentioning that he ate a lot of toast in high school, but he added,

> Because toast is easy - what do they call it, 'consumable' (...). But there's a lot of repetition too. (Own translation, Adam)

Adam believed that the content in high school was easy in the sense that it had a lot of practice and repetition, and that all the knowledge he gained during his high school years can be easily recalled and applied when required. However, with the stacked wooden beams symbolising university, it is different:

> […] You learn something more specific - it's not as easily 'consumable' because it's somewhat harder. It's also at a higher level." (Own translation, Adam)

Thus, a week into his studies, Adam envisioned that the academic level is higher and more challenging to get through at the university.

Figure 3. From the left, Frida's photos of high school (mother and child) and university (man with bike), and Eric's photos of high school (ruler) and university (bridge in city)

Frida chose a picture of a mother and a child with a book for 'high school' and that of a man with a bicycle over his shoulder on his way over a mountain for 'university'. The picture of the mother and child symbolises high school in Frida's view because

> [...] you're held by the hand and it's presented to you in a nice way. I mean, positively meant - it's served on a silver platter. (Own translation, Frida)

For the picture of the man, the bicycle and the mountains, she explained that, at the university, you are responsible for your own learning. She described the analogy that, if you want to climb a mountain, you have to prepare appropriately for the journey and you are solely responsible for crossing the mountain. Frida feels that the responsibility now lies largely on her shoulders, whereas before, she felt guided by her high school teachers. Eric chose a picture of a ruler for the word 'high school' and that of an arch for the word 'university'. He explained this by saying that he perceives high school as a place where one learns several fundamental mathematical concepts in small pieces, one thing at a time. Then, at university, one must have all these concepts in their repertoire and be able to combine them into a comprehensive understanding. In this way, one can build bridges between cities.

Among peers:

In interviews with third-term students, both Dennis and Ginna described their mathematical skills as 'almost sufficient'. In a notable contrast, their peer Frida revised her self-assessment, labelling her skills as 'somewhat sufficient'. This variation in self-efficacy among students is closely tied to their experiences in

university classes. The challenges encountered in difficult topics or courses significantly impact students' self-perception of mathematical ability.

For example, first-term student Adam reflected on his journey in mathematics, as follows:

> When I was in high school and elementary school, I would consider myself really good at mathematics because I was the one in class that people would approach when they had problems with math […] and I easily got 12 [red: the highest grade]. And then I came to university, and suddenly I became like; wait, I can't figure this out. (Own translation, Adam)

From the perspective of the affective field, this illustrates a moment of repulsion, where prior positive self-efficacy is destabilised by new social comparisons and increased academic demands. In high school, being among the best in class often boosts Adams self-efficacy. However, upon entering university, he is confronted with the realisation that he are no longer the top student, facing more challenging material.

Affective field in the transition from high school to university mathematics:

Viewed through the lens of the affective field, the transition from high school to university mathematics emerges as a process in which multiple affective factors are simultaneously at play, though not with equal visibility in the empirical material. Across the students' visual representations and interview accounts, certain aspects of the affective field, particularly beliefs about mathematics and self-efficacy, appear more explicitly articulated. At the same time, these more visible factors are closely entangled with other, less explicitly foregrounded affective factors. Emotional responses such as excitement, frustration, or uncertainty tend to surface in relation to shifts in perceived demands, social comparison with peers, or changing expectations regarding what it means to engage with mathematics at university. Similarly, students' motivation and interest are not presented as stable traits but unfold dynamically in relation to their evolving beliefs about mathematics as a discipline. The empirical material suggests that values and attitudes, while less directly verbalised, operate as underlying orientations shaping how students interpret both high school and university mathematics. For instance, valuing efficiency or clear guidance appears to underpin students' descriptions of high school mathematics, whereas emerging values related to independence and conceptual understanding become more salient in accounts of university mathematics. These value shifts are not always named explicitly by the students but can be inferred through their metaphors, photo choices, and reflections on responsibility and learning practices. In this way, the affective field during the transition is characterised by reconfigurations in how different affective factors relate to one another. Moments of attraction and repulsion, such as increased interest coexisting with lowered self-efficacy, or growing appreciation of mathematics accompanied by feelings of insecurity, illustrate the dynamic nature of the affective field as students navigate new academic expectations.

DISCUSSION AND CONCLUSION

This study set out to explore how students' affective orientations toward mathematics change during the transition from upper secondary school to the first year of university mathematics, as expressed through photos and interviews. Positioned within research on transitions in higher education and mathematics education, the findings contribute to a growing body of literature emphasising the central role of affect in students' experiences of becoming university mathematics students (Di Martino et al., 2023; Gale & Parker, 2014; Thomas et al., 2010). Consistent with previous research, the results confirm that the transition to university mathematics involves more than cognitive adjustment.

Viewed through the affective field (Schindler & Bakker, 2020), the transition emerges as a process of reconfiguration rather than linear development. Certain affective factors, like beliefs about mathematics and self-efficacy, are particularly visible in students' visual and verbal accounts, while other affective factors such as emotions, values, and attitudes operate as underlying, entangled elements that shape how these more explicit orientations are formed and experienced. A key finding concerns the shift in students' beliefs about mathematics itself. For many students, mathematics moves from being understood primarily as a tool for solving problems in upper secondary school to being experienced as a complex and differentiated scientific discipline at university. This shift is closely connected to changes in self-efficacy and emotional responses. Such tensions appear to be constitutive of the transition experience and resonate with holistic perspectives on affect that foreground simultaneity and dynamism (Grootenboer & Marshman, 2016; O'Keefe et al., 2017).

Methodologically, the study contributes by demonstrating the potential of photo-elicitation (Rose, 2016), for exploring students' affective orientations toward mathematics. By inviting students to select and interpret photos the method enabled access to affective experiences. The combination of visual representations and group interviews supported a holistic analysis of affective orientations and aligns with calls for methodological approaches capable of capturing the complexity of affect in mathematics education (Schindler & Bakker, 2020). At the same time, the study's findings must be interpreted in light of its scope and design. As an exploratory case study with a small number of participants, the results offer analytically grounded insights into students' affective experiences of transition in university mathematics. Moreover, not all aspects of the affective field could be traced with equal depth. The affordances of photo-elicitation appeared to foreground students' beliefs and self-efficacy more strongly than, for example, values or attitudes, which are likely to operate more implicitly and over longer timescales. Despite these limitations, the study offers a nuanced contribution to research on the transition to university mathematics by foregrounding students' own visual and narrative articulations of their affective orientations.

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Learning strategies and affective responses at the transition to university mathematics

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The transition to university mathematics involves changes in curriculum and teaching with potential associated changes for student engagement and attitudes. In September 2024, the Observatory for Mathematical Education surveyed 723 students prior to starting mathematical sciences degrees at universities in the UK, with a follow-up survey conducted six months later after the first semester. Students report greater dissatisfaction with teaching and experience a drop in mathematical confidence during the first semester. Meanwhile, perceptions of mathematics and learning strategies are mostly unchanged from school, except for increased use of AI. Female students are more likely to report a negative transition phase than male students.

Keywords: secondary-tertiary transition, learning strategies, affective response, rite-of-passage, quantitative pre- and post- design.

INTRODUCTION

The secondary-tertiary transition in mathematics is an unsettling experience for many students, which may lead to drop-out from university or reduced progress in academic attainment (Geisler et al., 2023). In the UK, students intending to study university mathematics typically only study 3 A level qualifications (2 of which can be mathematics) between ages 16 and 18 so have less breadth than students in many other education systems. This may contribute to university drop-out being less pronounced in the UK compared to other countries (Chen, 2013). Nevertheless, there are concerns that some demographic groups (e.g. female students) are under-represented in post-compulsory mathematics (Brignell et al., 2025). Examination of the secondary-tertiary transition may alleviate inequality in university participation and attainment.

Student difficulties in the transition phase have been variously categorised as epistemological-cognitive, sociological-cultural, and didactical (De Guzman et al., 1998); individual, social and institutional (Gueudet, 2008); and mathematics, approaches to study and institutional support (Hochmuth et al., 2021). The more recent review by Di Martino et al. (2022) noted an increased examination of socio-cultural and affective issues in the literature alongside the cognitive and epistemological lenses. An anthropological perspective is to view the secondary-tertiary transition as a rite of passage with, chronologically, separation (leaving school), liminal (in between school and university) and incorporation phases (first year of university) through which cognitive conflict, conceptual changes and culture shock arise as expectations and experiences are reconciled (Clark and Lovric, 2009).

There are comparatively few large quantitative studies on mathematical transitions in the recent literature. In a US study, Rothrock and Gay (2025) reported that first year

mathematicians generally have positive mathematical identity and self-efficacy and these constructs are correlated with prior attainment. Prior attainment, along with attendance, was also a strong predictor of successful mathematical transition for engineering students at a German university (Eicher & Gradwohl, 2021). Meanwhile Meyerhofer et al. (2024) used a repeated measures design to create latent profiles of students. Again, better prior attainment corresponded with more adaptive profiles which remain relatively stable in the first two months of study at an Austrian university.

Many theories have been proposed in the literature for the secondary-tertiary transition. For example, Meyerhofer et al. (2024) uses expectancy-value theory (EVT), widely used in education studies, where student achievement, choices and engagement is explained by expectancies for success (ability-beliefs related to self-concept and self-efficacy) and subjective task values (attainment, intrinsic, utility and cost). It should be noted that the data in the present study was collected as part of a much wider observational study of UK mathematical education for ages 4-25 which did not seek to evidence any one transition theory. Nevertheless, the findings regarding student behaviours and attitudes are organised below using the lenses identified by De Guzman et al. (1998) and Di Martino et al. (2022), but this large, cross-institutional, national data could be interpreted *a posteriori* to offer clues in support of EVT or other theories.

Students due to commence a mathematical sciences degree were surveyed in September 2024 (liminal phase) with a follow-up survey conducted six months later (incorporation phase). At this system-scale, the study focused on changes in student perceptions of mathematics, their strategies to learning, engagement with teaching, and their affective responses to the secondary-tertiary transition. Self-reported demographic data enabled a comparison by sex, university A level Mathematics entrance requirement (highest is grade A*, followed by A, B, or lower) and expected first year attainment classification (first-class, upper-second, lower-second, third-class). More precisely, the study asks

1. What epistemological, cognitive, didactical and affective difficulties do students report at UK universities during the secondary-tertiary transition?
2. Which student groups are most likely to have a negative transition experience?

METHODOLOGY

The first survey was conducted in September 2024 immediately before students started their degree and was distributed via email to all students (approximately 6,800) who had accepted an offer to study an undergraduate degree with at least 50% mathematical content in the UK. This includes single and joint honours programmes involving mathematics or statistics. The second survey was conducted between February and March 2025 for all individuals who had completed the initial survey. For most respondents, this represented the midpoint of their first year of university.

Survey 1 received 1503 responses, of which 1003 were valid with 65 UK universities represented. Invalid responses were due to blank or near blank entries, repeat responses from the same email address or ineligible courses. Survey 2 was sent to 990 students,

who had consented to receive the email, with 775 responses. Of these, 723 were still studying a mathematical sciences degree, with others excluded because of course transfers (10), deferred studies (19), no longer at university (6), or were blank or near blank (17). On completion of the second survey participants could opt-in to receive a monetary voucher. The gender and ethnicity of respondents was checked to be similar to those of all UK undergraduate mathematics students (Severn, 2025).

Survey 1 consisted of 52 items including questions on confidence in mathematics (6 items), nature of mathematics (10 items), prior education and demographics (8 items), preferences for learning (15 items) and expectations of learning mathematics at university (13 items). Confidence in mathematics items were adapted from the Fennema-Sherman scales (Fennema & Sherman, 1976). The nature of mathematics items were adapted from the Conceptions of Mathematics Questionnaire (Crawford et al., 1998). Expectations of learning mathematics items were translated into English and adapted from research into success factors in mathematical transition (Bengmark et al, 2017). Survey 2 was specifically designed to enable longitudinal analysis with most items repeated from survey 1 with only minor adjustments where necessary.

RESULTS

Epistemological

There was large support for statements which conveyed mathematics as an 'exploratory' subject in terms of solving problems multiple ways, new ideas, creativity, and helping us to understand the world (Figure 1). There was less support, although still reasonably large, for mathematics being 'fragmented' in terms of rules, equations, calculations, definitions and facts. Despite the formal introduction of proof, more rigorous definitions of calculus, and greater emphasis on coding, for example, the first semester of university did not greatly change students' perception about the fundamental nature of mathematics with 69% of students perceiving the 'mathematics at university follows on directly from what I learnt at school or college'. That said, 93% of students acknowledged they had to write mathematics more precisely.

The 'exploratory' view of mathematics is held by nearly all sub-groups of students. There are more notable differences, though, in who holds a more 'fragmented' view of mathematics. For example, in Survey 2, 74% of female students and 65% of 'home' students say mathematics is about calculations compared to 54% of male students and 39% of international students, respectively, with similar differences present in Survey 1. Similarly, in Survey 2, 51% of students at the universities with the highest entry requirements report mathematics is about calculations compared to 75% at other universities, while 39% of those on track for first-class outcomes think mathematics is about remembering and using definitions, formulas and facts compared to 65% of those on track for third-class outcomes, with similar differences observed in Survey 1.

Similarly, the learning strategies students reported using after the first six months at university are very similar to school or college (Figure 2). Students place great emphasis on solving problems quickly to measure their understanding. Interestingly, a

majority of students believe that they 'learn maths best when I work on my own' even though they also believe they 'learn a lot from explaining maths to others'. Approximately three-quarters of female students are likely to report needing an explanation from the teacher when learning something new compared to half of male students, but these proportions do not change between the surveys. Likewise, after six months of university, of those reportedly on track for a third-class degree, 81% rely on teacher explanations compared to 54% on track for a first-class degree.

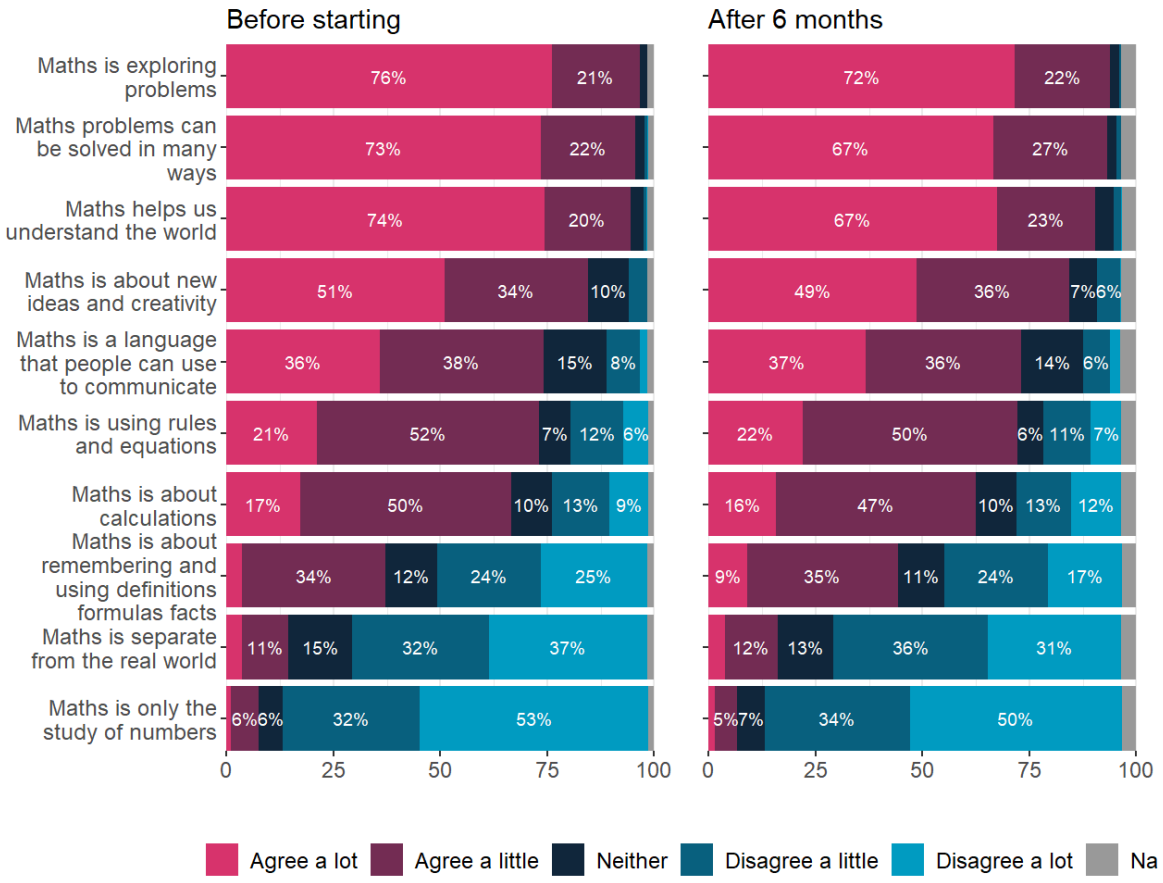

Figure 1: The agreement ratings across multiple statements on the nature of mathematics for survey 1 (left) and survey 2 (right) (N=723).

Cognitive

This study did not directly investigate the cognitive processes involved in processing more formal and abstract mathematics at university. However, an overwhelming 89% of students reported university mathematics as being 'much more difficult' compared to school or college, with the percentage reporting they have to work hard to do well in mathematics also increasing (see Figure 4). This points to an increased cognitive load and might explain the increase in the number of students reporting it is 'easier to memorise techniques rather than understand them' during the first semester (Figure 2).

Notably, 81% of female students in Survey 2 report they have to work hard to do well in mathematics compared to 72% of male students. Likewise, more female students (24% in Survey 1, 39% in Survey 2) find memorisation easier than understanding

compared to male students (17% in Survey 1, 26% in Survey 2); and, similarly, for those on track for a third-class degree (51% in Survey 2) compared to those on track for a first-class degree (23%).

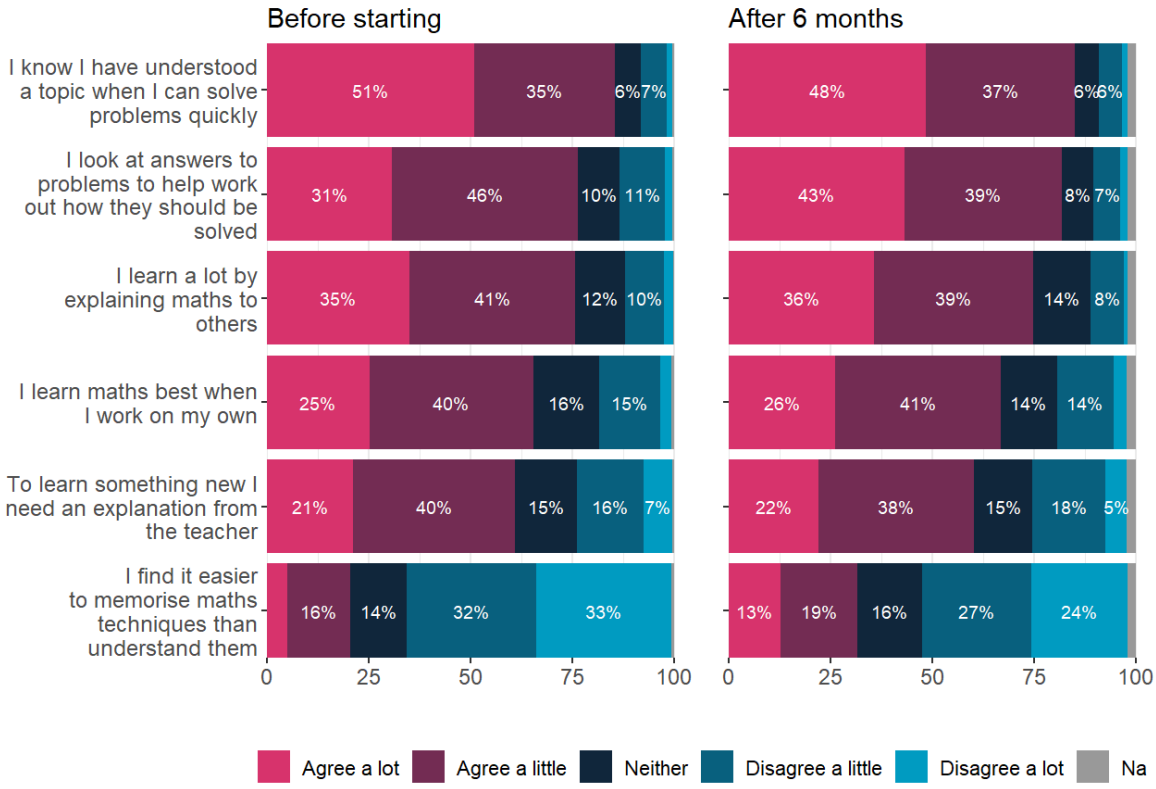

Figure 2: The agreement ratings across multiple statements investigating student learning strategies for survey 1 (left) and survey 2 (right) (N=723).

Didactical

While 58% of students prefer the mathematical content at university compared to the school or college curriculum, only 38% prefer the style of mathematics teaching at university compared to 43% who prefer the style of school or college teaching. One source of complaint is the pace of learning, with 58% reporting lectures move on to new mathematics too quickly and half reporting lectures often contain too many new ideas. Over a quarter of students do not feel they need to attend all timetabled teaching to do well. The most common reasons for not attending were not finding the sessions useful, preferring to study on their own, fallen behind with studying, and poor timekeeping. Female students, compared to males, are 13 percentage points more likely to report lectures move on to new mathematics too quickly and 19 percentage points more likely to feel the need to attend all timetabled teaching sessions. Overall, 80% of students said they had received feedback, but this was highly variable between institutions. Of those that had received feedback, only 26% strongly agreed that the feedback was useful.

Despite this apparent dissatisfaction with teaching, there is little difference in how helpful students find learning resources compared to school or college (Figure 3).

‘Working through problems or exercises’ and ‘working through past paper questions’ are almost universally seen as helpful ways to learn mathematics. The biggest change in the first six months of university is the use of artificial intelligence. Prior to starting, less than a fifth of students reported ‘using AI tools’ as useful in their learning of mathematics compared to 58% after the first six months of university. Although female students were less inclined to find AI tools useful before starting, there was no difference between the sexes after the first semester. However, students on track for a third-class average were more likely (72%) to find AI tools helpful compared to those on track for a first-class average (52%), while the opposite is true regarding using mathematics software to visualise data or plot graphs. Of those on track for a first-class average in Survey 2, 42% were confident in coding and computing in Survey 1 compared to only 14% of those later expecting a third-class average.

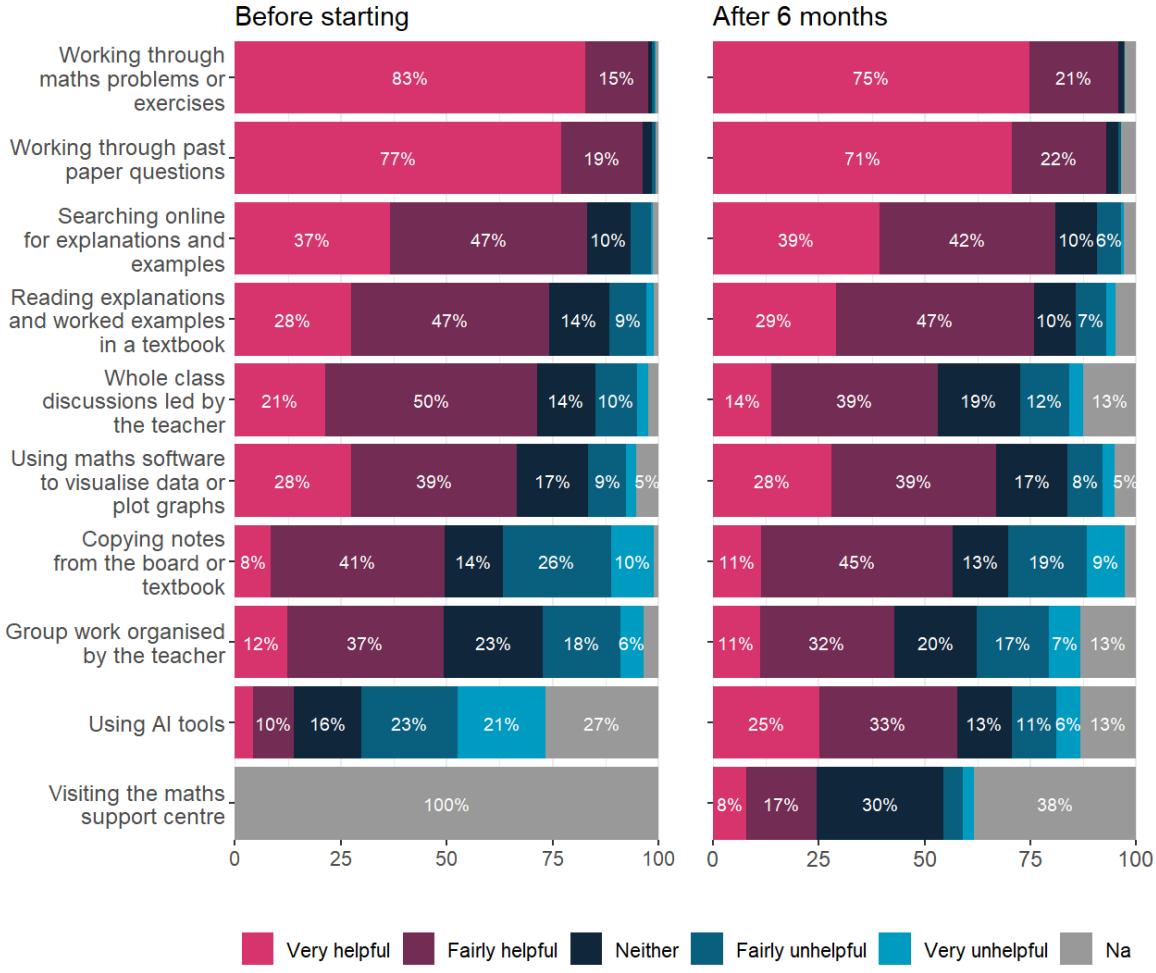

Figure 3: The agreement ratings of how helpful students found different resources when learning mathematics for survey 1 (left) and survey 2 (right) (N=723).

Affective

Nearly all students start their university mathematics degree confident in their mathematical ability and reporting that the discipline comes naturally to them, but these measures drop after six months of university (Figure 4). Having enjoyed the privileged

position of being one of the best at mathematics in their school or college, only a minority think they are among the best at university, recognising that the peer group against which they compare themselves has changed dramatically. These changes in mathematical confidence, though, are not experienced uniformly. Female students report a larger drop in confidence, with 76% of female students and 85% of male students still reporting confidence in mathematics after six months. At universities which typically require the highest A* grade in A level Mathematics, 78% were confident in mathematics after six months compared to 85% at universities that typically require grade A and 89% for institutions requiring lower grades.

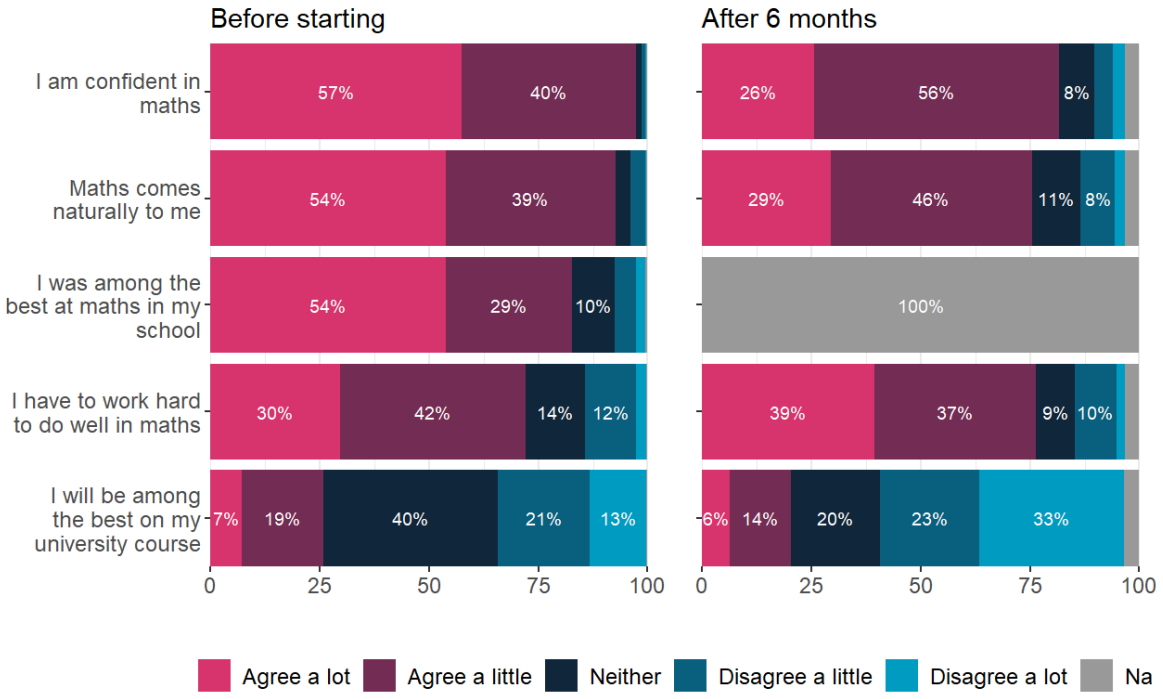

Figure 4: The agreement ratings across multiple statements measuring confidence in mathematics for survey 1 (left) and survey 2 (right) (N=723).

Unsurprisingly, after six months at university, students' reported confidence in mathematics was correlated with their expected first-year mark. Those who think they are on track for a first-class average are twice as likely to report confidence in mathematics compared to those on track for a third-class average. However, these expected first-year outcomes cannot be predicted by confidence levels, or whether mathematics comes naturally, prior to starting.

CONCLUSION

Despite the formalisation of mathematics at university, the study finds only minimal changes in students' perception of mathematics and their strategies for learning between school and university. The preference for working through past paper questions, looking up answers and solving problems quickly suggests an assessment driven approach carried over from high-stakes examinations at A level. This disconnect

between pragmatic mediation and lecturers' desire for deeper understanding, epistemic mediation, has been reported before (Gueudet & Pepin, 2016).

Continuing to value attainment contrasts with a reduction in self-efficacy. Students report insufficient time to process new mathematics, an increased reliance on memorisation over understanding and not finding feedback useful. The lower self-efficacy amongst female students is consistent with other studies which note females are more self-critical regardless of ability (e.g. Ganley & Lubienski, 2016). While some students gave a preference for independent study as a reason to miss timetabled sessions, the fact attendance was positively correlated with expected first year classification suggests, somewhat analogous to Quéré (2016), students not managing the resources available to them rather than self-efficacy.

The reported 'jump' in difficulty at the secondary-tertiary transition is reminiscent of the cognitive discontinuity reported in many studies (e.g. Tall, 1991) although with little curricular discontinuity reported, deficit approaches to knowledge gaps are less needed. Instead, students point to dissatisfaction with didactic approaches. That said, students at universities with higher entry requirements were 20 percentage points less likely to report the curriculum follows on from school or college.

Similar to other studies (e.g. Mayerhofer et al., 2023), there is a negative impact on student confidence following transition. A greater decline in confidence for female students, and their need to work harder may be connected to less sense of belonging and stereotype threat, but could be mitigated by promotion of mathematical ability as an acquired, rather than fixed, trait (Good et al., 2012). Student self-perception was also patterned by attainment, consistent with Geisler et al. (2023).

This study is timely in detecting a rapid increase in AI usage during the first semester. Among university students, AI usage is linked to high workload, time pressures and lower academic performance (Abbas et al., 2024). In contrast, this study shows using specialist mathematics software is positively linked to academic outcomes. Bergsten and Jablonka (2019) note the impact of technology on transitions in mathematics education warrants further investigation.

The data were collected as part of a wider monitoring exercise of UK mathematical education, but the results could be interpreted in relation to theoretical frameworks. For example, regarding EVT, the data shows lower student achievement is strongly correlated with lower expectancy for success (as measured by confidence), self-efficacy (e.g. reliant on teacher explanations), utility value (e.g. holding a fragmented view of mathematics) and higher cost (e.g. frustration with the pace of teaching). For all these constructs, there were more negative measures for female students than males, while the prevalence of assessment-related learning strategies suggests attainment value is universally high.

In summary, this large quantitative study across UK universities finds, in terms of the lenses identified by De Guzman et al. (1998) and Di Martino et al. (2022), that the *cognitive* jump and negative *affective* responses may be related to students failing to

recognise the *epistemological* change, nor adapt their learning strategies to different *didactical* approaches. In addition, results are strongly patterned by *socio-cultural* factors such as sex and type of university.

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Prestructuralist and Structuralist Praxeologies: The Case of Connectedness and the Intermediate Value Theorem

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On the basis of the Anthropological Theory of the Didactic (ATD) and structuralist praxeologies grounded in Bourbaki's methodology this paper analyses how connectedness and the Intermediate Value Theorem (IVT) mediate transitions from real to abstract analysis through pre-structuralist and structuralist praxeologies. The aim is to identify continuities, discontinuities, and ruptures organized in the bachelor pathway from analysis to topology. Towards this end we reconstruct institutional praxeological models distinguishing punctual and local praxeologies and tracing structuralist transitions. We observe continuity via shared task families and concepts as well as a rupture via abstraction and an intra-theoretical focus in topology with some unmatched praxeologies.

Keywords: transition across university, real analysis, abstract analysis, structuralist praxeologies, Anthropological Theory of the Didactic.

INTRODUCTION

Recent research into the teaching and learning of analysis at university, such as that conducted by the INDRUM network (Trigueros et al., 2021, chap. 5), has primarily examined the transition from calculus to real analysis, focusing on students' first exposure to formal mathematical reasoning. However, there is growing interest in how analysis connects with more advanced fields such as topology, functional analysis, and complex analysis. Topology courses, for instance, build upon students' earlier understanding from introductory analysis courses and experience with formal definitions and proofs. This advanced knowledge may, in turn, help students reinterpret elementary concepts from pre-calculus and calculus in new ways. Bergé (2008) emphasizes the significance of understanding the topology of the real line, especially the completeness property of real numbers, as a foundational step in the study of analysis. Educational research on topology still remains limited, as most work focuses on the use of diagrams or investigating the feedback that mathematicians provide on students' proofs (Kontorovich, 2021), rather than the interplay between analysis and topology.

Some university-level educational literature does attempt to bridge analysis and topology. For example, Cook et al.(2016) argue that topological concepts are foundational for many high school mathematics ideas, especially continuity and the Intermediate Value Theorem (IVT). They suggest that topologically motivated proofs can offer deeper insights than the more common epsilon-delta proofs and further argue that they are more succinct and elegant highlighting the potential for topology.

The use of connectedness in relation to the IVT is presented as a prime example, and they reference Munkres' (2000) topology textbook, where the third chapter uses both the IVT and the Extreme Value Theorem as primary motivations for developing the notions of connectedness and compactness.

However, existing literature lacks a systematic approach for understanding the scope of these examples and for critically examining the supposed simplicity and clarification that topology provides, both from an epistemological and didactic perspective. Our research addresses this gap by focusing on transition phenomena—continuities, discontinuities and ruptures—that occur at various educational "transition points". Specifically, we are interested in the transition from real analysis, as taught at university entry, to more abstract branches like topology and functional analysis, which are typically introduced in later undergraduate or early graduate studies. We refer to these advanced areas collectively as "abstract analysis", drawing a parallel to the term "abstract algebra."

Several studies on transitions within university mathematics education are cited in the INDRUM synthesis (Hochmuth et al., 2021). Some of these studies build on the model developed by Winsløw (2008), which is based on the Anthropological Theory of the Didactic (ATD) and aims to describe the transition from concrete to abstract reasoning in real analysis. Winsløw distinguishes between two components: concrete analysis (focused on specific classes of functions) and abstract analysis (dealing with axiomatic systems such as metric spaces and operators). While high school calculus primarily addresses practical-technical aspects of concrete analysis, university-level analysis introduces more complete praxeologies that include formal technological-theoretical components. These components draw on abstract analysis concepts providing rigorous justification for the practical aspects of the subject and organizing the underlying theory in a coherent way. The first transition is associated with the development of rigor and the learning of proof, while the second highlights the need to anchor new concepts in previously constructed praxeologies before applying them to more abstract tasks. The construction of institutional praxeological models, based on university syllabi and course documents, supports the relevance of this model as both a descriptive tool for didactic transposition (Bosch et al., 2021) and an explanatory tool for understanding student learning difficulties.

So far, the focus on abstract analysis as a collection of structures is often lost, and the dialectic between concrete and abstract reasoning—which impacts praxeologies—is not always clarified from an epistemological standpoint. The aim of this paper is to address these issues and provide new tools for the didactic study of under-researched branches of mathematics such as topology, while maintaining connections to the foundational fields upon which they are based. For this purpose we draw on the research program initiated by Hausberger (2018) around the concept of structuralist praxeology. Starting from an epistemological analysis of mathematical structuralism, Hausberger proposed a model in which the first transition is not mainly seen as a completion of praxeologies, but as the construction of structuralist praxeologies

according to the methodology outlined by Bourbaki (1950). These praxeologies evolve and generate tasks that focus mainly on concepts and structures, leading to a second, more abstract transition.

To test this model in the context of abstract analysis, we analyzed excerpts from a textbook (Bridges, 1998) used at the bachelor-master transition, chosen for its explicit didactic goal of showing how the concepts and theorems of abstract analysis illuminate real analysis, which is first recapitulated in view of its generalization. Thus, we demonstrated (Hausberger & Hochmuth, 2023) through a praxeological analysis of typical tasks from chapters on the real line and on metric spaces that abstract analysis teaching covers a wide range of structures that generalize real analysis, and that the abstract approach at the master's level is comparable to abstract algebra in terms of didactic challenges—particularly the development of a productive concrete-abstract dialectic. In our subsequent work (Hausberger & Hochmuth, 2024), we analyzed tasks related to the topological notion of connectedness in metric spaces and discussed their relationship with real analysis tasks from the perspective of Klein (Winsløw & Grønbæk, 2014). Unsurprisingly, the relationship with the IVT was prominent. However, we also observed that the generalization of the IVT to metric spaces was not problematized—other features of real line intervals, such as convexity or path-connectedness, could have been considered—and that the dialectic between concrete examples and abstract structures was not brought to life.

This paper seeks to further develop these results by applying our tools to data that can provide a clearer view of the development of structuralist praxeologies along the analysis pathway, from real analysis to topology. Given our previous work, we focus specifically on the relationship between connectedness in topology and the IVT in real analysis. We begin by presenting the theoretical framework that serves to state our research questions and sketch our methodology. The result section is dedicated to the praxeological analyses of the data and their discussion. Finally, a few conclusions are drawn together with further avenues of research.

THEORETICAL FRAMEWORK

The ATD, as outlined by Chevallard and Bosch (2020), aims to elucidate societies' relationships to the didactic, which encompasses all possible factors influencing learning. By adopting an anthropological perspective, the ATD seeks to embrace the didactic wherever it appears, with particular attention to institutional constructions of knowledge and the conditions established for its dissemination. The didactic phenomena examined in this article concern the impact of structuralism as a specific epistemology of mathematics on the teaching of real and abstract analysis and their interrelationships. For our purposes, we analyze the institutional relationship to real and abstract analysis during the bachelor's degree by examining several textbooks, which are considered products of institutional processes of didactic transposition and collectively cover the three years of undergraduate study.

Praxeologies

Within the ATD framework, the institutional relationship to a mathematical object (such as a concept in analysis) is analyzed through the set of praxeologies in which the object is involved at any component. Every human activity, according to the ATD, consists of the coordination of a praxis (know-how) and a logos (know-why), giving rise to the key notion of a praxeology. A praxeology is represented as a quadruple $[T/\tau/\theta/\Theta]$ that may be described as follows: The praxis block Π consists of a type of tasks T together with a corresponding technique τ (useful to carry out the tasks $t \in T$) and the logos block Λ comprises the technology θ, a discourse on the technique, and the theory Θ, the ultimate level of justification.

Structuralist and prestructuralist praxeologies

Mathematical structuralism, as developed by Bourbaki (1950), is a methodology that involves reasoning in terms of classes of objects, relations between these classes, and the properties of operations on structures (Hausberger, 2018). This approach allows specific properties of objects to be seen as consequences of more general theorems about structures. Conversely, generalizations are tested against specific objects, creating a dialectic between objects and structures.

In praxeological terms, the structuralist method involves moving from a praxeology $P = [T/?/?/\Theta_{particular}]$, where the appropriate technique is not always clear, to a structuralist praxeology $P_s = [T_g/\tau/\theta/\Theta_{structure}]$, in which the theory of a given structure guides the problem-solving process. This transition (type 1) leads to a structuralist praxeology that gains its rationale from its ability to solve concrete problems by leveraging the insights provided by structures. Effectively, it is a transition from a praxis $\Pi = [T/?]$ to a structuralist praxis $\Pi_s = [T_g/\tau]$ with a structuralist logos Θ.

Hausberger (2018) distinguishes two levels of structuralist praxeologies: At level 1, structures act primarily as vocabulary, and tasks are solved by checking definitions (e.g., verifying boundedness of a function between metric spaces by checking the definition); at level 2, techniques involve general results about structures (e.g., proving that every continuous function from a compact space into a metric space is bounded). In the process of developing level 2 praxeologies, abstract tasks (such as proving theorems about general classes of objects) are assigned to establish the necessary technology. Then the stage is reached when praxeologies based on such abstract task types are developed—this is called the type 2 transition, typically situated between bachelor and master degree programs. The model posits that the new purely abstract praxis Π' is anchored in reasoning with concepts originating from the logos of previously developed structuralist praxeologies. This connection between the two transition types is considered vital for a sound and properly motivated development of abstract analysis.

In early analysis courses, generic real functions are considered, but the domain and/or codomain are typically limited to real intervals or $\mathbb{R}^n$, rather than truly generic sets equipped with additional structure. Abstract tasks might involve proving variants or

refinements. Because these praxeologies do not make explicit reference to structures like metric or topological spaces, they are considered pre-structuralist with respect to those frameworks.

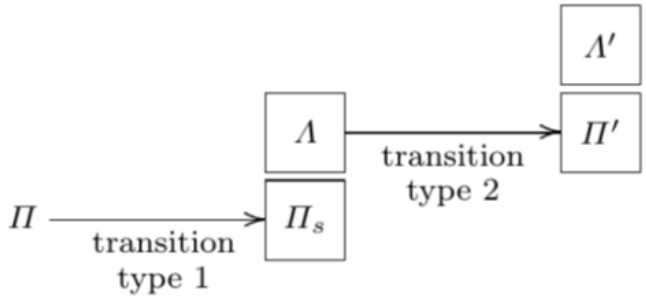

Figure 1: Structuralist transitions

A quadruple [T/τ/θ/Θ] is called a punctual[1] praxeology (PP). Several PPs can be unified by a common technology θ to form a local praxeology (LP), and LPs can be unified at the theoretical level Θ to create a regional praxeology (RP). Chapters on compactness or connectedness in textbooks typically correspond to LPs, and all LPs related to topological notions in metric spaces can be unified into a regional praxeology. The unification of PPs into LPs and RPs is complex for both didactic (e.g., a spiraling approach may keep some structuralist aspects implicit) and epistemological reasons (e.g., different structures generalize different aspects of the real numbers).

RESEARCH QUESTION AND METHODOLOGY

We assume that the structuralist method, in the Bourbaki sense, is prevalent in academic mathematics. However, it is less clear how this method appears in teaching materials and how it is processed, implemented, or neglected during didactic transposition by teachers. Our research methodology is based on the theoretical framework and praxeological modeling outlined earlier, with particular attention to identifying structuralist aspects in these models.

In previous work (Hausberger & Hochmuth, 2023), we demonstrated that empirical data from textbooks could be modeled praxeologically to reveal structuralist aspects, particularly in the context of compactness. We further developed this approach with a focus on teacher training and Klein's ideas (Hausberger & Hochmuth, 2024). In this article, we focus on the concept of connectedness and, in this context, on the IVT. Our analysis draws on both scholarly topological knowledge and didactically transposed versions of this knowledge. More precisely, we use Munkres' (2014) textbook as a source for modeling the teaching of connectedness in topology, and Peng's online lecture notes (Peng, 2023a, 2023b) for studying praxeologies related to the IVT in real and multivariate analysis.

[1] Punctual and also local refer here to organizational aspects of praxeologies rather than topological properties.

Thus, our research question is: How is the transition from analysis to topology - specifically regarding connectedness and the IVT - organized throughout the bachelor's program, and what continuities or ruptures can be identified?

RESULTS

In Chapter 3 of Munkres' book, titled by "Connectedness and Compactness", the concept of connectedness is presented as the appropriate notion for generalizing the IVT to arbitrary topological spaces[2]. More precisely, the topology course on connectedness provides an opportunity for the development of a local praxeology composed of six punctual praxeologies, unified by a technological discourse organized around the definition of connectedness (and its variant, path-connectedness), the structuralist properties of stability, their consequences for solving equations via the generalized IVT, and the concept of the linear continuum in relation to the real line. This corresponds with the historical development of connectedness and its relationship to the continuum.

	Type of tasks	Technology	Theory
P_1	prove that a topological space is not connected	definition of a separation	Topology: definition of open and closed sets, limit points, topological space, homeomorphism
P_2	prove that a topological space is connected	definition and structuralist properties of connectedness	
P_3	prove that a given ordered set is a linear continuum or not	definition of linear continuum, supremum	
P_4	prove that a subspace of a topological space is path-connected or not	definition and structuralist properties of path-connectedness	
P_5	prove that given topological spaces are homeomorphic or not	connectedness as a topological invariant	
P_6	show that an equation $f(x) = 0$ admits a solution (f a given continuous function between topological spaces)	(generalized) IVT	

Figure 2: Praxeologies related to connectedness in Topology

Moreover, our analysis of excerpts from the textbooks by Tao Lee Peng (2023a, 2023b) lead to the identification of a punctual praxeology in the first year, centred on the IVT for univariate functions[3], and a local praxeology in the second year, in the case of multivariate functions[4], composed of five punctual praxeologies. The first three are unified by a technological discourse around path-connectedness, which allows the application of the IVT to path-connected domains in $\mathbb{R}^n$. Subsequently, the notion of connectedness is developed, replacing the theoretical framework for applying the IVT with a more general one that is tied more closely to the structure of a topological space.

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2 Let f : X → Y be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between f (a) and f (b), then there exists a point c ∈ X such that f (c) = r.

3 Given a continuous function f : [a, b] → R, then for any w between f (a) and f (b) there is a c ∈ [a, b] with f (c)=w.

4 Let D be a path-connected subset of R^n, and let f : D → R be a function defined on D. If f is continuous, then f (D) is an interval.

The didactic choices made by Lee Peng can be understood as part of a movement toward generalization and abstraction, progressing from real analysis in the first year to topology in the third year—essentially, the emergence of structuralist thinking in analysis. Referring to the tables in the figures 2 and 3, praxeologies related to the IVT in univariate and multivariate cases unfold into the corresponding praxeologies in the topology course, while others lead to related but distinct praxeologies. This connection with the structuralist praxeologies of topology allows to identify the praxeologies of analysis as pre-structuralist with respect to the structure of a topological space. Notably, some structuralist praxeologies in topology have no pre-structuralist precursors in our model, and conversely, some pre-structuralist praxeologies from analysis are not continued in topology (for example, those focused on convexity).

	Type of tasks	Technology	Theory
P'_1	prove for $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ the existence of some $x \in D$ with $f(x) = c$.	IVT for univariate real continuous functions	Real Analysis, in particular completeness of R, univariate real continuous functions
P'_2	prove that a subset in $\mathbb{R}^n$ is convex or not	definition of convexity	
P'_3	prove that a subset in $\mathbb{R}^n$ is path-connected or not	definition and pre-structuralist properties of path-connectedness, convexity implies path-connectedness	
P'_4	prove the existence of certain points in the context of multivariate real functions and path-connected sets	IVT for multivariate real continuous functions on path-connected sets	Multivariate Analysis: $\mathbb{R}^n$ (complete finite dimensional normed vector space), multivariate real continuous functions
P'_5	prove that a subset in $\mathbb{R}^n$ is connected or not	definition and pre-structuralist properties of connectedness, path-connected implies connected	
P'_6	prove the existence of certain points in the context of multivariate real functions and connected sets	IVT for multivariate real continuous functions on connected sets	

Figure 3: Praxeologies related to IVT and connectedness in Analysis

The order of development is essentially reversed: the topology course begins by establishing the framework of connected spaces (the relevant notion from a topological perspective) and develops theoretical consequences before presenting applications (often limited to those treatable using the elementary IVT). In contrast, the analysis course builds up to abstraction by moving from convexity to path-connectedness and then to connectedness, and from the IVT in real analysis to its multivariable versions. The IVT appears in several forms (see footnotes) and gives rise to distinct praxeologies. The relationships between analysis and topology praxeologies can thus be interpreted as evidence of a certain continuity in the learning of analysis, tied to the development of the structuralist method and reinforced by teachers' didactic intentions.

Our model emphasizes the importance of type 1 transitions, where non-structuralist practice gives rise to mathematical problems that are greatly facilitated by adopting a more general perspective—thus motivating generalization and abstraction. For example, solving equations of the form f(x)=0, where f is a continuous real function

defined on a domain D, is well-served by the IVT when D is an interval. Rather than posing new problems that challenge the limits of this technique, the textbook by Munkres presents the structuralist generalization of the IVT within the topological framework as the natural extension of the classical result. Similarly, Lee Peng's examples often assume connectedness as a given, introducing it as an essential property for extending the IVT to multivariate functions without further problematization. Path-connectedness is initially emphasized as a more intuitive concept, while connectedness is introduced at the end of the chapter to address the theoretical question of necessity.

As a result, the dialectic between objects and structures does not operate at the level of the praxeology linked to the application of the IVT, but rather at the level of examining the properties of connectedness (and its variants), in order to determine the connectedness of specific domains using effective techniques. This inquiry is carried out in the structuralist spirit, considering properties preserved under set-theoretic operations, images under continuous mappings, and so on. Objects (i.e., specific examples) are used to guide intuition in constructing proofs or to produce counterexamples.

This allows us to situate the topology course within a type 2 transition, whereas the first- and second-year analysis courses are primarily engaged in type 1 transitions, developing pre-structuralist praxeologies in relation to the structure of topological space. We interpret the existence of these pre-structuralist praxeologies as a didactic device designed to facilitate type 1 transitions and to provide continuity. In our theoretical framework, we hypothesize that these praxeologies serve as foundational supports for the development of structuralist praxeologies in topology.

CONCLUSION

Our praxeological analyses of scholarly topological knowledge (in a didactically transposed version) for connectedness and for the IVT identifies for Topology a local praxeology around connectedness/path-connectedness, stability properties, and generalized IVT, organized structurally before applications. For Analysis we mainly find prestructuralist praxeologies, i.e, IVT in $\mathbb{R}$ and $\mathbb{R}^n$ evolve from convexity to path-connectedness to connectedness, anchoring later topological notions. Thus, it shows that tasks in analysis courses are mostly contextualized, while those in topology are mostly abstract, even when they concern similar types of tasks. This difference further supports the distinction between pre-structuralist and structuralist praxeologies and informs our understanding of how the transition from analysis to topology is managed in undergraduate mathematics education. On the other hand, the topology course ultimately aims to present a coherent theoretical corpus, organized in the structuralist spirit. The concrete examples encountered in multivariable analysis do not play a central role in the exposition of the topology course, especially since the topologies considered often go beyond those derived from the Euclidean norm. This leads to a rupture associated with abstraction and generality, characteristic of type 2 transitions, and is accompanied by the emergence of intra-theoretical questioning.

It is also worth questioning the extent to which pre-structuralist praxeologies contribute to the development of a structuralist framework. With respect to connectedness, the early presentation in multivariate analysis seems to serve primarily as acculturation. Furthermore, technical difficulties in writing proofs of connectedness in multivariate analysis often stem from the lack of explicit introduction of topology and subspace topology. In other words, the formalism of topological spaces serves to simplify proofs, affecting the logos level of earlier praxeologies. This remains largely implicit in textbook commentary, and the motivation for fully structuralist praxeologies is ultimately a didactic issue. Overall, we observe continuity through shared task families and concepts, discontinuity through abstraction and intra-theoretical focus in topology, with some unmatched praxeologies. Pre-structuralist work provides insufficient motivation for complete structuralism. An explicit topological formalism might simplify earlier proofs – and thus reveal starting points for new didactic designs and future research.

In summary, our praxeological model offers a framework for analysing the transition from analysis to topology courses with a focus on the development of the structuralist perspective. However, we consider it essential to examine the values of didactic variables in more detail in order to investigate the dialectic between objects and structures at the level of the main types of tasks emphasised by teachers and thus substantiate these claims. Moreover, the systematic epistemological analysis of this dialectic may provide a useful starting point for future didactic engineering projects aimed at specific teaching goals.

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Discrepancies between teachers and students and negotiation of the didactic contract in the first year of university

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The didactic contract is a dynamic process that unfolds during interactions between students and their teacher. Can certain discrepancies between non-specialist students in their first year of university and their mathematics teacher hinder the negotiation of a shared contract? We first examine the literature, and classify the discrepancies identified in previous works according to the elements of the didactic system. A case study on a tutorial on sequences allows us to actually observe these different types of discrepancies. Rules of the didactic contract are sometimes negotiated along the interactions between the teacher and her students and the discrepancies are overcome. Nevertheless, other discrepancies remain, at least for some students.

Keywords: Secondary-Tertiary Transition, Non-specialist students, Calculus, Didactic contract, Case study

1. INTRODUCTION

The changes experienced by students during their transition from secondary school to university can be considered as changes of the didactic contract (Brousseau, 1997). But different perspectives on the didactic contract exist, and this can lead researchers to develop different interpretations of first-year students' difficulties (Gueudet & Doukhan, 2025). In this paper, we consider the didactic contract to be the outcome of a negotiation process between the students and their teacher. We argue that this perspective can be productive, in particular to avoid considering that a single didactic contract exists at university. Pinto and Koichu (2023) have evidenced that university teachers now acknowledge the diversity of students, and are willing to adapt their practices to this diversity. We therefore consider that different didactic contracts could coexist in the same class.

Nevertheless, we also claim that some discrepancies between teachers and students must be overcome to negotiate a shared contract. We use here the term "discrepancies" with a very general meaning, that will be refined along our work: discrepancies between students' expectations and teachers' actual practices; between students' and teachers' beliefs, amongst others. We study these discrepancies, and their consequences for the negotiation of the didactic contract in the context of mathematics courses for first-year non-specialiststudents. We assume indeed that they may be

particularly numerous in this case, and we are working within a project [1] that aims, in particular, to identify them. The research questions we study here are:

Which types of discrepancies exist between first-year non-specialist students and their teachers? Are some of them overcome by the negotiation of the didactic contract? Do some of them hinder the negotiation of the contract?

2. THEORETICAL FRAMEWORK AND BACKGROUND

Didactic contract

The concept of the didactic contract has evolved over time in the work of Brousseau and others. The different definitions can lead in particular to different interpretations of the secondary-tertiary transition in terms of didactic contract (Gueudet & Doukhan, 2025). In this paper we retain the following definition:

> "the set of reciprocal obligations and 'sanctions' which each partner in the didactical situation imposes or believes him/herself to have imposed, explicitly or implicitly, on the others, or are imposed, or he/she believes to have been imposed, on him/her with respect to the knowledge in question. The didactic contract is the result of an often implicit 'negotiation' of the mode of establishing the relationships among a student or group of students, a certain milieu and an educational system." (Brousseau & Warfield, 2003, p. 7)

We choose to view the contract as a set of expectations and obligations that may evolve and be negotiated between the students and the teacher. This perspective should allow us to move away from viewing the university's didactic contract in mathematics as a stable and unique set of rules to which novice students must adapt. Moreover, this definition emphasizes the importance of the relationships between all the elements of the didactic system (Brousseau, 1997): the students, the teacher, the knowledge in question and the milieu (objects with which the subject interacts in a situation). It is closely linked to the approach proposed by Herbst and Chazan (2012), who study the didactic contract as negotiation processes in what they call the instructional triangle.

The didactic contract negotiation processes involve reconciling the expectations and mutual obligations that the partners in the teaching relationship believe they have, thus reducing the discrepancies. In order to understand these processes, we firstly investigate which such discrepancies have been identified in the research literature for first-year non specialist students.

Background on discrepancies between students and teachers

Only a limited number of studies explicitly highlight discrepancies between teachers and first-year non-specialist students. We observed that these studies identify discrepancies concerning the different elements of the didactic system: the knowledge at stake, the teacher, the students, and the milieu. In fact the identified discrepancies can often be seen as relating to interactions between two or more elements. However, for the sake of brevity, we have opted here for a simplified classification where the categories are the four elements of the didactic system.

Discrepancies concerning the knowledge at stake

Some discrepancies concern the perspective on the discipline and its learning. Several papers evidence a gap between a conceptual (on the teacher's side) and a procedural (on the students' side) perspective on mathematics. This is pointed out for example by Ni Shé *et al.* (2017b) in a paper regarding students' difficulties. For most mathematical topics, students claimed they had more difficulties in applying procedures than in understanding. In contrast, most lecturers think that students' difficulties are due to a lack of conceptual understanding, even when they are able to apply the procedures. As a possible explanation of this mismatch, the authors propose that students and lecturers attribute different meanings to "understanding", whether they adopt a conceptual or a procedural perspective. Another mismatch linked with the perspective on the discipline concerns the characteristics of a mathematical discourse. Jablonka *et al.* (2017) asked lecturers and students to rank four texts as more or less mathematical, and interviewed them to understand their choices. The results showed that some low-achieving students had difficulties in recognizing a university mathematical discourse, selecting as more mathematical the texts that were more accessible for them.

Ni Shé *et al.* (2017b) highlight a discrepancy in the perception of difficulty of different mathematical topics. A typical example is basic algebra, which is according to teachers the most difficult topic for students, yet rarely mentioned as difficult by students themselves. A gap concerning high school mathematical content appears in Azrou (2023), regarding the learning of proof. Lecturers consider that university students have learnt how to deal with proof, particularly in geometry in secondary school. In contrast, students feel incompetent and powerless in writing proofs, particularly when it involves contents that were not met before in secondary school, such as complex numbers.

Discrepancies concerning the teacher and teaching practices

Regarding teaching practices, the literature identifies discrepancies between teacher's intention and students' experience. An example is provided by Bridoux et al. (2024), regarding the usefulness of details. The teacher intends to present mathematical details of the new concepts to foster the students' understanding. But the students' experience is the opposite: they feel lost in too many details and examples. Moreover, the pace of the course is too fast and the content too dense for them; and the lecture leaves no room for interacting with the teacher. Similarly, Viirman and Nardi (2017) found that biology students were lost when their mathematics teacher proposed a modelling activity in order to motivate them. The activity appears too complex and students seem unable to give sense to it. Another discrepancy is described by Azrou (2023) regarding teaching and learning proof. Students expect their math teachers to teach them how to produce a good mathematical proof, while the teachers believe that writing a proof is acquired through practice and does not need to be taught as such.

Discrepancies concerning the students and their activity

The literature also shows discrepancies between students' activity during lectures and what the lecturers expect them to do. A typical example is note-taking (Meehan et al.,

2017): a lecturer expects her students to write down what she explains orally, but she realises that they only write down what stands on the board.

Discrepancies concerning the milieu: resources and their use

Many authors highlight discrepancies between the resources suggested by teachers and those actually used by students. Pepin and Kock (2020) compare students' use and orchestration of resources with two lecturers' recommendations. One of the lecturers provided a home-made reader with all necessary mathematical content and important exercises, and a few complementary resources that were used by students according to the lecturer's guidance. The other lecturer provided a large amount of online and paper resources that students could pick from according to their needs, but gave no indication on how to orchestrate them. Students had difficulties navigating through them, which the authors interpret as an unclear didactic contract, leading students to create their own study path. Another mismatch regards Internet usage for mathematical problems. Aguilar and Puga (2020) investigated the way students use the Internet as a source of help to solve mathematical problems. They found different usage types, dominated by mastery-oriented usage (instrumental help-seeking), with some limited dependency-oriented usage (executive help-seeking). However, the authors point out that some lecturers ban online resources and discourage their adoption among students. Ni Shé *et al.* (2017a) describe the types of resources used by first-year undergraduate students vs. those recommended by lecturers. While handouts are viewed as helpful by students and lecturers, students also appreciate websites and online videos, which seem to them easier to understand than books, whereas resources recommended by lecturers include books, and technology-enhanced resources they have developed themselves, rather than freely available online resources. Finally, Ni Shé *et al.* (2017a) also point out that students appreciate resources where the solutions or even detailed answers are explicit, while lecturers generally prefer resources where complete solutions are unavailable.

Drawing on these previous works, we designed an empirical study to investigate the presence of these different kinds of discrepancies in class, and their consequences in terms of negotiation (or not) of a shared didactic contract.

3. METHODS

For this exploratory data collection, we asked several colleagues in our universities about the possibility to record one of their lessons for first-year non-specialist students during the first months of their courses. We supposed indeed that the didactic contract was not yet stabilized. Maud, a mathematician strongly committed to pedagogy and working in a middle-size French university, accepted. She proposed that we observe and record one of her sessions for students enrolled in a "Biology and computer science" program; this session was dedicated to sequences and their limits.

Since the COVID-19 pandemic, Maud designs and posts online videos presenting the solution of each exercise of her tutorials. In class, her students are invited to work in groups and to discuss the exercises on blackboards. Maud walks and visits the groups;

she never writes solutions on the board. Maud has more than 20 years experience in teaching at university. However, it is her first year teaching this course, for which she is responsible for two tutorial groups (the lecture is taught by another teacher). We observed a three-hour tutorial session on sequences (limits, arithmetic, geometric and arithmetico-geometric sequences) during the first semester 2024-2025 (November). Before the tutorial, the students followed a two-hour lecture on sequences. For some students this subject is not new: indeed in grade 12, those who had chosen the "maths specialty" (6h a week) or "maths complements" (3h a week) courses had already learned about sequences and their limits.

We met Maud the week before for a one-hour interview (IT1) with three parts: (1) her teaching of first-year students in general, (2) her teaching in this specific course, and (3) the tutorial we were going to observe. We also interviewed her after the tutorial about how it went (IT2). One week before the observed session, we visited Maud's course, presented our project and asked for volunteers for an interview after the tutorial. Two students, Greg (ISG) and John (ISJ), volunteered. Both are in their second attempt at university, having attended other programs the previous year without success. Both had also followed the "math complement" option in grade 12. We interviewed them and collected their written work. Our data also includes the exercises sheet (provided by the lecture teacher); the text of the control about sequences after the end of the chapter; and Greg's and John's productions for this control.

In the transcribed interview of the teacher, we identified her views on mathematics; her pedagogical choices; her beliefs and expectations concerning the students' activity in class and out-of-class. We did the same with the students' interviews, and confronted them with the teacher's interview, searching for discrepancies. Then we classified these discrepancies according to the four categories presented above. We also compared the declarations of the teacher and of her students with what happened during the tutorial, the exercises sheets proposed and the students' notes. Moreover we transcribed selected episodes of the tutorial (TU) in which discrepancies were observed, focusing on whether a shared didactic contract was negotiated to overcome them.

4. RESULTS

Discrepancies concerning the knowledge at stake

Concerning the content taught in grades 11-12, Maud declares at the beginning (TU):

> Maud: [speaking to the whole class] This chapter contains revisions for those who took maths or maths complements last year.

Indeed the 'maths specialty' and 'maths complements' official curriculum mentions "limits of sequences". Nevertheless, in "maths complements" the students do not learn a formal definition of limit. The first exercise of the tutorial is (figure 1):

Do the following sequences have a limit? If so, compute it.
a) $u_n = 1 - \frac{1}{n^2}$ b) $u_n = 2^n - n$ c) $u_n = (-3)^n$

Figure 1: First exercise of the tutorial about sequences

A group of students working on the board calls Maud when she passes by. They ask her at which point they must compute the limit. Maud is quite surprised, and says that the limit of a sequence is always in $+\infty$, but she does not provide a justification for this statement. The data suggest that Maud's expectations regarding prior knowledge did not align with what some students had actually learned in grade 12. They learned that "a sequence (u_n) has a limit l in $+\infty$, if the terms u_n can be as close from l as we want, when n is large enough." But they were never taught that one does not compute the limit of a sequence for example in 0, and Maud does not find a way to explain this. In his interview Greg declares:

> Greg: Sometimes there are things that we are supposed to know, so they are not explained again.

In Greg's notes that we collected after the tutorial, we observed that he computed for each sequence its limit in $+\infty$ AND its limit in 0. Here the discrepancy lies between what Maud thinks they learned in high school and what they actually learned. In the post-lesson interview, Maud declared about the text of the exercise:

> Maud: I think it would be better to rephrase it. Here, we're doing maths as tools, I will ask them to calculate a limit at $+\infty$. So, I will change the text of the exercises.

During the lesson, on several occasions Maud tells the students that they cannot provide a rigorous justification, and that she only expects some arguments supporting their answers. She does not expect her students to learn conceptual mathematics, contrary to what has been suggested in some of the previous studies mentioned above.

Discrepancies concerning the teacher and teaching practices

About pedagogical practices, both students declare that they feel left to their own in this tutorial, and more generally at university. They say that they sometimes ask Maud for support during the tutorials, but are disappointed by her answers because she does not provide direct advice. We consider that this reflects a discrepancy between the teachers' support expected by the students (most probably resulting from the secondary school teaching practice) and the support they actually receive at university. They feel in particular unsure about the justifications expected from them; this is directly linked with Maud's specific practices. Greg compares university with high school:

> Greg: In high school, we saw the justifications because the teacher redid the exercises.

There is no shared contract rule in the class about the justifications to provide, e.g. to claim that a sequence is divergent. Indeed, during the tutorial Maud provides as much feedback as she can to each group, but she does not write on the board the expected solution: this solution is available in her videos. If the students do not watch her videos out of class they do not have access to examples of solutions expected from them. This is directly linked with the discrepancies concerning the milieu (here the resources), we discuss this in the corresponding subsection.

Discrepancies concerning students' activity

Several aspects in our data reveal discrepancies concerning students' activity. Maud declares (IT1) that some of the students do not master elementary mathematics, like fractions as they are taught in grade 9, and that this creates an obstacle for her teaching of first-year content. She expects the students to work by themselves out of class to develop their fluency with elementary computations (fractions, algebra), and recommended websites for this aim. John acknowledges that he needs to review some basic mathematics, but he did not start at the time of the tutorial observed (ISJ).

Concerning written notes taken by the students during the tutorial, Maud considers that they are under students' responsibility, according to their needs and personal organisation. Her primary aim is that students during the tutorial are collectively engaged in searching the solution of the exercises. During the tutorial, we observed that John was often in different groups on the blackboard, but he never wrote himself on it. When we interviewed him, he admitted that he took no written notes, and added:

> John: I understand the exercise perfectly well orally. But as soon as it's written down and I have to write something myself, it's more complicated for me.

This discrepancy concerning written notes does not completely coincide with the one identified in the literature, since Maud does not expect her students to take written notes during the tutorial. It concerns here students' responsibility expected by the teacher: Maud thinks that the students are mature enough to organize their work and decide if they need to take written notes or not. But John would probably need support (or constraints) to write mathematics.

Discrepancies concerning the milieu: resources and their use

Concerning the resources proposed by the teacher compared those used by the students, the discrepancy identified in the literature is clearly present in the case we study. Maud makes a video for each exercise in the tutorial showing the solutions (the video shows Maud's hands writing down the solution to the exercise as she comments aloud). At the beginning of the year, Maud recommends watching the videos; she regularly reminds the students to do so. But in his interview, Greg admits that he never watches the videos, and declares at the same time that he often searches the Internet for maths resources. John declares that he watches Maud's videos after the tutorial, and also other

videos. He does not seem to give Maud's videos any special status among those he watches; moreover, when watching videos at home he does not take written notes.

5. DISCUSSION

In the case we studied, like in the literature, we have identified discrepancies between the teacher and her students concerning the four elements of the didactic system: the knowledge at stake, the teacher, the students and the milieu (here the resources).

Concerning the knowledge at stake, here the limits of sequences, Maud overestimated what at least some students learnt in high school. Because of this, some students were not able to fulfil their responsibility in class (solving the exercises) since they did not know where they had to compute the limit. We observed here the negotiation of a didactic contract's rule: Maud explained that "the limit of a sequence" is the limit in $+\infty$; moreover in further exercises, she changed the text and wrote "compute the limit of the sequence in $+\infty$". This choice might be questionable for a more conceptually-oriented teacher; but Maud considered that she had to teach mathematics as a tool for these non-specialist students; she avoided theoretical considerations.

Concerning students' activity, Maud wants her students to be actively involved during the tutorials, working individually or in groups on the exercises. The students seem to adhere to this request: "the students actively work on the exercises during the tutorial" is a didactic contract rule shared in Maud's class, negotiated before our data collection. Maud explained (IT1) that she advises students to leave the room if they do not work during the tutorial. To encourage active participation, she avoids writing the solutions on the blackboard: preventing students from simply waiting and copying them. Maud makes a video for each exercise, and she expects the students to watch her videos and take notes at home. This is a central aspect of her practice, which allows her to let students choose whether or not to take written notes during class, since she expects them to do so while watching the videos at home.

For the two students interviewed, this leads to discrepancies creating obstacles that seem difficult to overcome and hinder the establishment of a didactic contract rule shared with Maud. Greg does not watch Maud's videos. He is actively involved during the lesson, working mostly by himself, but after the tutorial his written notes contain numerous errors. John also participates actively during the class, but he does not take notes and does not write at home when he watches the videos either. Greg and John expect more guidance and feedback from Maud during the tutorials. Nevertheless they do not explicitly ask for more support, and finally there is no shared contract rule negotiated between them and Maud concerning this aspect.

We will further investigate discrepancies between teachers and students, and the didactic contract negotiation processes. In the literature, only a few studies explicitly address such discrepancies, so we will complement them by considering studies from which discrepancies can be inferred (for example studies evidencing that students were

surprised by the teacher's expectations when they started at university). We have already collected data in other classes and we analyse it, in particular to see if we find the same discrepancies and similar consequences for the didactic contract. We hypothesize that the discrepancies concerning resources will also be observed in other cases. It is possible that in most classes the didactic contract does not incorporate rules concerning students' work with resources out of class. How could such rules be negotiated? The discrepancies concerning the support from the teacher, and thus interactions between students and teachers (which are essential, for the establishment of a shared didactic contract!) also seem especially difficult to overcome.

We retain from these first steps in our study that an approach in terms of discrepancies between the teacher and the students and their classification in four categories corresponding to the four elements of the didactic system enables a detailed understanding of potential obstacles for the negotiation of a shared didactic contract with some students. This allows us to move beyond the vision of a single didactic contract developed through interactions between teachers and their students. Negotiating the didactic contract can also include a negotiation of specific rules for certain students. We claim that this is especially necessary, if university teachers are now to acknowledge the diversity of students.

NOTES

1. Project EUS-BC (Entering University Studies – Beliefs and Contracts), ANR-24-CE41-0456-01

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Pre-service Secondary School Teachers' Collective Productive Struggle in a University Mathematics Content Course

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Productive struggle is central to learning mathematics. In most research, productive struggle has been framed in individual cognitive terms which often hides the socio-emotional aspects of struggle. In this study we introduce the notion of collective productive struggle as a way to foreground the positive role of social interactions and mutual support that make struggling feel normal and worthwhile. Our research study is situated in the context of a university mathematics content course specifically designed for pre-service teachers (PSTs). Drawing on the emergent perspective, we address the ways that PSTs report on engaging in collective productive struggle and the extent to which they aspire to create a learning environment that embraces collective productive struggle in their future teaching and why.

Keywords: differential equations, collective productive struggle, emergent perspective, design-based research.

Productive struggle is central to mathematics learning, yet there are different views about its meaning and enactment (Warshauer, 2015; Warshauer et al., 2021). A conventional view emphasizes productive struggle as an individual cognitive effort where a person makes sense of mathematical problems in order "to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 387). While this view is widely used, it tends to neglect the socio-emotional aspects of productive struggle, such as building confidence and creative inclusive spaces. Recent work has extended the more individual cognitively-oriented view by incorporating socio-cognitive and social dimensions (e.g., Jarry-Shore & Richardson, 2025; Nusser, 2022) and framing mistakes and errors as essential opportunities for growth, identity formation, and enjoyment (e.g., Wessman-Enzinger & Gerstenschlager, 2023).

Although some empirical studies have investigated in-service and pre-service teachers' (PSTs') experiences with productive struggle in the context of methods courses or in primary and secondary school settings as teachers supporting students' productive struggle, very little, if any, studies have investigated PSTs' own productive struggle in the context of their undergraduate mathematics classes. In fact, Young et al. (2024) systematic review identified only 11% of studies, related to how productive struggle impacts the teaching and learning of mathematics, took place in post-secondary settings. Specifically, for the studies in which participants were PSTs, the majority

were conducted in mathematics teaching or methods courses (e.g., Rahman, 2022). Young et al. (2024) also argued that there lacked a "clear explanation as to how the productive struggle was operationalized and measured" (p. 492). Our work aims to contribute to the operationalization and theorization of productive struggle.

Our research study is situated in the context of a third- or fourth-year university mathematics content course, Inquiry Oriented Dynamical Systems and Modeling (IODSM), specifically designed for secondary mathematics PSTs. IODSM focuses on university mathematics and engages PSTs in making connections between IODSM content and secondary mathematics. IODSM is grounded in the four pillars of Inquiry-Based Mathematics Education (Laursen & Rasmussen, 2019). In this approach, students engage deeply in a series of coherent and challenging tasks in a highly collaborative setting. Consequently, there are ample opportunities for PSTs to engage in productive struggle through collaboration and peer-to-peer interaction.

To highlight the social dimensions of productive struggle, we introduce the term *collective* productive struggle. The modifier collective is meant to foreground the positive role of social interactions and mutual support that make struggling feel normal and worthwhile. While the feeling of struggle is something that an individual experiences, in an ideal classroom setting, struggle is shared with peers in ways that are mutually supportive. Our notion of collective productive struggle highlights the ways in which uncertainty can be more productive when it occurs in an intellectually safe and collaborative social environment where learners share their reasoning, learn from their classmates, normalize the right to be wrong, and find value in sharing initial ways of thinking, even if those ways of thinking are revised or discarded. From this point of view, the collective part of productive struggle speaks to the central role of social interactions.

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Our theoretical framing of collective productive struggle draws on the Emergent Perspective (Cobb & Yackel, 1996), which coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and a sociocultural perspective based on symbolic interactionism (Blumer, 1969; Rasmussen et al., 2024). A primary assumption from this point of view is that learning is both a process of active individual construction and a process of enculturation (Cobb & Yackel, 1996). Meaning, growth, or changes in an individual's thinking, beliefs, and values are inseparable from the norms of the classroom community. Symbolic interactionism is central to the emergent perspective as it highlights that it is through social interaction that individuals create, negotiate, and modify meanings (Blumer, 1969). Drawing on this foundational perspective, an individual's experience of struggle and their valuing of this struggle as productive is inseparable from the classroom norms and social interactions. These norms and social interactions include supporting and normalizing students sharing their reasoning, however tentative, and listening to others' reasoning to both understand someone else's thinking and to expand or even change one's own thinking. In this way, collective productive struggle characterizes the ways in which the classroom

community through their interactions with others embraces and values uncertainty, confusion, and tentative reasoning.

For PSTs, their experiences as learners with collective productive struggle may also inform their future teaching and how they support productive struggle in their future students. We will answer the following research questions: 1) *In what ways do PSTs report on engaging in collective productive struggle in the IODSM course?* and 2) *To what extent do PSTs in the IODSM course aspire to create a learning environment that embraces collective productive struggle in their future teaching and why?* The first research question addresses the call for further theoretical development of productive struggle, and the second research question addresses the importance of collective productive struggle for PSTs' future teaching.

METHODS

Our overall study leverages a design-based research approach (Design-Based Research Collective, 2003) at three universities in the United States (US). Data for this analysis comes from the first two semester long iterations of IODSM at two of these three universities. University A is a large Hispanic-serving Institution (meaning that at least 25% of all undergraduates are of Hispanic background) in the Western US with 32 IODSM students and University B is a predominantly white institution in the Southern US with 7 IODSM students. The IODSM course, taught by the fourth author at University A and the first author at University B, is intended for PSTs but open for students who are not planning to teach. IODSM takes a qualitative, graphical, numerical approach to differential equations or dynamical systems as we are focusing on quantities changing over time. We cover first order and systems of differential equations. We also explicitly make connections to content from secondary school such as function translation, proportional reasoning, and polynomial roots. Both classrooms were set up with students arranged in groups of 3 or 4 and every group had access to a shared whiteboard at their table or on a nearby wall (see Figure 1). As part of the broader design-based research project, we collected classroom video recordings of every class session at University A with two cameras.

Figure 1: IODSM Class at University A.

Data sources for this analysis include end of class reflections and an end of semester teaching autobiography. End of class reflections, which were assigned after every class and due prior to the next class, asked students to respond to the following prompt: "Who, either in small group or whole class discussion, said something that influenced your thinking and what did they say? Please provide a little bit of detail." All end of class reflections were used to answer Research Question (RQ) 1. The teaching autobiography, which was used to answer RQ1 and RQ2, invited PSTs to reflect on their experiences in IODSM as well as their beliefs and values as it relates to learning and teaching mathematics. Students were also asked to think about their future teaching (or future professional self if not planning to teach) and how they see themselves teaching secondary mathematics to their future students. Length of submissions ranged from 2-3 single-spaced pages.

As end of class reflections could have been about any aspect of class, we first filtered for reflections that addressed collective productive struggle. The following three criteria, which were not determined a priori but rather the result of explorative, open coding, serve as our operational definition of collective productive struggle. These three criteria are: (1) a reflection needed to contain some form of struggle (e.g., uncertainty, it's okay to be wrong, confused, confidence in saying what you think), (2) a reflection needed to include indication that it was productive (i.e., acknowledgement of the benefit of the struggle in some way), (3) a reflection needed to be situated with others in the class (i.e., they needed to acknowledge social interactions with another student or students).

After developing our operational definition of collective productive struggle, we then coded all end of class reflections per our three criteria. To develop a shared interpretation of the three criteria, we first selected approximately 20% of the end of class reflections to code as a full team. We then subdivided the remaining reflections to be coded by three teams of three people. After each team member coded individually, we had an average of 75.4% agreement. Each team then discussed all discrepancies until 100% agreement was reached. In all from both universities, there were a total of 773 submitted end of class reflections and 270 (34.9%) of those showed evidence of collective productive struggle per our three criteria. At the student level, on average, 26.1% (25.0% at University A and 32.0% at University B) of a student's end of class reflections showed evidence of collective productive struggle.

38 of the 39 students submitted teaching autobiographies. Similar to how we narrowed down the data set of end of class reflections, we began by identifying autobiographies that included reflections on their learning experiences with collective productive struggle and/or envisioned a learning environment that embraces collective productive struggle for their future students. This winnowing of the data resulted in 24 autobiographies that addressed collective productive struggle. Each autobiography was read and coded by two researchers. They then engaged in descriptive analysis (Miles et al., 2014) where they leveraged open coding together as a pair, discussing disagreements as they emerged. This coding led to the development of themes related

to their aspirational future teaching (e.g., creating safe and comfortable spaces, normalizing mistake making in their future teaching).

RESULTS

Collective Productive Struggle: RQ1

In IODSM, PSTs identified collective productive struggle in various ways. For example, some struggle was focused on content or understanding (e.g., expressing uncertainty, changing mind or perspective, expending effort, being confused). Other forms of struggle were about established norms (e.g., it's okay to be wrong, it's okay to disagree) or social-emotional aspects (e.g., confidence in speaking up). Struggle was identified as productive if in some way the end of class reflection acknowledged that it was beneficial for the students. Productive struggle was identified as *collective* if it acknowledged social interactions in some way with other student(s).

Two very common struggles identified by the students were "I did not get it right away" and "it's okay to be wrong." For example, Rohan said (**emphasis** added):

> When I demonstrated what would happen to our solution curve y if we changed its initial condition, **I initially believed** I had it right (lifting it up and twisting it to the right). **It wasn't until I discussed it with my group and let them clarify what should have happened that I finally understood** the correct outcome. Since our differential equation does not depend on time t, the slope field for it would have vectors pointing in the same direction for a given value of y. When I lifted and twisted the solution curve to the right, I accidently changed the slope of that curve, causing it to no longer align with the slope field. **It's not fun to be wrong in front of the class, but we got to remember that making mistakes is a part of the learning process.**

Similarly, students expressed that at first they "did not understand" something but that collaboratively working with their peers was the catalyst for their growing understanding. For example, Onyx said (**emphasis** added):

> I thought [one table] had a really great way of explaining why exactly the transformation of x^2 via $(x+3)^2$ will shift the graph to the left. **In all honesty, I had no idea how to answer that question** when it was asked on the survey we took at the beginning of the semester. As mentioned in class, it was just a rule that we had learned and then implemented from then on out. Even in our individual groups, **I was having a hard time coming up with an explanation for why this happened**. So, when that group said to think about it as the transformed function being "ahead" of the other and needing to have its input "reduced" by 3 in order for the output to be the same, **I felt like I actually understood for the first time what it was that that transformation did**. Using their explanation, I thought about it a bit more after class and decided that imagining the transformed function "compensating" for the changed input in order to produce the same output is **what makes the most sense intuitively for me**.

In both of these examples there is an expression of struggling with the content but that social interactions with other students lead to collective and productive resolution. Sometimes the collective productive struggle was about the norms of the class and not specific to content. For example, Liana said that a conversation with a peer "allowed

[them] to clarify with each other while also letting [them] feel heard. I think today showed me that it is okay to disagree and it can show a different perspective and less what is right and wrong." Another common collective productive struggle that was identified by the students was confidence (lack thereof or burgeoning) in speaking up but that working together supported that confidence. Jasmin said (**emphasis** added):

> One thing that influenced my thinking in class was when we were doing peer review and one of my group members was checking my work on my homework. I appreciate that they helped me see different perspectives on the problems that I didn't see myself previously. **I was also able to ask questions without feeling bad about myself because they were helpful and nice about it.** I was confused on some of the questions so asking one of my group members about them was actually helpful.

Here Jasmin has expressed that her struggles were specific to her social-emotional experiences, but that IODSM students collectively supported her in not feeling bad about herself. In this way, her struggle was made productive by the social interactions with other students in the class, emphasizing the critical role that the collective plays in supporting productive struggle.

Per the end of class reflections, students described the IODSM learning environment as a space where confusion, mistakes, and uncertainty were shared openly among peers and how they made these struggles productive through collective efforts. We also observe similar themes in the autobiographies. Students repeatedly emphasized how collaboration helped transform struggle into a collective resource. Emily explained that the course made her "comfortable with being very wrong before I figure[d] out the right answer", and "never once have I been berated for having a wrong answer, told I was wrong without explanation, or discouraged away from a path I was taking." She further emphasized that (**emphasis** added):

> Also, working with other people who are okay with being wrong has given me the confidence I need to realize that you learn when you are making mistakes, not when you know the answer to everything. **I've learned to be comfortable with confusion and with asking questions (that I used to think were "stupid") because I know that the payoff of my confusion is learning something new**… The group collaboration in this class is the main reason I've been able to keep up… because I'm working with other people who are just as confused as I am.

Here, Emily demonstrated clear evidence that she valued struggling with her classmates. Heidi similarly noted that the course showed her "not every answer has to be perfect, and it is okay to ask for help." Stephen also shared that once the class developed "a classroom culture that is built on working together where no one is nervous to share their ideas," it became easier to "struggle through a difficult ill-defined question." These experiences highlight the powerful role of social interactions which normalizes error and sustains students through challenging mathematical reasoning.

Collective Productive Struggle in Future Teaching: RQ2

PSTs' learning experiences with collective productive struggle strongly shaped the classroom environments they hoped to create as future teachers. The desire to recreate a safe, comfortable, collective environment for their future students appeared across 16 of the 24 autobiographies. Many explicitly linked their learning experiences to the environments they wanted to cultivate as future teachers. For example, Heidi wanted her future students to "feel confident trying their hardest to solve a problem but if they give the wrong answer, it is okay, and we can learn from it." Emily aspired to offer the same learning experiences she had to her future students, because "the struggle is worth it if my students can experience meaningful learning." Laurel similarly sought to "establish a classroom environment where students feel comfortable asking questions, taking risks, and improving." Karim articulated his commitment to a classroom where students "feel safe to try new things, share their thoughts, and learn from their mistakes," noting that such comfort encourages students "to take risks and try new things." Moreover, Jasmin argued that "feeling comfortable in a learning environment is important as well because it helps a student not fall behind if they do not feel comfortable to ask questions when they feel stuck or confused on a question." Liana wanted her "classroom to be a space where students feel seen and heard, where they can ask questions, make mistakes, and learn through exploration." In her future classroom, she wants to "...normalize struggle and provide the kind of encouragement that tells students, 'You are capable of doing this. Let us work through it together'." Kendall highlighted that "also similar to what [Dr. A at University A] does in his classrooms, I want to create an environment where wrong questions and answers are never shamed but also encouraged."

PSTs also described specific ways they intended to design their future classrooms for collective productive struggle. Some emphasized structured group roles (reporter, spy, moderator, starter), whiteboards, and public reasoning as core mechanisms. Jasmin and Liana both planned to use regular group work and student presentations (IODSM classroom design features that they reported to help them learn from different perspectives) to ensure their future students could collaboratively reason, revise ideas, and build confidence in sharing uncertainties in whole class discussions. Jude viewed rotating roles as essential because they ensured "everyone's voices are being heard" and allowed students to contribute in multiple ways. Ann planned to use the specific group role of reporter to guarantee that "all voices in the classroom were heard," helping students become comfortable sharing ideas and engaging in mathematical discussions.

PSTs also demonstrated an emphasis on socio-emotional support for their future students' potential struggles. For example, Ash explained his goal of wanting to create a classroom where students feel "curious, safe, and challenged," aiming not only to teach mathematics but "to break the cycle of fear and pressure" he once experienced. Skye similarly emphasized building an environment where "no one felt ashamed or

embarrassed" when unsure, and "we all worked together to help explain to each other and made sure that our peers felt seen."

Underlying these aspirations were clear reasons grounded in PSTs' own learning experiences. Many described how collaborative sensemaking helped them recognize that uncertainty is a normal part of mathematics, that multiple perspectives deepen understanding, and that collective effort makes challenging problems more manageable. These reasons are similar to what we found from students' reports of engaging in collective productive struggle (RQ1). For example, Jude wrote (**emphasis** added):

> By creating a non-hostile environment, I aim to have students think of math as more of a fun and understandable subject rather than a difficult and confusing one… **I want to create an environment where students can freely experiment with their ideas without the fear of being judged.** This is a core element that I would like to build around as it helps make math a more fun and engaging experience as well as making it a topic that they can do for fun rather than because the teacher told them to. **It is also less frightening and scary to ask one of your peers for help rather than asking the teacher who sometimes may not be available.**

Jude commented on the fun and meaningfulness of mathematics that collective productive struggle can offer, imagining his future students using peers as resources that minimize fear. Rohan discussed how his group transformed confusion into clarity by pooling complementary ideas, leading him to conclude that "collective reasoning often reveals solutions that are inaccessible to individual problem solvers." He then described how "the syllabus' explicit acknowledgment of students' 'right to make errors and revise thinking' created a culture where imperfection became a pathway to growth" led him to learn to "[reframe] mistakes as learning opportunities." He envisioned replicating this in his future classrooms by "normaliz[ing] productive struggle rather than penaliz[ing] it." These learning moments helped PSTs see collective productive struggle as beneficial and necessary for deeper understanding, confidence, and persistence, which motivates them to want to create a similar learning environment for their future students.

CONCLUSION

Addressing calls from Young et al. (2024), we aimed to contribute to the operationalization and theorization of productive struggle, by introducing the notion of *collective* productive struggle. We operationalized collective productive struggle through three criteria: (1) there exists a form of struggle, (2) there is an indication that the struggle is productive with some kind of learning benefits, (3) the productive struggle is situated with others in the class, which reflects our theoretical perspective that elevates the role of social interactions.

Our results indicated that PSTs' struggle in a mathematics class may not be just an individual experience that one must persist through, but rather PSTs may engage in struggle collectively and work to resolve it together. We found a variety of ways in

which PSTs struggled in IODSM. First, the struggle may have been related to the content itself (e.g., that they did not get the math right away). These are struggles that are well reported in the literature (Hiebert & Grouws, 2007). However, we also found that by realizing "it's okay to be wrong" in collective settings (which is a sociomathematical norm, Cobb & Yackel, 1996), PSTs' embraced new ways of engaging in mathematics that they were not used to. In fact, the socio-emotional aspects of struggle were very prevalent. PSTs reported that engaging in collective productive struggle supported burgeoning confidence in speaking up about tentative ways of reasoning about mathematics. These positive social interactions also helped form an intellectually safe environment to learn mathematics where struggling felt normal and worthwhile.

In terms of aspirations to create a learning environment that embraces collective productive struggle in their future teaching, PSTs identified specific teaching practices (e.g., group roles, asking students to share reasoning publicly) from IODSM that they aim to use in the future. Importantly, PSTs often noted that normalizing mistake making is a central way to engage in collective productive struggle because they experienced benefits like building confidence and revising their thinking as learners. These learning experiences also motivated PSTs to aspire to create a similar learning environment for their future students, but of course we do not know if these aspirations will be realized (or not). The impact of these and other undergraduate experiences on their actual practice is a topic for future analysis.

Contributing to the theorization of productive struggle that embraces the importance of social interactions, our work begins the operationalization of collective productive struggle. Moving forward, we will next analyze classroom video data from University A to examine how students' interactions supported each other and how the instructors nurtured an environment that embraced collective productive struggle, informing course design and implementation broadly. Long-term plans include following PSTs into classrooms to observe the ways in which they themselves can support collective productive struggle as a teacher.

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From mathematics to pedagogy and back again: Secondary teachers learning real analysis through co-tutoring

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This paper examines how pedagogical explorations of tertiary mathematics instruction can foster secondary mathematics teachers' learning of real analysis. Within a school-university partnership, ten practicing teachers co-tutored students transitioning to proof-based mathematics. They collaboratively prepared for tutorials in discussions that alternated between mathematical and pedagogical inquiry. Analysis of these alternations revealed three productive types of exploration: critiquing mathematical expositions, unpacking learning opportunities, and uncovering instructional norms. These explorations suggest that pedagogical engagement with tertiary instruction can serve as a mechanism by which pedagogical reasoning and mathematical sensemaking are woven together and become mutually reinforcing in proof-based domains.

Keywords: Real analysis, teacher preparation and professional development, transition to proof, school-university partnerships.

INTRODUCTION

In 2024, ten experienced secondary mathematics (SM) teachers joined a first-year, proof-based real analysis course at a research university in Israel as co-tutors and pedagogical consultants. This atypical school–university partnership was designed to leverage the complementary expertise of university mathematics (UM) instructors and SM teachers in order to better understand students' transition to proof (TTP) and to collaboratively develop TTP-attentive instructional practices within the course. Over the course of one semester, the teachers prepared for tutorials, co-facilitated them alongside UM instructors, and participated in regular reflection meetings focused on students' learning, TTP-related challenges, and instructional decision making.

Most participating teachers had prior – but distant – experience as students of real analysis, yet no experience teaching it. At the same time, they brought forms of expertise typically unavailable in university mathematics instruction, including deep familiarity with students' ways of thinking, common misconceptions, and pedagogical strategies for supporting sensemaking. Their engagement in the course therefore created a setting in which mathematical and pedagogical perspectives and practices were brought into sustained dialogue within an authentic tertiary instructional context.

This study examines how such dialogue unfolded in teachers' collaborative work, focusing on moments in which teachers' discussions alternated between mathematical and pedagogical inquiry, and on how these alternations supported their engagement with and reinterpretation of real analysis. Specifically, this paper investigates: *How do pedagogical explorations of tertiary instructional situations prompt and shape secondary mathematics teachers' mathematical engagement with real analysis?*

THEORETICAL BACKGROUND

The discontinuities between secondary and tertiary mathematics education are a well-documented and persistent challenge (Wasserman et al., 2023). Students transitioning to university encounter substantial shifts in what counts as legitimate mathematical activity, including new expectations regarding proof, rigor, and the use of definitions. At the same time, prospective and practicing teachers often experience a second discontinuity when attempting to relate their university mathematics studies to the instructional realities of school teaching. These challenges have frequently been associated with a structural *fragmentation* between mathematical and pedagogical components of teacher education, which are typically organized within separate institutional and disciplinary contexts. Such fragmentation can lead to conflicting messages about doing, learning, and teaching mathematics. (Ehrenfeld & Brady, 2025).

A common response to this fragmentation has been to enrich university mathematics courses with pedagogical tasks that engage teachers with school-based scenarios, carefully designed to illustrate connections between proof-based mathematics and SM teaching (Lai et al., 2024). While such approaches have shown promise, they also face limitations. In some cases, pedagogical work faces resistance from UM instructors and remains external to the mathematical activity of the course, thereby reproducing the separation it aims to address. In others, practical implications of such connections to instruction remain superficial or difficult to articulate. This is particularly pronounced in proof-based domains of UM, where the utilization of mathematical insight to inform instruction is often tacit, indirect, and mediated by nuanced balancing of professional obligations and epistemological differences (Pinto & Cooper, 2022, 2023).

An alternative perspective conceptualizes the secondary–tertiary divide not primarily as a curricular gap, but as a *boundary* between distinct professional communities, each characterized by its own norms, practices, and evaluative frameworks (Pinto & Cooper, 2022, 2023; Pinto & Lai, 2025). From this perspective, the challenge is not only to connect mathematical content across levels, but to create structured opportunities – boundary encounters – for meaningful interaction between communities of teachers who engage with mathematics in different ways. Such boundary encounters can make differences in practice visible and available for reflection, creating conditions for mutual learning rather than one-directional transfer of knowledge (Pinto, in press).

Within this perspective, pedagogical engagement with instructional situations in tertiary contexts may serve as a productive site for such interaction. When teachers engage with such situations, they simultaneously revisit central concepts of real analysis while reflecting about how these ideas are communicated, justified, and learned. These situations can therefore function as *boundary objects* – shared reference points for collective inquiry that enable teachers to alternate between mathematical and pedagogical reasoning, potentially reinforcing both (Pinto & Cooper, 2022, 2023).

This study examines these movements through the lens of mathematical and pedagogical discourse (Sfard, 2019), distinguishing between utterances that address

components of mathematical theory (e.g., definitions, proofs, and their properties) and utterances that address relationships between mathematics, students, and instruction. This discursive perspective provides the analytical lens through which alternations between mathematical and pedagogical inquiry are identified in the data and interpreted with respect to how pedagogical explorations of university instruction situations and the challenges students face in the transition to proof can encourage and support teachers in revisiting and reinterpreting mathematical ideas from real analysis.

METHODOLOGY

Research setting

This study was conducted in the context of a first-year undergraduate real analysis course at a large research university in Israel. The course is offered twice each academic year and serves approximately 800 mathematics and computer science majors. It introduces students to proof-based mathematics through the traditional definition–theorem–proof format, covering the construction of the real numbers, sequences and limits, continuity, and differentiation and integration. Faculty members regard the course as students' primary gateway to formal mathematical reasoning.

Teaching is distributed across multiple instructional formats. Students attend faculty-led lectures (6 hours weekly), recitations led by PhD students (2 hours), and tutorials led by MSc students (2 hours). In the context of this study, SM teachers joined the tutorials as co-tutors alongside the graduate student instructors. The lecturers were responsible for the design of the course, including the tutorial tasks, but did not regularly participate in the preparation or reflection meetings analysed in this study. Assessment is uniform across sections and includes weekly assignments, a midterm, and a final exam. Despite students' strong backgrounds, approximately half fail the course on their first attempt, with most passing after retaking the exam or the course.

Participants

The study was part of a school–university co-learning partnership involving SM teachers, university instructors (faculty and graduate students), and a mathematics education researcher (the author), who facilitated the collaborative meetings.

Ten experienced SM teachers participated in the project. This paper focuses on four teachers – Joy, Ariel, Jordan, and Coral (all pseudonyms) – selected for the richness and recurrence of their contributions across the dataset, and for their diverse academic and professional backgrounds. Joy holds an M.Sc. in mathematics and was in her first year of school teaching after several years of tutoring real analysis at another university. Ariel and Coral each hold graduate degrees in mathematics education and had approximately ten years of teaching experience in upper secondary school (ages 15–18). Jordan holds a B.Sc. in mathematics and had nearly 15 years of experience teaching secondary mathematics. All participants regularly taught advanced high-school mathematics, including preparation for matriculation examinations, but had no prior experience teaching proof-based university mathematics.

Data collection and analysis

To prepare for the tutorials, the teachers first worked on the tutorial tasks individually and uploaded their solutions to a shared Google document, along with suggested topics for group discussion. They then participated in 90-minute preparation meetings, where they compared their solutions and collaboratively explored the mathematical and pedagogical topics they had identified. After co-facilitating the tutorials, the graduate student instructors and teachers took part in 60-minute reflection meetings, where they discussed student difficulties, pedagogical strategies, and instructional decisions. Both the preparation and reflection meetings were facilitated by the author, video-recorded, and fully transcribed. Additional data include bi-weekly reflections teachers wrote on their most memorable teaching moments and learning experiences.

The analysis proceeded in two stages. First, utterances in the meetings and written reflections were coded according to discursive perspective outlined in Section 2, distinguishing between *mathematical* and *pedagogical* utterances (Sfard, 2019). Given the focus of the discussions in the dataset on the tutorial tasks, these categories were further elaborated and refined by adopting Rezat and Sträßer's (2012) socio-didactical tetrahedron, which underscores the role of tasks in mediating the relationships between teacher, students, and mathematics, as second theoretical lens. Accordingly, utterances were labelled as pedagogical when they were concerned with the relationships between mathematical content and the students, the tutorial tasks, or teachers. These included, for example, students' interpretation of and engagements with the tasks, the mathematical ideas tasks may foreground, or pedagogical strategies for guiding students' work. Contributions that can be interpreted simultaneously in mathematical and pedagogical terms – for example, "this definition is better for proving than for intuition" – were categorised as *boundary* utterances. These were analytically central as indicators of transitions between discourses (Pinto & Cooper, 2022, 2023). Coding was iterative and refined through repeated comparison across data sources.

The second stage examined sequences of interaction in which speakers alternated between mathematical, pedagogical, and boundary utterances, focusing on how these utterances were related – for example, whether they questioned, supported, elaborated, or reframed one another. Particular attention was given to recurring explorations involving multiple participants and to moments identified by teachers as especially meaningful. This analytic approach allowed us to trace how pedagogical explorations of instructional situations prompted, supported and shaped mathematical engagement, and to identify patterns in how teachers moved between mathematical and pedagogical reasoning in their discussions.

FINDINGS

This section presents three types of pedagogical explorations that proved especially productive in prompting alternations between mathematical and pedagogical inquiry. Each type is illustrated through a representative instructional scenario and analysed in terms of how pedagogical reflection can support teachers' mathematical engagement.

Explorations of the exposition of real analysis

The SM teachers frequently questioned lecturers' choices regarding the sequencing and presentation of definitions, theorems, and proofs. A key example arose in the first week, when students encountered two equivalent definitions of the *supremum* of a set:

Definition: Let $\emptyset \neq A \subseteq \mathbb{R}$, *then* $u \in \mathbb{R}$ *is the least upper bound (supremum) of A if:*
(1) u *is an upper bound of A*
(2) $if\ u'$ *is an upper bound of A then* $u \leq u'$.

Corollary: Let $\emptyset \neq A \subseteq \mathbb{R}$, *then* $u \in \mathbb{R}$ *is supremum of A if and only if:*
(1) u *is an upper bound of A; and*
(2) for every $0 < \varepsilon \in \mathbb{R}$, *there exists* $a \in A$ *such that* $u - \varepsilon < a$.

Joy, drawing on her prior tutoring experience, suggested reversing the order in which the definitions were introduced, arguing that the corollary might be more accessible to students who may still be struggling with the making sense of upper bounds – let alone reasoning about the set of all upper bounds. By contrast, the corollary involves reasoning with only one upper bound at a time and may feel more accessible.

This observation initiated an extended exploration in which teachers alternated between pedagogical and mathematical considerations. Some teachers emphasized the importance of intuitive grounding, while others highlighted the operational advantages of definitions that facilitate proof. For instance, Jordan objected to Joy's reasoning. He framed the corollary as an *operational* definition, arguing that mathematicians favour them since they simplify the proving of the objects' properties, yet pedagogically they are problematic since they do not convey to students what the objects actually are. Ariel mediated between these positions. She agreed with Jordan that the definition is preferable at first, as students might "struggle to put together the words" the definition as presented in the corollary; but she also agreed with Joy that a deeper understanding of supremum requires hands-on engagement with concrete numerical examples.

This exploration functioned as a boundary object and opened a line of inquiry into the mathematical roles and pedagogical affordances of equivalent definitions. The teachers articulated distinctions between definitions that support intuition and those that support proof and reflected on how these functions are related. For example, consideration of an alternative approach – motivating the supremum as a natural generalization of the maximum – revealed a circular dependency in the formal development of the concept, making explicit a structural feature of the theory that had previously remained implicit.

In later reflections, teachers identified this exploration as particularly meaningful. Viewing definitions through students' perspectives enabled teachers to recognize and articulate subtle differences between school and university mathematics, including the role of formal equivalence, the interplay between intuition and formalization, and the ways in which definitions organize mathematical reasoning – observations they had not fully engaged with as undergraduate students. In this case, pedagogical reflection on exposition served as a mechanism for making mathematical structure visible.

Explorations of the learning opportunities and challenges tasks afford

A second type of exploration centred on the pedagogical affordances of tutorial tasks. Although lecturers did not provide explicit teaching goals, teachers treated the tasks as purposeful and sought to infer their intended learning trajectories. This was particularly evident in discussions of "prove or refute" tasks, which posed a range of challenges – and thus learning opportunities – for students and teachers alike, as Jordan's observed:

> These tasks seem to emphasize finding counterexamples. Was there any discussion about that? How does one explain how to look for counterexamples? Obviously, there isn't a systematic algorithm, but maybe we can at least outline some helpful heuristics?

A representative example came from a task focused on the generalization of the relation "$\leq$" to non-empty sets, which was included in the 2nd tutorial:

Task 2: Let $A, B, C \subseteq \mathbb{R}$ *be non-empty sets. Prove or refute:*

a. $A \leq A$
b. If $A \leq B$ *and* $B \leq C$ *then* $A \leq C$
c. If $A \leq B$ *and* $B \leq A$ *then* $A = B$

This generation of "$\leq$", which is defined by $A \leq B$ *if and only* $a \leq b$ *for every* $a \in A$ *and* $b \in B$, is central in the course to early discussions of the completeness of the real numbers. It confronts students with a familiar symbol whose meaning and properties now diverge from what they know from school mathematics. When applied to non-empty sets, the relation remains *transitive* (2b) but is not *reflexive* (2a) and while it is *antisymmetric (1c),* the hypothesis "$A \leq B$ *and* $B \leq A$" implies A and B are singletons.

Teachers initially approached tasks by anticipating student difficulties – challenges in interpreting notation and quantified statements, or uncertainty about what counts as a valid proof or refutation. For example, they briefly considered how extending the relation to include the empty set would generate paradoxical cases, highlighting the sensitivity of formal definitions to seemingly minor conditions.

These pedagogical concerns prompted detailed mathematical engagement with the definitions and properties involved. For instance, in analysing whether and in what sense the relation is reflexive, teachers confronted the discrepancy between students' intuitive interpretation of "$A \leq A$" and its formal definition. They anticipated that students would interpret $A \leq A$ as obviously true due to the false assumption that it means either 'smaller' or 'equal'. Another potential confusion the teachers identified concerned the absence of explicit quantifiers in the statements, which led them to reflect on whether there is a difference between "for *every* set" and "for *any* set". Coral linked her own engagement with the task directly to students' anticipated difficulties:

> Even though I understood what the correct answer is, my intuition resisted it. That really illustrated for me the kind of difficulty students might experience.

These reflections shifted the discussion from merely describing student misconceptions to analysing the mathematical structure and heuristics that underlie them. For instance, the teachers noticed that some students interpreted the instruction "prove or refute" as

an invitation to choose freely between the two. Conversely, another common assumption was that the sets *A, B* and *C* are given and fixed, even if unknown, because they are defined outside the statements – students were surprised they were allowed to define freely these sets when constructing a counterexample.

These observations invited the teachers to reflect on how they approach "prove or refute" tasks, and to make explicit general heuristics grounded in definitions. Ariel described a strategy of translating unfamiliar objects into familiar ones via definitions, applying known properties, and then returning to the original context. In formulating this heuristic, she explicitly connected proof strategies in real analysis to analogous practices in school mathematics, thereby constructing a bridge between domains:

> This strategy should not surprise them – similar things happened in high school. For example, each time they learned to solve a new class of equations, we looked for a simple way to transform them into equations they already knew how to solve.

Some teachers considered students' likely misconceptions about the generalized "≤" relation might distract them from the task's central purpose, as they inferred it – learning to use definitions in constructions of proofs and refutations. They suggested an alternative route – to instead generalize "<" and obtain a relation on sets that would still be *irreflexive*, *asymmetric*, and *transitive*, thus retaining the properties students would expect. Others defended the use of a counterintuitive definition, noting that confronting such discomfort can foster meaningful learning, as Ariel explained:

> There's something in this definition that challenges our intuition – it doesn't feel natural to us […] encountering this kind of definition at this initial stage, when we want to redefine everything, to make a sort of a reset […] I think it's very sound didactically.

These explorations began to surface broader epistemological questions about proof, such as what constitutes a valid argument or a counterexample, but these remained embedded in teachers' attempts to interpret the pedagogical affordances of tasks. Teachers reflected on students' assumptions – for example, that proving a few cases suffices – to clarify the role of generality and logical necessity in proof.

In this type of exploration, pedagogical attention to students' engagement with tasks consistently motivated teachers to unpack definitions, analyse properties, and articulate proof strategies. These inquiries intertwined mathematical analysis, pedagogical reasoning, and teachers' reflections on their own mathematical engagement.

Explorations of instructional and epistemological norms

A third type of exploration emerged when teachers compared their solutions or reflected on situations from the tutorials and encountered differing interpretations of instructional and epistemological norms in the tutorials. These situations often involved tensions between intuitiveness and formal rigor, or between understanding and formal expression. One such debate arose in the discussion of Task 2. It concerned the legitimacy of a refutation that did not include an explicit counterexample:

> *$A \leq A$ is clearly not true in general. If there exists $a < b \in A$ then it is disproven.*

Teachers disagreed about whether such reasoning should be accepted in tutorials, leading to a discussion about the pedagogical and mathematical purposes of formal mathematical writing. Ariel initially resisted the lecturers' emphasis on precision over intuition, viewing it as conflicting with her values and beliefs as a teacher:

> For me, the main thing is that students understand! Proper writing is something one learns more gradually and in context, like a language. [But, in time,] I realized that […] at this stage, students must learn to notice subtleties, and formal writing really helps with that.

Similarly, Coral observed that as a teacher, she views writing and understanding as separate things – "*first* you understand, and *then* we look at how to write". However, reflected on her own experience as a student, she recalled that difficulties in writing proofs revealed gaps in understanding that were not apparent at an intuitive level. These reflections compelled some teachers to re-evaluate the relationship between understanding and formal expression, not as a sequence but as mutually constitutive.

Teachers also analysed instructional moves observed in tutorials, such as a graduate instructor's suggestion that students that still do not understand a definition will "gaze at it". While initially perceived as evasive, this move was later reinterpreted as positioning students as independent sense-makers rather than passive recipients of explanations. In these explorations, pedagogical disagreements exposed implicit norms governing mathematical practice. By attempting to interpret and evaluate these norms, teachers engaged with underlying mathematical values – such as rigor, generality, and justification – and reconsidered their own assumptions about mathematics teaching.

DISCUSSION

Research has consistently shown that teachers often experience a disconnect between mathematics and pedagogy when engaging with proof-based mathematics. In contrast, such a disconnect was not evident in the preparation and reflection meetings analysed in this study. Rather, teachers' discussions regularly alternated between mathematical and pedagogical inquiry in ways that were mutually informing. These findings address the research question by showing how pedagogical explorations of tertiary instructional situations can prompt and shape teachers' mathematical engagement with real analysis.

Across these three types of exploration identified in this study – critiquing expositions, unpacking learning opportunities, and examining instructional norms – a common pattern emerges: pedagogical reflection functions as an entry point into mathematical sensemaking and vice versa. Rather than engage with mathematical and pedagogical reasoning in isolation, teachers' attempts to interpret instructional situations, anticipate students' reasoning, and evaluate pedagogical decisions were intertwined with their efforts to unpack definitions, analyse properties, and articulate proof-related heuristics. These alternations are not incidental but constitute a central mechanism through which mathematical engagement and pedagogical reasoning become mutually constitutive.

From the perspective developed in this paper, these processes can be understood as instances of boundary-oriented dialogue across the secondary–tertiary divide. The

instructional situations that anchored teachers' discussions – definitions, tasks, and classroom interactions – functioned as boundary objects that expose tacit norms and mediate transitions between pedagogical and mathematical reasoning, revealing both productive tensions and opportunities for synthesis. This movement made visible distinctions between school and university mathematics, for example in the roles of intuition, formal equivalence, generality, and justification, enabling teachers to engage with such distinctions as objects of reflection and potential sites for transformation.

Importantly, these explorations were not limited to identifying differences between domains. Rather, they created opportunities for relating and coordinating these differences. For example, teachers connected proof strategies in real analysis to familiar practices from school mathematics, reconsidered the relationship between intuitive understanding and formal expression, and articulated how definitions organize mathematical reasoning. These findings suggest that engagement with tertiary instructional situations can support what may be described as reciprocal learning: pedagogical reflection on instructional situations reshapes mathematical understanding, while mathematical considerations inform pedagogical interpretation.

These results contribute to ongoing efforts to address fragmentation in teacher education (Ehrenfeld & Brady, 2025; Wasserman et al., 2023). Much prior work has sought to connect mathematics and pedagogy by introducing school-based tasks into university courses. This study suggests that mathematical engagement in proof-based domains may depend less on integrating pedagogy into mathematical coursework, and more on designing settings in which pedagogical responsibility for authentic instructional situations generates sustained engagement with mathematical meaning. In such settings, pedagogical discussions did not displace mathematical rigor nor compete with mathematical exploration with but instead deepen and enhance it.

At the same time, the study highlights the value of positioning SM teachers as pedagogical experts within university mathematics instruction. Their perspectives made visible aspects of students' engagement and of instructional design that are often less accessible to university instructors and created opportunities for dialogue that challenged implicit norms and assumptions. This points to the potential of school-university partnerships not only for supporting schoolteacher learning, but also for informing research and development of tertiary instruction. While we do not present this partnership as a prescriptive model, several features appear consequential. First, the use of shared instructional situations as objects of joint inquiry supported sustained engagement with both mathematical content and student reasoning. Second, the structured iterations of preparation, enactment, and reflection created feedback loops between pedagogy and mathematics. Third, the absence of rigid role hierarchies in the meetings enabled teachers to engage critically with university instruction and to articulate their perspectives as legitimate contributions to the dialogue.

More broadly, these findings support approaches to the secondary-tertiary divide that treat it not solely as a problem of curricular alignment or student preparation, but as a site for collaborative inquiry across professional boundaries. By creating structured

opportunities for dialogue between communities, such approaches can leverage differences in practice as resources for learning. In this sense, the study contributes to a growing body of work that reconceptualizes discontinuities in mathematics education not only as challenges to be resolved, but also as opportunities for mutual professional development and for rethinking relationships between mathematics and pedagogy.

ACKNOWLEDGMENT

This study was supported by the Israel Science Foundation (grant 2015/23).

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Klein's second discontinuity: a case from basic probability

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We consider an exercise on probability which, as we argue, is situated between introductory university courses in probability and the mandatory teaching of probability in high schools when that teaching goes as far as the binomial distribution. We analyse Chinese and Danish bachelor students' solution to the first part of the exercise and how their (mostly incorrect) solutions can be explained given their undergraduate experience. The case illustrates that probability constitutes a somewhat radical case for the second discontinuity problem as identified by Klein.

Keywords: probability, binomial distribution, negative binomial distribution, Klein's second discontinuity

INTRODUCTION

Klein (2016) is credited for having formulated, as early as 1908, a crucial and complex problem for mathematics teacher education, which concerns the gap between university mathematics education and content knowledge of relevance to mathematics teachers (cf. also Winsløw & Grønbæk, 2014). The problem, known as *Klein's double discontinuity*, concerns the gap between, on the one hand, university mathematics courses, making up a major (sometimes the only) content requirement for secondary mathematics teachers, and on the other hand, the mathematical content knowledges required from students and teachers in secondary school. The "double" nature of the problem is due to two transitions being involved: first, from being a student in upper secondary school to being a student at university, and then from being a student at university to becoming a teacher at secondary school. The latter transition is therefore called *Klein's second discontinuity* and is what will concern us here, in the special case of probability.

This special case was not substantially considered by Klein, for the simple reason that formal probability theory (and, more broadly, stochastics) was only progressively introduced as mandatory elements of school and undergraduate mathematics in the second half of the of the 20th century (cf. Rolfes et al., 2023, p. 1328, for the German case). For instance, Klein (2016, vol. 3, p. 174) writes that the "method of least squares, which belongs to probability, is beyond the scope of our considerations". And while elementary probability theory has historical roots in the 16th and 17th centuries, the forms of stochastics (based on random variables etc.) that are taught in upper secondary school, dates from the early 20th century. At university, probability theory – when based on set or even measure theory – is often considered as a part of (or at least close to) pure mathematics. Nevertheless, stochastics at secondary level is usually less formal and, especially when it comes to statistics, a somewhat separate chapter, which leaves teachers (with their mathematical background) less confident (cf. Huber et al., 2024; Ingram, 2022).

In this paper, we will stay with the subject of elementary probability. There are several small-scale studies of teachers' and teacher students' knowledge in this area, but few of them consider Klein's discontinuity problem. One exception is the paper by Derouet et al. (2018), which addresses the links between integral calculus and the study of continuous random variables. Our focus here will be less on how probability theory is presented, and more on the extent to which students acquire a "higher viewpoint" on high school probability during their mandatory university course(s) in probability.

CONTEXT

Naturally, Klein's second discontinuity may play out differently in different institutions and educational systems. As part of a wider comparative study on Klein's double discontinuity in probability, we experimented the following item with undergraduate students in China (n = 27) and in Denmark (n = 16):

> An ordinary die is rolled until "six" has been obtained three times.
>
> a) What is the probability that the die must be rolled at least 5 times?
>
> b) What is the most probable number of times the die is rolled?

We invite the reader to do the exercise by whatever means (s)he likes.

The students in our samples were in their second undergraduate year. The Chinese students did the item about halfway through their mandatory probability course, shortly after studying discrete distributions including the negative binomial. The group of Danish students had passed the mandatory course in probability about 1 year ago.

In both countries, the binomial distribution is a central topic in high school. At university, many other distributions (including the negative binomial) are covered in the mandatory course on probability and statistics which future teachers must attend at university. A main general difference in how probability is taught in the two countries is that in Denmark, computers are extensively used both in high school and university; students are taught to use tools like Maple or R to perform computations of standard distribution functions. In China, computations are performed by hand and computers are not allowed at exams; being able to calculate quickly and precisely is a significant factor in succeeding.

We return later to how the two student groups did with the exercise above, given this context and background; our focus is primarily on the probabilistic modelling, while calculations techniques are briefly mentioned. To carry out a systematic analysis of the (many) modelling strategies that appeared, and to formulate precise research questions, we need a theoretical framework.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Probability tasks such as a) and b) above, require a form of mathematical modelling which is specific to analysing random phenomena. One must relate an (*a priori*) non-mathematical context, such as rolling a die, in terms of a probability theory. In lower secondary school it is often a question of identifying a sample space and a way to

compute the probabilities that are related to the task. In upper secondary school, it is often linked to defining a stochastic variable, its distribution, and the range asked for. In both cases, the modelling results in reducing the given task $\tilde{\boldsymbol{t}}$ with a task $\boldsymbol{t} \in \boldsymbol{T}$ where $\boldsymbol{T}$ is a familiar type of task from probability theory from the institution (in the sense of ATD, cf. Barbé et al., 2005). In the ATD, such mathematical activity is described in terms of praxeologies, which consist of a type of task, a technique, and a logos block (technology and theory). For our experiment, the institution is university and more specifically undergraduate mathematics, but when the students become teachers, they will be working with upper secondary mathematics.

This reduction or modelling of $\tilde{\boldsymbol{t}}$ in terms of a $\boldsymbol{t} \in \boldsymbol{T}$ is sometimes far from trivial. Explicit assumptions often need to be formulated, related to the information given by the original task but now based on theoretical elements (discourse or *logos,* in ATD) from probability theory. In this way, a point praxeology $\widetilde{\boldsymbol{\Pi}}$ (in the sense of ATD, cf. Barbé et al., 2005) is modelled by another praxeology $\boldsymbol{\Pi}$, belonging to the domain of probability. Specifically, the logos block provides the necessary justification for the techniques used in the modelling process.

In the example above, $\widetilde{\Pi}$ involves both practices and logos which are foreign to probability, such as rolling a die (with the material aspect of a die, techniques and rules for how to roll it, etc.) but also some elements known from Π (here, mostly the notion of "probability" itself). Some elements of the modelling are likely to be routine, like translating the rolling of an "ordinary die" into a probability to obtain the event "six" in any particular roll, the independence of the outcomes of different rolls, etc. Both teachers and students in high school will have experience with such modelling, summarised loosely as the process

(*) $\qquad (\tilde{t} \in \widetilde{\Pi}) \rightarrow (t \in \Pi)$

where, more precisely, t must belong to a type of task from Π for which one or more techniques are known. In many cases, the modelling can be done in many ways, both more or less correct, as in any modelling task.

A similarity between probabilistic and deterministic modelling (like describing a mechanical system with classical means) is the possibility of experiments related to the task $\tilde{t}$, to validate the probabilistic solution to t within Π (or, more subtly, by some simulated version of $\widetilde{\Pi}$). Here, we could simply repeat the experiment of rolling a die until "six" is obtained three times and note the number of rolls required in each experiment; the absolute or relative frequencies (in statistical logos) can then be used to obtain estimates for the answers to both a) and b), if the experiment is repeated many times. But this is quite uncommon in undergraduate mathematics and would lead us into statistics. It does not directly address the question of whether the probabilistic modelling (*) is correct or not; and unless solvers have dice or simulation tools at hand, it could not even be used to assess the solution to t.

Then the task of validating the solution based on (*) comes down to validating (*) itself, including the assumptions made and the structural similarity of $\tilde{t}$ and t.

We are now ready to formulate our research questions:

RQ1. How do students carry out (*) in the case of $\tilde{t}$ = a) as cited above?

RQ2. How could the findings on RQ1 be related to students' experience at university?

Note that for lack of space, and because of the similar modelling required, we will not go into a detailed analysis related to b) here but only offer a few remarks on it.

METHODOLOGY AND A PRIORI ANALYSIS

We first discuss two correct solutions to the exercise above: one which is at least possible when Π comes from undergraduate mathematics as attended by our samples, and one which is possible with Π in upper secondary schools (and teachers there) in both countries. As we shall later see, there are several alternatives, and the answer to RQ1 proceeds by analysing students' modelling of the situation. The solutions are based on two ways of modelling the problem; we omit the often-tacit assumption that the rolls are independent Bernouilli trials with probability 1/6.

Model M0 (negative binomial distribution). It consists in reformulating $\tilde{t}$ slightly, still within $\widetilde{\Pi}$, as the probability of getting "something else than six" at least two times before the third six. Then, replacing "six" by success and "something else than six" by "failure" we have a task of the standard type of task which is used to introduce the negative binomial distribution (cf. Blitzstein & Hwang, 2019, p. 130). In other words, with $X \sim \mathrm{Nbin}(3, \frac{1}{6})$ we are looking for $P(X \geq 2)$ which (using the definition of N$b$in or a tool) turns out to be 425/435 ≈ 0.9838.

Model M1 (binomial distribution). Here $\tilde{t}$ is reformulated a bit more thoroughly, by using the following reasoning (within $\widetilde{\Pi}$): we only get to throw a fifth time if we have not already had the third six in the first four rolls, so we look for the probability of getting less than three "six" in the first four rolls. To find that probability is indeed a standard case of the binomial distribution (again, identifying "six" with success): if X is the number of "six" in four rolls of a normal die, then $X \sim \mathrm{bin}\left(4, \frac{1}{6}\right)$. Then the answer to a) is $P(X \leq 2)$.

We note in passing that b) can be solved by generalising one of the above approaches to find the probability p_n that the die is rolled exactly n times before the third six occurs and then pinpointing the maximum of value of n where p_n is maximal. It turns out that $p_{12} = p_{13}$ and that this is maximal. The "pinpointing" is greatly facilitated by instrumented techniques, and observing (for instance) that $p_n = \frac{1}{6} \cdot P(Y = 2)$ where $Y \sim \mathrm{bin}\left(n-1, \frac{1}{6}\right)$; however, realising that the mean value of Y is 2 exactly when $n = 2$ can reduce the need.

To answer RQ1, we analyse the modelling carried out by students, extending the inventory of models (begun above) progressively, as we focus on the types of task (with known techniques) that students use to model $\tilde{t}$ within Π. Students' attempts of modelling is then related to the scope of their previous experience of probabilistic modelling, and when erroneous, failures to recognise how it needs to be adapted here.

RESULTS

We first analyse the models appearing in answers produced by the Danish students and then complete it by adding further models produced by the Chinese.

Models leading to correct answers (Danish students)

Six of the Danish students found the correct result. Two of them more or less used M1. The "more or less" lies in explicitly introducing the random variable X: while both use the Maple-technique inherent in performing the calculation $bincdf(4, 1/6\,, 2)$, only one uses random variable notation, but both provide a reasonable verbal explanation corresponding to M1.

The remaining four students do not refer to the binomial distribution or its implementation in Maple, but got a correct answer by two alternative models.

Model M2: Complementary probability, multiplicative rule. Three students examine the problem of finding the probabilities of getting exactly three "six" in three and four rolls, considering that we must roll the die at least five times if these two events do not occur. Two of the three explicitly note that one cannot obtain three "six" in just one or two rolls. All three find the probabilities of getting six three times in three rolls by ad hoc computations, with partial details given. For instance, one computed the calculation $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot 3$ for the case of 4 rolls, explaining that the multiplication by 3 comes from the possibility of "arranging" the "arbitrary third number" in "three ways". The number $\frac{5}{6}$ suggests implicitly that the third number should not be 6, and "three ways" that it must not be obtained in the last roll.

After having computed the two probabilities, the three students subtract the two probabilities obtained from 1 and offers this as the answer to a), with no explanation of this last technique (which in Π would relate to the rules $P(A \cup B) = P(A) + P(B)$ and $P(A^c) = 1 - P(A)$). The main difference lies in notation: two use random variable notation with insufficient or no explanation of what the random variable is. The student who does not use random variable notation, but instead $P(roll \geq 5\ times)$ and the like, is in fact more explicit than the two who use undefined symbols.

Model M3: Complementary probability, counting cases of four rolls. The last of the six students implicitly assumes that the die continues to be rolled after three "six" are obtained, and models $\tilde{t}$ as t: compute the probability that less than 3 "six" are obtained in any four rolls of a die (while not stopping, as assumed in $\tilde{t}$, whenever a third "six" is obtained). The student first computes the probability of getting at least 3

"six" in four rolls, as follows. There are $6^4 = 1296$ possible outcomes (implicit use of independence). The student lists the outcomes with at least three "six" in four rolls:

$$666x, 66x6, 6x66, x666, 6666 \text{ where } x \in \{1,..,5\},$$

and states that there are 21 of these. It is concluded that the possibility of getting at least 3 "six" in four rolls is $21/1296$, and then that $1 - 21/1296$ is the probability that fewer are obtained (so 5 rolls are needed). The last step draws implicitly on the technique from Π based on $P(A^c) = 1 - P(A)$. The result obtained is indeed correct. but the model implicitly overcounts rolls in terms of the original situation.

Models leading to wrong answers (Danish students).

The ten students in this group used five different models of $\tilde{t}$, all except one based in some way on the binomial distribution.

Model M4: Binomial, at most 2 "six" in 5 rolls. Two students used a variant of M1, where $\tilde{t}$ was modelled as finding the probability of getting 2 or less "six" in 5 rolls - confusing "rolling at least five times" with "the game does not end in 5 rolls". Both compute $bincdf(5, 1/6, 2)$ in Maple.

Model M5: Complementary probability, mistaken use of binomial 1. Five students consider the number of "six" obtained in the *n* rolls, and just as in M2, they consider that $\tilde{t}$ can be rephrased as find the probability that 3 "six" are *not* achieved in three or four rolls. But unlike M2, they consider that the number of "six" in a fourth roll is simply binomially distributed so that the probability of getting 3 "six" can be computed by $binpdf(4, \frac{1}{6}, 3)$ in Maple. However, this includes the event of getting four "six" already in the first three rolls, in which case no fourth roll would happen; so, this probability is slightly overestimated. They obtain the result

$$1 - binpdf(3, 1/6, 3) - binpdf(4, 1/6, 3).$$

Model M6: Complementary probability, mistaken use of binomial 2. One student uses a slight variation of M5: for reasons not apparent in the response, the student also subtracts $binpdf(5, 1/6, 3)$, reflecting a mismodelling of the question ("we must rolls at least five times" does not exclude ending the game in 5 rolls) combined with the M4 "overcounting" of complementary events.

Model M7: Complementary probability, mistaken use of binomial 3. One student interprets $\tilde{t}$ as asking for the probability that at most 2 "six" are obtained in 5 rolls, like in M1 and M4; but this is further elaborated in Π through the computation

$$P(X \geq 3) = 1 - P(X \leq 2) = 1 - bincdf(5, 5/6, 2).$$

The symbol X is not explained, but from the context it is likely meant to be the number of "non-six" in five rolls. The model thus leads to use the binomial distribution to find the probability of *not* obtaining at most two results different from "six" in 5 rolls. Logically this is equivalent to obtain at least three results different from "six", that is, at most 2 "six" in five rolls.

Model M8: Getting "six" three times in exactly five rolls. Finally, the last of the 10 students apparently misunderstood $\tilde{t}$ as asking for the probability of getting exactly 3 "six" in 5 rolls. This would not even give the probability of ending the game in five rolls, but the conclusion of the student

$$P(\text{rolling at least five times}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$$

is not further explained. We note that the binomial distribution does not appear in any way in this model, nor is the ordering of results taken into account.

Models leading to correct answers (Chinese students)

Among the 27 Chinese students, 17 obtained a correct answer.

Four students solved the problem by more or less explicitly use of M0. Three of them explicitly recognised at the beginning that the task required the negative binomial formula and wrote the general probability function

$$P(X = m) = \binom{m-1}{k-1} p^k (1-p)^{m-k}$$

Although one student did not write down the standard formula, the solution indicates it by the form of the computation. All four students identified that the relevant values of $m = 3{,}4$, and substituted these into the formula to obtain the required probabilities:

$$P(X = 3) + P(X = 4) = \binom{3-1}{3-1}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 + \binom{4-1}{3-1}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1.$$

Two applied M1 described above. It is worth noting, however, that their approaches differed slightly. One student calculated the probability directly by summing the three relevant binomial terms: $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$, where X denotes the number of "six". The other student computed the same probability via the complementary event, writing $P(X \leq 2) = 1 - [P(X = 3) + P(X = 4)]$.

Finally, a total of 11 students arrived at the correct answer by using the complementary model M2, making it the most frequently applied approach among the correct answers.

Models leading to wrong answers (Chinese students)

Among the ten incorrect responses, one student's answer used M5, and one M8. We also found a variant of M8, probably caused by a misinterpretation of the original task:

Model M9: Binomial, but game ending in exactly five rolls. Three students misinterpreted the task as the game ends at the exact fifth roll. This misunderstanding was evident in the expressions they wrote, such as: $\binom{4}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$.

The rest of the 10 incorrect answers reflect a variety of errors in using correct models:

Model M10: Negative binomial used on mistaken model.

Two students recognised the negative binomial distribution can be used but misinterpreted the stopping event or its complementary event.

One student modelled the task using a negative binomial distribution but calculated the probability that the third "six" occurs on the fourth or fifth roll:

$$P(X \geq 5) = P(X = 5) + P(X = 4) = \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + \binom{3}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

The first equality is evidently wrong while the rest is correct if interpreting X as the number at which the third "six" occurs.

Another student misinterpreted the complementary event as the situation in which the second "six" occurs exactly on the fifth roll: $\binom{4}{1}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$.

Model M0': Negative binomial, with correct model but other errors. When computing the probability that all three "six" occur within the first three rolls, and when computing the probability that two "six" appear in the first three rolls, one student used incorrect combinatorial terms (without further explanation): $\binom{2-1}{2-1}$ and $\binom{3-1}{2-1}$. The pattern shows that the student attempted to use the negative binomial structure but misidentified the combinatorial term.

Model M2': M2, but with various errors. One student applied model M2 correctly while using explicit combinatorial coefficients but made an error in the calculation of the one of these. The correct combinatorial factor should be $\binom{3}{2} = 3$, but the student used an incorrect $\binom{2}{3} = \frac{1}{3}$. This indicates that the student recognized the event structure but did not really connect the binomial coefficient with the number of ways to arrange two "six" in the first three rolls. Another student used M2 correctly, except that in the calculations, $\frac{1}{3}$ appears instead of $\frac{1}{6}$ as the probability of obtaining a "six" in each roll.

Summary of students' models for a)

Table 1 summarizes our analysis of the students' modelling of a).

Models used	**Chinese students**	**Danish students**
Correct answers	M0(4), M1(2), M2(11)	M1(2), M2(3), M3(1)
Incorrect answers	M0'(1), M2'(2), M5(1), M8(1), M9(3), M10(2)	M4(2), M5(5), M6(1), M7(1), M8(1)

Table 1. Summary of our analysis in terms of models.

We notice that in both cases, the *ad hoc* computation of complementary probabilities (M2) was the most frequent correct solution in both groups, rather then based on the school-to-university bridge M1. And that the negative binomial distribution does not appear at all in the models proposed by the Danish students, while it appears (often with various mistakes in identifying the relevant parameters) among 7 Chinese

students. The latter phenomena can maybe be ascribed to the Chinese students' having met in more recently (in their course), while M2 – typically, relying on simple direct computations – is made possible by the choice of relatively small numbers in the formulation of a). This, overall, also makes the effect of different access and familiarity of instrumented techniques far less important than it could otherwise have been.

DISCUSSION AND CONCLUSION

From the point of view of the university courses, the use of Model M0 is evidence of having learned something new with respect to high school probability teaching. We cannot say if the presence of M0 among the Chinese students, and absence with the Danish students, is due to the larger distance in time from the mandatory probability course, or if less emphasis was put on examples based on the negative binomial distribution in the Danish course. There are in fact no signs the Danish students ever encountered that distribution, and while some Chinese students were able to use it correctly, others made mistakes in the actual calculation (by hand).

However, it must be emphasised that M0 and M1 both involve some interpretation of the event "rolling at least five times": that the third "six" (success) only appears in five rolls or more (M0); or that no more than two "six" appears in the four rolls. The first is evidently more direct, and indeed M1 appears only with two students in each group – which is to us a relatively surprising observation, given the supposed familiarity from high school with the binomial distribution. Both of these "expected" models are generalizable to solve b), but in fact, only a few Chinese students were able to make such a generalization.

Most students who got a correct solution to a) used more *ad hoc* arguments like M2, which due to the low numbers involved can be used even when restricted to manual computation (Chinese group). A frequent trait of the answers, only partially visible in the above analysis, is a lack of explicit justification of the formulae written – especially when no explicit distribution is involved (while in fact, an *ad hoc* re-creation of a binomial probability is carried out).

For future teachers, learning a wealth of new distributions – not to be taught at high school – appears altogether much less pertinent than the capability of constructing a model based on the binomial distribution and other elements of secondary mathematics, and explaining it clearly. Here, the access and capacity to use instrumented techniques would be of smaller significance than thorough training in non-standard modelling of problems which can be solved with such elements. Most students used the binomial distribution directly or indirectly; but it is frequently combined with poorly explained *ad hoc* models, albeit in some cases ending with a correct result.

To conclude, a plausible strategy for reducing the second discontinuity in probability could be to equip students with a wide range of experiences of non-standard modelling with relatively elementary means, and with a focus on explicit justifications and assumptions being made (cf. Kazak & Pratt, 2022). In the ideal case, this could then

lead to more students producing model M1 for a), and generalize it to also solve b). The relative success of the Chinese group – mostly with M2-based solutions – could strengthen that hypothesis if we could document our impression that they do relatively more practical work with situations calling for non-trivial probabilistic modelling. In our current comparative work, this hypothesis will be further investigated.

Another open question is whether the exclusive use of manual techniques (as in China) remains relevant for preparing teachers for high schools. In many countries (also in China), instrumented techniques have become more prominent in view of more advanced applications, not least in statistics; yet they may (as in other areas of mathematics) lead to backdraws when it comes to acquiring basic theoretical knowledge about probability and connecting it with practical problems. We find some support in the data for this hypothesis as well.

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Designing inquiry-based mathematics tasks to support institutional transition between secondary and university levels

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Keywords: transition, inquiry-based learning, task design, dialectics of inquiry.

BACKGROUND AND CONTEXT

We are working on a project to develop mathematics teaching materials for inquiry-based learning (IBL). One reason IBL is necessary is its potential to address the transition to university mathematics. While extensive research has been conducted on transition issues (cf. Hochmuth et al., 2021), this study focuses specifically on the transition from secondary school to university. Research within the ATD framework indicates that the core challenge in this transition lies not so much in changes of mathematical rigour or content, but rather in differences in approaches to mathematics and in cultural aspects (Artigue, 2022).

For example, secondary mathematics learning tends to be a form of repetitive practice and memorisation of given tasks and solutions. Thus, the praxeology of secondary school mathematics may be rich in tasks and techniques, but technologies and theories may not be fully referenced. Artigue (2022) highlights this point from an ATD perspective: proofs might become the "cherry on the cake" in secondary mathematics. Conversely, in IBL, learners tend to aim to understand the underlying principles of phenomena–why things are as they are–to produce their own results. Therefore, in terms of praxeology, technology and theory are emphasised, which is closer to the practice of university mathematics. In other words, through IBL, the approach to mathematics can be changed, facilitating the transition to university mathematics.

The authors have previously guided several students' mathematics inquiries within teacher training faculties. However, the participants were third- and fourth-year students. Thus, in this project we propose IBL tasks designed to facilitate the transition to the university approach to mathematics within first-year university education.

A TASK EXAMPLE

The distinctive feature of this project lies in developing IBL mathematics materials for university students majoring in mathematics, from a mathematician's perspective, with continuity with secondary mathematics. Here, we present an example of a topic that possesses such continuity and potential for development into university mathematics.

Linear functions are fundamental functions studied at secondary school level. Consider the following question: what common properties do the following four linear functions share?

$$y = x + 4, \qquad y = 2x + 3, \qquad y = 3x + 2, \qquad y = 4x + 1$$

It is not apparent from the equations alone, but if one plots their graphs, it is clear that all the graphs pass through a fixed point (1,5). Moreover, it is easily explained that if the sum $a + b$ is constant, then the graph of $y = ax + b$ always passes through a fixed point. Then, what happens if the difference $a - b$ is constant? What if the quotient a/b is constant? Or the product ab is constant?

Indeed, experimentation reveals that when the sum, difference, or quotient is constant, the corresponding graph always passes through a fixed point, while the case of the constant product shows a complex pattern; the corresponding graph lines are all tangent to a fixed parabola. What kind of nature would an inquiry aimed at understanding this phenomenon have? And what possible conclusion might they reach?

This phenomenon can be understood as a property of the map F, which assigns the line $y = ax + b$ to the point (a, b). When the sum, difference, or quotient of a and b is constant, (a, b) moves along the straight line. This collinearity leads to the concurrency of the corresponding lines. On the other hand, when ab is constant, (a, b) moves along a hyperbola. The fact that the hyperbola is a conic curve induces the property that the corresponding lines are tangent to a fixed conic curve. Learners could use dynamic geometry software to experimentally make such conjectures. Once a conjecture about the properties of collinearity and concurrency is formulated, its proof would be accessible. Within the framework of the dialectics of inquiriy in ATD, this can be considered as the *dialectics of conjecture and proof.* Furthermore, viewing each phenomenon as the property of one map F constitutes an instance of the *dialectic of objects and structures*. Moreover, closer examination reveals that even when (a, b) moves along a straight line, the corresponding line may become parallel without passing through the fixed point. This can also lead to activities of proof and refutation. Thus, this topic can serve as a source for rich exploratory activities.

Moreover, the correspondence of F leads to the concept of duality between points and lines in projective geometry, potentially offering a pathway to deeper learning in university-level mathematics.

ACKNOWLEDGEMENT

This study is supported by Joint Research Aid from Joint Graduate School in Science of School Education, Hyogo University of Teacher Education.

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Prospective teachers' knowledge of prisms: Institutional perspective

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Keywords: The teaching and learning of specific topics in university mathematics, Transitions to, across, and beyond university mathematics, Geometric solids, Prospective teachers, Anthropological theory of the didactic (ATD).

INTRODUCTION AND THEORETICAL BACKGROUND

Knowledge of geometric solids develops across all educational levels: from recognizing concrete models in preschool, to describing their properties in primary school, applying measurement and coordinate geometry in secondary education, and finally abstracting, using and generalising these concepts at the university level. With their uniform cross-sections, prisms exemplify solids generated by translation and regions used in Fubini's theorem. They are also canonical examples of polytopes—key building blocks in topology, numerical mathematics, and many STEAM fields—and serve as test cases in higher mathematics, making them essential for students to master.

However, previous studies recognised multiple issues with the knowledge of geometric solids. Students and even teachers struggled with differentiating two- and three-dimensional shapes, interpreting solids regardless of shape, size, and position, and applying invariance properties (Weigand et al., 2025; Erdem & Man, 2023).

Given that teacher education occupies a strategic position in the institutional circulation of knowledge—linking scholarly mathematics with mathematics to be taught—this study examines differences in prospective primary school and secondary mathematics teachers' knowledge of geometric solids, focusing on prisms and the increasing complexity of their definitions: from static visual descriptions of bounding surfaces, through their dynamic generation by translation as extrusions of polygons (or even rounded regions), to a formal definition using the cross-product.

In ATD, knowledge is understood as an institutionally relative construct, modelled through praxeology, which comprises a practical block (*praxis*) and discursive block (*logos*), and is perceptible through verbal, symbolic, or graphic ostensive objects (Bosch & Chevallard, 1999). We constructed a reference epistemological model (REM) of prism-related knowledge based on the literature and curricular requirements across educational levels and used it as an analytic tool (Chevallard & Bosch, 2014) to interpret students' responses about prisms.

METHODS AND DATA

Participants in the study were 182 university students with diverse upper secondary backgrounds (vocational or general), enrolled at different university levels

(undergraduate or graduate) and preparing for different teaching careers (primary school or secondary mathematics).

Students completed an online questionnaire with eight items addressing prism knowledge across educational levels: one multiple-choice item using symbolic ostensive objects on the equations of a prism and a pyramid in a coordinate system (Q1, administered only to prospective mathematics teachers); four multiple-choice items using graphical ostensive objects on shapes, positions, and projections of prisms (Q2), polyhedron nets (Q3), two-dimensional representations of prisms (Q5), and polyhedra with prism-related properties (Q6); two multiple-choice items using verbal ostensive objects on statements about prisms (Q7) and on formal and informal definitions of prisms (Q8); and one open-ended item asking students to define a prism (Q4). Inquiring into students' knowledge across different backgrounds, and combining different ostensive objects with specific discourses, enabled us to identify institutional perspectives and variations in ostensive perspectives.

PRELIMINARY RESULTS AND IMPLICATIONS

Preliminary results indicate that students recognise prisms mainly in prototypical positions and shapes (upright solids with convex polygonal bases), and tend to relate discourses about prisms to static visual ostensives (described verbally as solids with certain bases and faces). Even future secondary mathematics teachers rarely define prism through a dynamic or genetic approach-as solids with constant cross-sections or generated by translation-and often exhibit typical issues with knowledge of prisms.

The poster presents our REM, an interpretation of students' knowledge of prisms across institutions in relation to this model, and implications for teacher education in preparing future teachers to support students through transitions within and across institutions.

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Initiating dialogue between the ATD and Commognition in the context of university mathematics education research

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This paper initiates a theoretical dialogue between the Anthropological Theory of the Didactic and Commognition in university mathematics education research. Using a networking theories approach, we review the basic tenets of the theories with the aim of identifying potential complementarities. We propose areas for mutual enrichment, including using praxeologies to structure discourse, employing discursive analysis to operationalize the logos block, and utilizing institutional constraints to explain discursive shifts.

Keywords: Commognition, ATD, networking theories, discourse, praxeology.

INTRODUCTION

A 2014 special issue (SI) of Research in Mathematics Education on "Institutional, sociocultural and discursive approaches to research in undergraduate mathematics education" (Nardi, Biza et al., 2014) showcased several (then) relatively novel theoretical approaches to UME research, including the Anthropological Theory of the Didactic (ATD) and the theory of Commognition (ToC). Since the publication of the SI, both ATD and ToC have become widely used in research on UME. Taking INDRUM and the Topic Working Group 14 on UME at CERME over the last five years as a sample, ATD or ToC were used in a large percentage of papers (as much as 42% at INDRUM 2024). ToC was primarily employed in studies examining the practices of university students and teachers (e.g., Karavi & Schüler-Meyer, 2025). However, there were also papers on topics such as task design or course development (e.g., Hooper et al., 2025). ATD, on the other hand, was used for epistemological analysis of different forms of documents, such as textbooks, tasks and their solutions, exams and curricula (e.g., Hitier & González-Martín, 2023); as a tool for educational design or for studying ecological conditions of teaching and learning processes (e.g., Hochmuth & Peters, 2024). There were also a few ATD papers that drew on interview data to study, for instance, didactic choices or views on mathematics (e.g., Derouet et al., 2024). Judging from this summary, there appears to be a level of complementarity between the frameworks, as they address partly similar topics but pose different questions and draw on distinct forms of data.

Still, so far, there has been very little theoretical dialogue between the two perspectives, even though already in the 2014 SI, Winsløw et al. hinted at the potential value in complementing the institutional perspective of the ATD:

> "In relation to ATD, several complementary approaches to UME appear in the other four papers in this Special Issue, not least in terms of methodologies and theories for fine-grained analyses of variations related to individuals and small group interactions within

UME contexts. One challenge for future research is to relate such analyses with explicit reference models for the mathematical praxeologies which are, or could be, involved in these types of interaction" (Winsløw et al., 2014, p. 109).

In the same SI, Nardi, Ryve et al. (2014, p. 196) mention the need for "scaling up" the unit of ToC analysis to encompass, for instance, history, culture and social institutions.

There have been some attempts to network ATD with other theoretical perspectives, including APOS theory (e.g., Florensa et al., 2024) and the Theory of Didactic Situations (TDS) (e.g., Barquero et al., 2022). The collaborations with APOS have mainly taken the form of parallel analyses (e.g., Florensa et al., 2024), because of the significant differences observed between the approaches regarding their objects of study, research questions, empirical bases, and research ends considered. In contrast, the points of contact with the TDS approach have been more fruitful, leading to the incorporation of notions such as "adidacticity" from TDS into ATD analyses (Barquero et al., 2022). Work on networking ToC with other frameworks has not been as extensive but includes, for instance, the aforementioned study by Hooper et al. (2025), drawing on the theory of Communities of Practice.

This paper attempts to take Winsløw et al. (2014) up on the challenge by developing in more detail an argument for how ATD and ToC can serve as complementary perspectives for gaining a deeper understanding of phenomena related to UME, and investigating the potential for networking that extends beyond parallel analysis.

THEORETICAL FRAMEWORK(S)

This study can be situated within the broader framework of networking theories (Prediger et al., 2008) as a means to address the calls for collaboration and to explore the potential complementarities between two of the most relevant approaches in the field of UME. The networking framework is grounded in four principles: valuing theoretical diversity, recognising each theory's specificity, identifying connections between theories and their results, and developing theory use to inform practice better. More specifically, we position the study at the intersection of contrasting and combining approaches, laying the groundwork for further research investigating the relationship between the two theories (see Fig. 1) (Prediger et al., 2008).

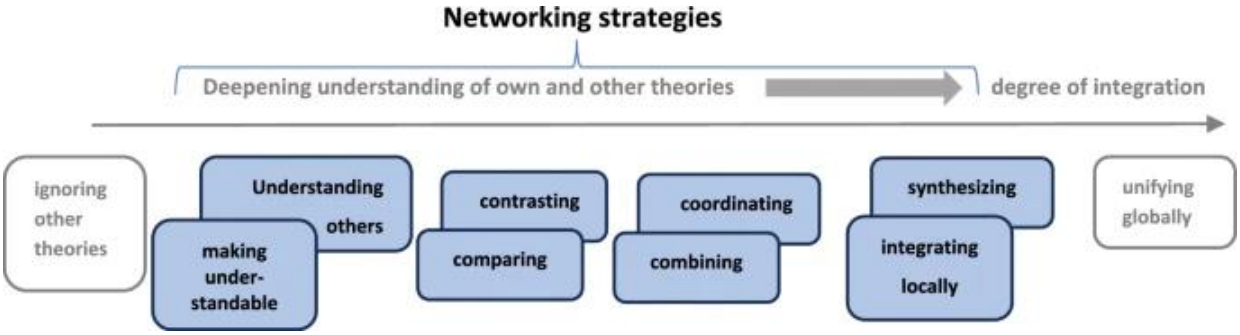

Figure 1: Networking strategies (Prediger et al., 2008, p. 170).

In an attempt to open up for dialogue between the ATD and other theoretical frameworks, Gascón and Nicolás (2019) presented an outline of the research and teaching ends of the ATD, considering it an unavoidable step to make these assumptions explicit prior to any possibility of collaboration. In the following sections, the most relevant research aspects of each approach are presented.

Basic assumptions of ATD

The study by Gascón and Nicolás (2019) provides a detailed analysis of these assumptions, which we will present in a highly summarised way. One of the main assumptions of the ATD is that any human activity (and in particular those related to mathematics) can be described in terms of praxeologies (Chevallard, 1999). A praxeology takes the form of a quadruplet [T/τ/θ/Θ], where T refers to a type of task to perform, τ to a technique that allows the completion of the task, θ to a technology, that is, a discourse explaining and justifying the adequacy of the technique, and Θ to a theory that provides the rationale for the technology. These four components form two blocks: [T/τ] is the practical block that describes tasks and ways to solve them (know-how), and [θ/Θ] is the knowledge or logos block that describes, explains, and justifies what is done (Hitier & González-Martín, 2023). A second assumption is that of the institutional relativity of knowledge; the sub-theory of didactic transposition illustrates this assumption. Regarding the unit of analysis, the ATD assumes a macro-didactic perspective, meaning that the minimal unit of analysis should include not only an isolated mathematical exercise but also all the elements that allow for analysing the institution producing the knowledge, the school institution, and the study community. Indeed, the ATD explicitly pays attention to how the different institutional levels, from society all the way down to the particular school, classroom and topic of study, influence and co-determine each other. This broad view on the unit of analysis contrasts heavily with many other approaches, where the analysis is often limited to punctual teaching experiences.

Considering these assumptions, among other details in Gascón & Nicolás (2019), the research questions within the framework of ATD are related to the genesis, development, teaching-learning, use and spreading of any kind of knowledge, as described in terms of praxeologies, through institutions. Consequently, the expected results from the research within the ATD would include explanations and laws concerning certain didactic phenomena, and assessments of the efficacy of certain means to achieve a previously given teaching end.

Basic assumptions of ToC

Human activity is fundamentally social: “patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual” (Sfard, 2008, p. 78). It follows that communication, as the collective form of thinking, is central to any study of human activity, and specifically the study of teaching and learning processes. Communication should be understood in participatory terms, as a specific form of collectively performed patterned activity. Different forms of

communication are called discourses and are distinguished by four characteristics: the words and visual mediators used, the narratives formulated and endorsed, and the routines engaged with (ibid., p. 133-135). The basic unit of analysis for the commognitive researcher is discursive practice. Every communicational act belongs to a specific (or perhaps to several partially overlapping) discourse. Any discourse helps create a discourse community, consisting of those individuals capable of participating in the discourse. Of course, any one individual is typically a member of many discourse communities. Moreover, discourses are constantly developing and should not be seen as static. "Still, human ways of communicating are often distinct enough to justify talking about discourses and their communities as reasonably well-delineated wholes" (ibid., p. 91). Discourses can be, but need not be, following the demarcations of academic disciplines or school subjects, but they develop in particular settings to fulfil particular needs. Discourses are governed by rules, but these rules can differ between settings. Moreover, the rules are of different types: object-level rules, which govern the behaviour of the objects of the discourse, and meta-level rules, which govern the behaviour of the discursants. Where object-level rules of mathematical discourse are typically dictated by the properties of the objects of the discourse, with little room for manoeuvre, so to speak, meta-level rules are typically context-dependent and evolving; normative, contingent rather than necessary, tacit, and constraining rather than deterministic (ibid., p. 200-208). Indeed, one major factor impacting learning is commognitive conflict, where different discursants act according to different meta-level rules (ibid., p. 296).

Mathematical objects are seen as discursively constituted, that is, they obtain their meaning and existence through the discourse. Generally, mathematics is an autopoietic system, meaning that it generates its own objects through processes of increasing abstraction (Sfard, 2008, p. 129). A mathematical object, as understood by a specific discursant, can be described through its realisation tree, a hierarchical structure where an object, for instance a function, is connected to its different realisations, in this case an algebraic formula, a graph, etc. Each of these realisations is in turn connected to its realisations, creating a tree-like structure (ibid., p. 165).

Learning is understood as individualising discourse, becoming ever more capable of communicating within the discourse, with others as well as with oneself. The teacher's role is to facilitate this process of discursive change. This involves both acting as the "voice" of the leading, canonical, discourse and providing the learner with opportunities to participate in the discourse on her own terms, with the goal of turning it into a discourse-for-her (Sfard, p. 285). Routines (repetitive patterns characteristic of the discourse) are central to any discursive activity. Indeed, an alternative way of viewing learning from a commognitive standpoint is as routinization (Lavie, Steiner & Sfard, 2019). Routines are seen as task/procedure pairs, and can be characterised according to their aims, where outcome-oriented routines are called explorations and process-oriented routines are called rituals. What is perceived as the task in a given situation, and what procedure should be enacted to solve it, is influenced by the

learners' prior experiences. Any routine enacted by an individual discursant is located somewhere on the ritual-exploration spectrum. One way of characterising learning is as de-ritualisation, that is, shifting one's routine use towards the explorative end of this spectrum (ibid., p. 167).

In the light of these assumptions, research questions in ToC studies often relate to characteristics of mathematical or mathematics teaching discourse and how these can serve to support or hinder mathematics learning, patterns in students' or teachers' discursive activity, or how various didactical phenomena can be understood in discursive terms. Hence, expected research outcomes include classifications of (aspects of) particular discourses, mapping of processes of students' discursive change, or theoretical explanations of aspects of teachers' and students' practices.

OPPORTUNITIES FOR THEORETICAL DIALOGUE

Reflecting generally on the frameworks, their underlying assumptions clearly differ, with ATD taking an institutional perspective and focusing on knowledge organisation in and across institutions where ToC instead takes a discursive perspective focusing on the shaping and development of practices of different discourse communities. Hence, aiming for a wholesale integration of the theories is probably not realistic and, in our view, perhaps not desirable. Rather, we will point out some areas where we see potential points of connection between the theories, and where notions and tools from one framework might serve to strengthen analyses conducted within the other. Building on our familiarity with the research conducted within the two frameworks, in what follows we highlight four such areas and exemplify with specific studies where we believe that augmenting the analyses with theoretical tools from the other framework might have produced further insights.

Understanding the structure of mathematical discourse

A limitation of ToC is the lack of efficient tools for describing and investigating the structure of mathematical discourse(s). The four characteristics of discourse serve to describe the discourse in certain ways, but they tell us little of the relationships between, for instance, different narratives and routines. The main tool for describing and analysing the relationships between different objects in mathematical discourse, the realization tree, is limited in its use by the fact that it is centred around one specific object. The tree structure suggested by it does not adequately model the structure of mathematical discourse which, although recursive and autopoietic, is perhaps better modelled using a network structure.

Recently, Weingarden and colleagues have done work developing the realization tree as an analytical tool. As one example, Weingarden (2024) employed the so-called Realization Tree Mediator tool in working with supporting prospective teachers in developing mathematical connections. The paper as such is very interesting, but there are limitations built into the very conceptualization of "connection" used: "Mathematical connections were conceptualised as *SNs about two so-called different*

realisations of a mathematical object" (p. 276, emph. in original). However, not all mathematical connections take this form. In teaching more advanced mathematics in particular, one is often interested in making connections between different mathematical objects that cannot in a meaningful way be viewed as realisations of the same more abstract object. This is where we argue that the notion of praxeology as a model of the structure of mathematical knowledge could be adapted for use. The praxeological model as such is not strongly dependent upon basic assumptions about what constitutes "knowledge". There is nothing in it that does not allow for conceiving of it as a model for how discourse around a specific piece of mathematics is organised. In this way it complements the ToC, and should be possible to adopt also in a ToC study.

Limitations of praxeological analysis: how does technology inform technique?

From the ATD perspective, the fact that the logos part of a praxeology is described in terms of "discourse around the techniques" in turn hints at how ToC could be brought to bear. In the praxeological model, the term discourse is used more or less in its everyday sense. However, we believe that it is perfectly possible to interpret it according to a ToC definition. This would open up for a type of analysis difficult to conduct using existent ATD tools, namely on how the different parts of a praxeology are connected. One of the dominant methodologies mobilised within the framework of ATD is the so-called praxeological analysis. This methodology is frequently used as a tool to systematically describe the epistemological conceptions of knowledge in a specific institution. The empirical bases are often textbooks, syllabi or students' productions, allowing the researchers to describe how knowledge is structured. However, the discourse of students and teachers is often not considered as a potential source for characterisation of the institutional epistemological conceptions.

We argue that discourse analysis from a ToC perspective can contribute to explicitly identifying elements in the logos of the praxeologies and *how* they connect to the techniques. In other words, more precisely how the technology and theory elements justify the selection, adequacy and use of a specific technique is often absent from the crystallised elements such as textbooks and students' productions. The incorporation of discursive analysis based on the ToC approach could shed light on these parts of the analysed praxeologies. Consider, for instance, the work by Hardy (2009). As part of this research Hardy investigated "students' perceptions of the knowledge to be learned about limits of functions in a college level Calculus course" (p. 341), drawing on interview data. Using tools from the ATD, Hardy was able to show that the students' spontaneous praxeologies included a mixture of mathematical, social, cognitive and didactic elements. However, her analysis had less to say about *how* these praxeologies developed. We believe that a fine-grained discourse analysis using ToC tools could contribute further insight in this regard, thereby enriching the description of this phenomenon.

A further point of possible mutual enrichment stems from relating the notion of praxeology with the more recent developments in ToC, where routines are seen as task-procedure pairs. In particular, the notion of "incomplete praxeologies" from ATD then

mirrors the one of "ritualized routine use" in ToC. More specifically, where incomplete praxeologies are praxeologies where the logos block is poorly structured or even absent, ritualized routines are understood as recurrent patterns performed mainly to conform to social norms and focused on the doing rather than on the expected outcome or its justification.

Knowing and learning: operationalizing the notion of personal relation

ATD lacks useful tools for investigating differences in how individual students/teachers engage with mathematics – the structuralist and institutional aspects of the theory run the risk of reducing people to the positions they occupy in a particular institution. Chevallard (2024, p. 4) describes in a very detailed way what is considered as *knowing something* within the ATD approach in the following terms: a person knows something better than another person if someone judges that the relation of the first to this object of knowledge is closer to the institutional relation than the relation of the second. The notion of "personal relation", intended to handle this limitation of the theory, remains hard to operationalize. In particular, the methodological identification of this personal relation is one of the aspects that remain underdeveloped.

From the ToC point of view, taking the notion of canonical discourse and how the discourse of learners aligns with it as a way of conceptualizing *knowing something* can enrich the ATD approach. In this way, the discursive perspective of ToC can potentially provide a clear methodology for describing these personal relations to a specific object and its closeness to the institutional relationship to the object.

Institutional factors shaping discursive shifts

Turning again to the ToC, studies of transition processes is another area where we see potential for notions from the ATD to enhance ToC research. In commognitive terms, transitions are often described in terms of discursive shifts, where crucial changes at the meta-level of discourse might create commognitive and other conflicts for learners (e.g., Nardi, Ryve et al., 2014). There is also research investigating, for instance, how aspects of the secondary-tertiary transition can be traced in the narratives of university teachers and tutors (e.g., Kontorovich & Ovadiya, 2023). However, there is no well-developed language in ToC for addressing how external factors at the institutional level might influence these shifts and narratives. Indeed, Kontorovich and Ovadiya mention the need for further study of how and why certain narratives concerning the source of students' transition problems "become part of the pedagogical discourses of beginning university teachers" (p. 141). Here, the notions of didactic transposition and levels of didactic co-determination might provide a language for describing and analysing the extra-discursive conditions and constraints that might contribute to shaping specific narratives or discursive shifts.

CONCLUSIONS AND FUTURE WORK

We have outlined several areas where ATD and ToC demonstrate the potential for mutual enrichment. However, this paper should be viewed as merely a preliminary step

toward exploring the networking of these two theories. We began by delineating the distinct assumptions of each approach, both in terms of epistemological starting points, objects of study, units of analysis and research questions. Although the frameworks differ significantly in some ways, at the same time they operate at an institutional and socio-cultural level, offering grounds for potential complementarity. Indeed, there are particular constructs within the theories that seem particularly promising in this regard. As argued above, the notion of praxeology as a tool for describing and understanding how knowledge is organised in a particular institution should be possible to adopt into a discursive framework, and the commognitive tools for characterising and analysing discourse should be possible to utilise in order to investigate the logos part of a praxeology, viewed as "discourse on the techniques".

Still, devising and conducting such studies would require careful reflection both in the planning and analysis stages, to ensure that the adopted constructs are employed in ways that stay true to their theoretical origins. Indeed, in this short paper we were not able to engage with this issue in any detail, and an avenue for further research would be to more closely investigate the compatibility of the theories in terms of epistemological and ontological assumptions. A further challenge to such collaboration lies in the nature of the data analysed. ATD studies often remain at a documental level, while ToC, although also applicable to documents, mostly engages with transcriptions of audio and video data documenting discourse as it enfolds in interaction. Hence, a study drawing on aspects of both frameworks will probably require gathering a fairly complex body of data.

Beyond these challenges, the paper identifies four avenues for collaboration: mathematical knowledge structure, the expansion of praxeological analysis, personal relations, and the study of transitions. Naturally, as this work is exploratory, the logical next step is to conduct empirical research where these theoretical connections are operationalized and tested on actual data.

ACKNOWLEDGMENT

The research reported in this paper was funded by the Swedish Research Council, Grant No. 2024-04698, and the grant PID2021-126717NB-C33 funded by Ministerio de Ciencia, Innovación y Universidades grant MCIN/AEI/10.13039/501100011033 "ERDF A way of making Europe"

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Pedagogically productive talk across the school-university divide

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Academic mathematics courses are often required for secondary teacher certification, yet growing evidence suggests these courses have a marginal impact on classroom teaching. We propose a novel approach to this so called "second discontinuity" by inviting experienced secondary teachers to engage with real analysis problems and explore the challenges they pose for students during their secondary-tertiary transition – the "first discontinuity". Such pedagogical discussions are considered productive if they frame teaching and learning challenges as actionable. For teachers, this includes not only reflecting on tertiary mathematics instruction, but also how to better prepare students for their transition to proof-based mathematics. Our findings indicate how revisiting the first discontinuity may help experienced teachers overcome the second.

Keywords: real analysis, double discontinuity, pedagogically productive talk, meta-didactic transposition.

INTRODUCTION

Making academic mathematics (AM) matter for the teaching of secondary mathematics (SM) has been a concern for education researchers for over a century. While AM courses play a significant role in many teacher preparation programs, empirical studies repeatedly show that teachers often fail to recognize the value of their academic mathematics studies, finding them irrelevant to their practice (Wasserman et al., 2023).

This was recognized by the mathematician Felix Klein, who coined the term *"double discontinuity"* (Kilpatrick, 2019). The first discontinuity concerns the challenges undergraduates face in their transition from school to university, as the mathematical content and epistemic approaches to doing and learning mathematics differ substantially from their prior school experiences. The second relates to the challenges novice teachers face when they enter the school system, as the requirement to align with traditional teaching methods inhibits their ability to relate and apply the mathematics they have learned to practical teaching tasks (Gueudet et al., 2016).

Guided by a gap metaphor, efforts to improve the relevance of AM coursework in teacher preparation programs have focused on tackling the second discontinuity by explicating connections between the learning and teaching of AM and SM (Wasserman et al., 2023). Approaches vary according to key points of connection (Wasserman, 2018). These include mathematical content connections, which emphasize how AM content can enhance a mathematically robust understanding of SM content; disciplinary practices and classroom instruction connections, which demonstrate how AM practices and AM content can be applied in SM teaching; and modeled instruction connections, which relate to AM instructional methods that may be applied in teaching SM.

Much of the research to date has focused on pre-service teacher preparation, often operating under the implicit assumption that students struggle with the transition to AM and possess limited experience or intrinsic motivation to connect their AM studies with SM instruction. However, although scarce, studies on practicing teachers who engage with AM coursework after gaining classroom experience suggest that they are better positioned to identify how AM content and disciplinary values apply to SM teaching (e.g., Even & Mytlis, 2024; Hoffmann & Even, 2019; Pinto & Cooper, 2022, 2023).

Motivated by these preliminary indications, our study aims to contribute to the line of inquiry on practicing teachers by investigating a novel approach to their engagement with AM. We provide teachers the opportunity to engage pedagogically with AM as instructors and to collaboratively explore, through group discussions, the challenges of supporting undergraduate transition to AM. This form of engagement aligns with changes in the field of teacher professional development, which are gradually shifting from lecture-style courses, where the instructor serves as the primary source of knowledge, toward approaches that promote group discussion and facilitate teachers' collaborative knowledge around problems of practice (Lefstein et al., 2020). We first argue, and then demonstrate, that this form of teacher engagement with AM may yield actionable insights for SM teaching. We investigate this in a unique school-university partnership, where a group of SM teachers act as co-instructors in real analysis tutorials, and collaboratively explore students' transition difficulties to proof-based AM. Findings may inform the design of sustainable teacher development programs.

THEORETICAL FRAMING

We frame our research in terms of the meta-didactical (MD) transposition (Arzarello et al., 2013). The *didactic* transposition refers to changes that AM must undergo to become a body of knowledge that is teachable in institutionalized schooling (i.e., SM). The word *meta* in the MD transposition indicates a focus on teachers. It is concerned with making the *didactic* transposition an object for teachers' learning, transforming it from an academic idea into an object for teachers' reflection. Learning occurs along *MD trajectories* –activities that create opportunities for teachers to reflect on teaching and learning with respect to the didactic transposition. In these terms, our research aims to investigate MD trajectories for teachers to connect AM and SM.

Our approach is basically Vygotskian, viewing learning as the internalization of elements of interpersonal dialogue. Hence, we aim to engage teachers in discussions that promote making connections between AM and SM. Not all discussions are conducive for learning. To analyse the productiveness of such discussions, we utilize Lefstein et al.'s (2020) framework of *pedagogically productive talk*. This framework identifies six criteria for pedagogical productiveness of teachers' talk: *(1) focused on problems of practice; (2) involves pedagogical reasoning; (3) anchored in rich representation of practice; (4) is multi voiced; (5) includes generative orientation towards students, learning, content, teaching, and problems of practice; (6) combines support and critique* (Lefstein et al., 2020, pp. 362-363). This framework was

developed for analysing teachers' informal on-the-job talk, yet it remains relevant for facilitated discussions.

The MD trajectory we investigate here begins with authentic problems of practice, when we invite teachers to discuss students' difficulties with real analysis problems. Talk that is pedagogically productive for AM, i.e., anchored in rich representation of student difficulties and involving pedagogical reasoning for overcoming them, is expected to encourage the teachers to seek the origins of these difficulties in the different, sometimes conflicting ways in which mathematics is taught and learned in AM and in SM courses. In elaborating distinctions and connections between AM and SM, and how these connections inspire alternative pedagogies, we expect teachers' talk to be pedagogically productive not only for AM but also for SM.

Hence, our research question: In what ways can secondary mathematics teachers' discussions on challenges of teaching and learning real analysis be pedagogically productive in the context of academic and secondary mathematics?

METHODOLOGY

Setting

The study was conducted within a school-university partnership aimed at exploring first-year students' transition into AM at a research university in Israel. The setting was a first-year, proof-based real analysis course, compulsory for mathematics and computer science majors. The course is considered challenging, with approximately 50% of students failing their first exam. The partnership comprised the course lecturers, six graduate students acting as teaching assistants (TAs), three mathematics education researchers (the authors), and ten SM teachers, recruited by the authors based on prior personal encounters and recommendations. This study focuses on five of the SM teachers with varied teaching experience and academic backgrounds: Joy (1 year, M.Sc. in mathematics), Barb (10 years, M.Sc. in mathematics), Jordan (15 years, B.Sc. in mathematics), Harry (over 30 years, B.Sc. in mathematics), and Sharon (over 30 years, B.Ed. in mathematics).

The teachers co-facilitated weekly two-hour tutorials alongside the graduate TAs. These sessions complemented lectures and recitations, each accommodating up to 15 students who worked on problem sets with instructor support. In preparation, teachers worked on the problem sets and attended a two-hour preparation meeting facilitated by the authors to review solutions and anticipate student difficulties. Following each tutorial, the instructional teams held a reflection meeting to share experiences and collaboratively explore students' difficulties, potential causes, and solutions.

In the week following each tutorial, teachers submitted written reflective reports, guided by a prompt that asked them to recall memorable moments, insights, and ideas that might influence their SM teaching. At the end of the semester, teachers submitted a final report in which they selected two problems from the tutorial problem sets that

were memorable, highlighted the tension between SM and AM, or influenced their personal teaching.

Data and analysis

Data for the study included fully transcribed video recordings of all preparation and reflection meetings, weekly written reflections, and final written reports.

Here, we restrict our attention to a problem that many of the teachers referred to in their weekly and final reports, suggesting it will be a rich data source.

Let $0 < a, b \in \mathbb{R}$, let $n \in \mathbb{N}$, show that $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$

In the course lecture, $\sqrt[n]{a}$ was defined for any positive real a and any integer n as the number that taken to the nth power (defined as repeated multiplication) gives a. Existence and uniqueness were proven, showing it to be well-defined. To prove $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ students should invoke the definition for $\sqrt[n]{ab}$, i.e., this is the (unique) number that taken to the nth power gives ab, and show that $\sqrt[n]{a} \cdot \sqrt[n]{b}$ satisfies this condition, i.e. show that $\left(\sqrt[n]{a} \cdot \sqrt[n]{b}\right)^n = ab$ based on the definitions of the operations.

In our analysis, we first filtered the data for segments that addressed the specific problem. Next, we sought exchanges that were pedagogically productive according to Lefstein et al.'s (2020) framework criteria and coded them accordingly. We then proceeded to analyse two facets of productiveness that we expected to drive teachers' connection-making:

Involves pedagogical reasoning - the use of evidence, explanations, and reasons to interpret classroom events and analyse and justify courses of action (Lefstein et al., 2020): In our context, we examine how teachers interpret student difficulties (both those anticipated before the tutorial and those witnessed during it), as well as how they justify or critique the course's instructional actions.

Includes generative orientations towards students, learning, content, teaching, and problems of practice… [framed] in a way that is actionable for the teachers involved (Lefstein et al., 2020): Normally, in the context of a proof-based real analysis course, a generative orientation would be directed toward the course itself (e.g. how to improve teaching to overcome students' difficulties). However, when SM teachers are involved, an actionable framing takes on a broader interpretation, including reference to actions that can be taken in school (e.g., tracing university students' difficulties to the way mathematics is taught and learned in school, and suggesting actionable ways to better prepare them for university). In our analysis, we include connections drawn between AM and SM, considering them "pre-*actionable*" for SM in the sense that they may have actionable implications even when not explicitly elaborated.

For each facet (*involves pedagogical reasoning,* and *includes generative orientations)*, we identified emerging themes using inductive thematic analysis (Braun & Clarke, 2006).

FINDINGS

During the preparation meeting, prior to the tutorial, the teachers anticipated that some students would not recognize what it is they need to show (i.e., that the left side complies with the definition of $\sqrt[n]{ab}$), how to show it (i.e., by raising the left side to the power n), or even why they need to prove the rules of exponents. However, they were surprised by the extent of the difficulty. As Jordan wrote in his final report, "In the tutorial, this problem surprised us […] almost all students were stuck on it, and by the end of class they were still struggling."

The teachers analyzed difficulties the students' exhibited. We identified 4 themes in their analysis – two epistemic (grounded in the mathematics) and two didactic (related to how mathematics is taught and learned). For each, we demonstrate some of the teachers' pedagogical reasoning and emergent generative orientations to SM and AM.

Epistemic theme 1: Unproductive prior knowledge

In school, knowledge of mathematics is often synonymous with knowing what procedure to apply in a particular situation. In contrast, knowing in real analysis is constituted in mathematical claims, which can be used once proven. This epistemic gap entailed two challenges. First, not recognizing a need for proof, as observed by Joy: "For some students, [the identity] was so trivial that they had no idea what to do." Sharon referred to this in her final report: "Seeing the students brought back memories of myself as a student, thinking, but I already know the rules, and it makes sense, so what is there to prove?" Once students accepted the need for proof, they sought and failed to find an appropriate procedure. In Harry's words: "They were mostly stuck on what I am allowed to do and what I am not allowed to do."

The second challenge triggered a generative orientation towards the AM course. Joy described how her efforts were aimed at shaping the students' learning. She encouraged students to review their lecture notes, search for claims about exponents, and, for each claim, ask: "Does this help me […] am I allowed to use it?"

Both challenges triggered reflection on similar challenges in SM. When triangle similarity theorems are proven in 10th-grade, this may cause confusion because students have already learned and used these theorems without proof in 8th grade and they may not understand why they need to prove something they already know. Sharon claimed that this confusion leads to "a constant tension of not knowing what can or cannot be used." She recognized this tension in real analysis as well, as students need to prove things they already know from high school. Sharon could not suggest teaching actions to ease this tension:

> This is something we, as teachers, [AM] students, and SM students, need to endure. This tension cannot be resolved by us telling them what to do. They need to confront the dilemmas themselves […] it is part of the learning process. There are no shortcuts.

Another unproductive facet of students' prior SM knowledge was identified as mathematical habits. Jordan observed that: "their difficulty was with the equal sign

[…] they reacted to it as if [the problem] was an equation and started performing algebraic manipulations." Hence, the prominence of solving equations in SM curriculum conditioned students to apply familiar procedures that were irrelevant for the problem at hand. This was also referred to as "instinctive muscle memory from high school."

Generative orientations towards the course in relation to this challenge focused on teaching moves. Teachers suggested that lecturers should discuss the double meaning of the equal sign, and TAs should clarify the learning goal of the problem and the skills required to solve it. They suggested some modifications to the tutorial problem set and proposed ways in which the TAs can guide the students.

The discussion triggered reflection on similar challenges in SM. Barb compared students' misinterpretation of the equal sign to her 8th-graders' confusion between identities such as $2(x+3)=2x+6$ and equations such as $2x+6=5x-9$. For her this similarity was "interesting to notice", yet she was unsure about drawing her SM students' attention to the double meaning of the equal sign, claiming that "it is not always the right thing to do, but sometimes it is."

Epistemic theme 2: Nature of knowing and understanding - Informal vs. formal

In the preparation talk, the teachers anticipated difficulties related to the role of formal definitions in mathematics. Harry demonstrated this in relation to rules of exponents:

> Rules of exponents are a good example of something that is very intuitive when dealing with natural numbers. There is a dichotomy between intuition and going by the definition. As soon as you go beyond the natural numbers, even exponent 0, and certainly beyond, you must think differently. You can no longer think intuitively. You need to think strictly in terms of definitions […] In secondary school, [the students] don't always understand this. They simply understand from what the teacher says that they can continue using the same rules for all numbers. And they do.

In the reflection meeting, one of the TAs identified undergraduate students' inexperience in working with definitions as the root cause for their observed difficulties. The participants agreed that this is a natural stage of learning, and Harry suggested that students need to learn through experience.

Generative orientations towards SM teaching varied. Harry, for example, concluded that he should consider placing an emphasis on reasoning and on working by definition when teaching the rules of exponents. In contrast, Sharon favored intuitive reasoning in school:

> I don't think it is possible to do it differently in secondary school. You need to adapt to different student [capabilities], not everyone is a mathematician. Furthermore, I think that nearly all of the students are not ready to understand the nuances of [proof by definition].

Didactic theme 1: Responsibility of lecturers and tutorial TAs

In university, students are responsible for their learning. Nevertheless, Jordan placed some responsibility on the TAs:

> For me, the obvious conclusion, when I saw the students and responded to their questions, was that this degree of struggling with a problem that is not very complicated mathematically necessarily indicates an instructional gap.

Jordan considered this to be an institutional issue:

> As a high school teacher, if most of my students are stuck on a problem, there is no doubt that this indicates a flaw in instruction. […] In contrast, the TAs, who clearly seem to be supported [by the lecturers], place all the responsibility on the students.

TAs often failed to provide support for tasks that students could not perform on their own. Jordan quoted a TA suggesting that a student "gaze at [the problem] for a while" until it made sense, explaining in the subsequent reflective discussion that "there's nothing to be done, they just need to learn it".

Generative orientations towards the AM course included teachers' implicit suggestions for the TAs to claim didactic responsibility by describing how they attempted to shape the students' learning (c.f. the section on naming skills and procedures).

Generative orientations towards SM were to better prepare students to take responsibility for their own learning. Jordan described how this could be done:

> SM teachers should prepare students, not only in terms of mathematical content [they need to know], but also in the way they will be expected to study. […] We need to include questions without titles […] so part of the challenge is to figure out what needs to be done.

However, Jordan was not certain that this was the right thing to do in SM. He stressed that this approach contradicts his didactical orientation, both in school and university. Furthermore, he claimed that such skills are not required for the matriculation exams and may interfere with the goal of optimizing students' graduation grades.

Didactic theme 2: Explicit naming of skills and procedures

The skills students need to draw on to solve a problem may not be obvious. In his SM teaching, Jordan attends to this explicitly:

> I begin the discussion on nearly every problem with a clarification on the type of question before us. […] The first thing students must understand, and hence I must explain, is the kind of activity they are expected to engage in to solve the problem.

Other teachers referred to this as the students' inability to "read" the problem and decode the context. Jordan and other teachers stressed the didactic importance of explicitly naming skills and procedures that students should be able to apply.

Generative orientations towards the AM course included suggestions to modify the instructions, to clarify the problem's didactic goal, and to name the required skills.

Generative orientations towards SM are evident in Jordan's connection to his teaching.

DISCUSSION

Our study examines a novel meta-didactical trajectory aimed at encouraging teachers to connect the AM they had studied with the SM they teach. We hypothesized that participating in pedagogical discussions regarding students' challenges in a proof-based real analysis course might induce generative insights into the teaching and learning of SM.

Our findings demonstrate that, in the context of the course, discussions (in preparation and reflection) were pedagogically productive in relation to the criteria outlined in Lefstein et al.'s (2020) framework of Pedagogically Productive Talk. Discussions *focused on problems of practice* – difficulties with the tutorial problem set and invoked rich *pedagogical reasoning* as participants offered various interpretations of observed difficulties, identifying epistemic and didactic challenges. Discussions were anchored in *representations of practice*: Preparation talks focused on the tutorial problem set, while reflection meetings, held immediately after the tutorial, analyzed authentic instructional experiences. Discussions were *multi-voiced*, leveraging the diverse expertise of SM teachers, graduate TAs, and mathematics education researchers, in ways that respectfully *combined support and critique*. These discussions fostered *generative orientations towards students, content, teaching, and problems of practice* in the real analysis course. This was evident in a sensitivity to the epistemic and didactic challenges of teaching and learning AM, which prompted specific suggestions for modifying the problem sets and refining tutorial facilitation. These suggestions often drew on teachers' SM experience.

Identifying epistemic and didactic gaps between SM and university as a significant source of students' difficulties led teachers to revisit their own role in preparing students for their future studies. This comprised epistemic practices - emphasizing reasoning and relying on definition, and AM didactic practices - fostering students' independence. Thus, their generative orientation to AM instigated a generative orientation to SM.

Teachers' generative orientations to SM teaching demonstrate all the key points of connection outlined by Wasserman (2018). The problem task established a *content connection*, as rules of exponents are addressed in both school and university contexts. Teachers' awareness and intention to emphasize working by definition highlighted a connection to *disciplinary practice*. Furthermore, recognizing concrete SM situations, such as teaching 10th-grade triangle similarity, as opportunities to introduce AM practices, such as proof, represents a form of *practice connection*. Finally, reflective thoughts on adopting AM didactic principles, such as fostering independent learners, serve as an instance of *modeled instruction*.

The MD trajectory we investigated was grounded in a real analysis course, and thus, its generative orientation to AM was highly actionable, suggesting concrete teaching alternatives inspired by connections to SM that were expected to enhance learning.

Regarding SM, teachers' generative orientations were less actionable. Rather, connections with AM suggested teaching alternatives that the teachers considered but, given their awareness of SM institutional complexities, were not quick to endorse. Thus, their generative orientation evoked dilemmas for their future SM teaching. Many of the teacher reports indicate that they had not yet decided on a definite course of action. Harry, for instance, wrote that he is "considering" putting an emphasis on proof by definition when teaching the rules of exponents. Jordan shared his reasons and concerns for adopting AM instructional methods. Nevertheless, the reports indicate that teachers gained insight into alternative ways to connect AM practices or instructional methods with their SM teaching that may serve them in their future teaching.

Much of teachers' generative orientations to SM was reported in individual written reflections, limiting multi-voicedness. Furthermore, they were highly individual, with each teacher drawn to a different point of connection between AM and SM (Wasserman, 2018). This constraint on pedagogical productiveness inspired a new MD trajectory, which was piloted in the following semester. The pedagogically productive discussions that took place served as a springboard for sharing individual generative orientations and the teaching dilemmas they entail. Findings from this follow-up research will be reported in due course.

To further contribute to the implementability of our findings, we wish to highlight some of the features of the particular problem we have discussed. While there were many problems that we could have analysed for the pedagogically productive discussions that they triggered, this seemed to have triggered richer discussions than many of the others. We attribute this to the following features: The problem had a superficial similarity to high school exercises, which invoked a comparison. Students were required to prove something that these high school students make free use of and take for granted. The required skill – proving based on definition – is not required for high school matriculation exam, yet does have some space in the curriculum – in geometry (where, for example, triangle and quadrilateral theorems are proven by the teacher based on definitions) and also in algebra, (in fact by some teachers in the very context of exponents of the real analysis problem – rules of exponents). A more thorough analysis of how particular features of real analysis problems can contribute to pedagogically productive discussions on high school mathematics will require further research.

ACKNOWLEDGMENT

This study was supported by the Israel Science Foundation (grant 2015/23).

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Interweaving mathematics – a discursive view on pre-service teachers navigating between university mathematics and school teaching

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In a university geometry course, pre-service teachers are presented with vignette-based exercise problems that aim to address Klein's second discontinuity by developing ways of drawing on university mathematics to benefit mathematical understanding of a teaching situation. We take a discursive approach to this endeavour, suggesting that it requires pre-service teachers to navigate between several discourses, which they perceive as quite challenging. Analysing examples from PSTs' written solutions to one exercise problem, we argue that interpreting their learning processes as an interdiscursive challenge can give new and meaningful insight into why exactly navigating these discourses is so demanding, and how it may be achieved.

Keywords: Klein's second discontinuity, discourse analysis, interdiscursivity, geometry, pre-service teacher education.

INTRODUCTION

In a review of research on the role of university mathematics in secondary teacher preparation, Wasserman and colleagues (2023) highlight two central challenges. Firstly, university programme structure often divides the different components of professional knowledge and their application to teaching situations into separate courses, which leads to a fragmented engagement with content that needs to be connected. Secondly, specifically regarding mathematical subject matter, pre-service teachers (PSTs) for secondary school are typically required to take courses dealing with abstract university mathematics, but the knowledge gained from them cannot be directly applied in teaching situations. This latter issue was already described by German mathematician Felix Klein (1908) as the second part of a 'double discontinuity' for PSTs in mathematics. Since then, a lot of thought and research has gone into understanding and trying to bridge Klein's so-called second discontinuity. The notion that knowledge about university mathematics will simply 'trickle down' to the school level and effortlessly become applicable there has also been refuted several times (e.g., Wu, 2011). Consequently, different models have been developed to describe the mathematical knowledge that secondary school teachers need. *School-related content knowledge* (SRCK; Dreher et al., 2018), for instance, formulates explicit beneficial ways for teachers to draw connections between school and abstract university mathematics. At the affective level, it has been observed that PSTs frequently develop negative beliefs on the coherence between university and school mathematics and the relevance of university mathematics courses to the teaching profession, and that specific teacher-oriented interventions can prevent this development (Eichler & Isaev, 2023).

Even though many interventions to bridge Klein's second discontinuity have been employed and reported as effective, more detailed insight is needed into what corresponding learning processes actually look like, how PSTs apply their knowledge to teaching situations, and how they can be supported in doing this already during their university education (Wasserman et al., 2023; Dreher et al., 2018).

PROJECT CONTEXT

Building on previous research, a new type of exercise problems for PSTs, called *interweaving problems*, is being designed and implemented at Heidelberg University in a university geometry course, consisting each week of four hours of lecture, a set of exercise problems, and a tutorial to discuss the exercise problems. The PSTs take this course near the end of their university education, after all other university mathematics courses and not long before entering teaching practice in secondary school. Each week, one of the exercise problems is an interweaving problem, intended to explore ways in which PSTs' knowledge about university mathematics can benefit them in teaching situations. Each problem is framed around a representation of a realistic teaching situation as a text-image vignette, and contains several scaffolding prompts to guide PSTs' engagement. Building upon the SRCK construct and teacher noticing related to the mathematical subject matter (cf. Sebők, 2023), the scaffolding guides PSTs to analyse the given situation from a subject matter perspective, directly drawing on specific mathematical content that the geometry lecture deals with in the corresponding week. Eventually, the scaffolding asks PSTs to imagine a possible reaction as teachers in the situation, informed by their previously gained insight (see Weith et al., 2025).

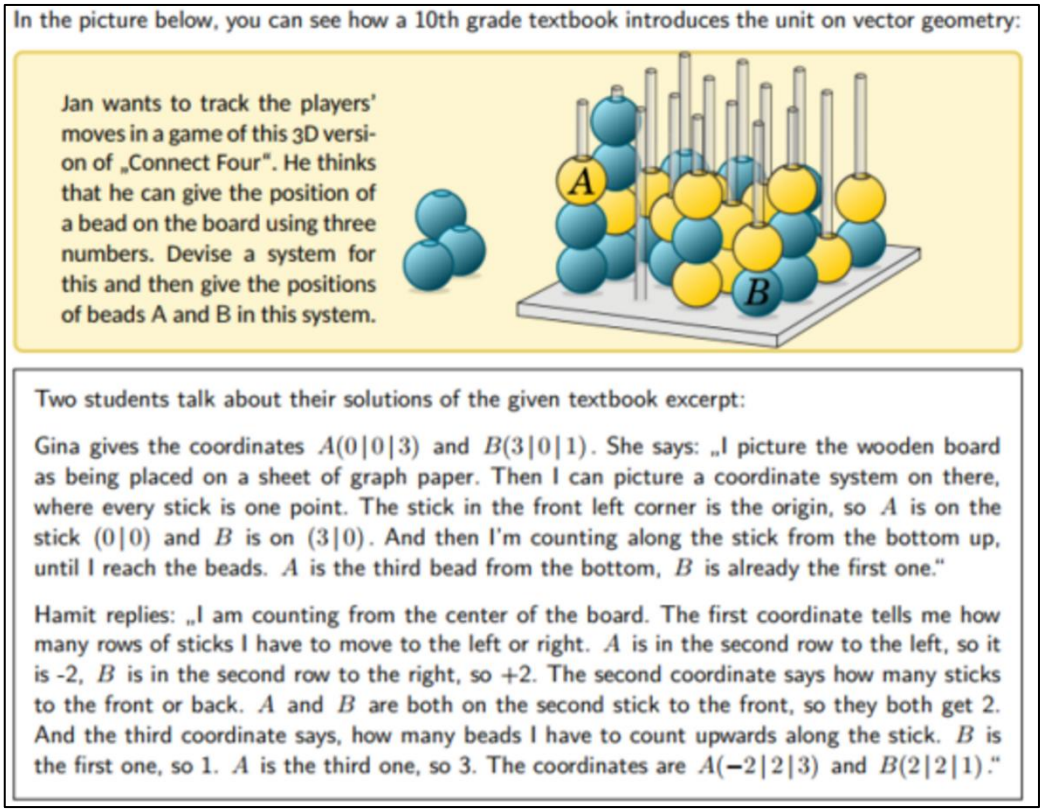

Figure 1: Interweaving problem vignette (translated from German by the first author)

Figure 1 shows the vignette for the interweaving problem that the PSTs solve in week 4 of 13 of the geometry course. It has been constructed to allow for the following observations: Even though Gina and Hamit both solve the textbook task correctly, their solutions differ in quality; Gina's solution is compatible with the Cartesian coordinate system, which the textbook excerpt is meant to lead up to, but Hamit's is not. It 'skips zero' and would lead to inconsistencies in vector arithmetic, e.g., when calculating distances. A teacher in this situation would be confronted with the challenge of explaining why Hamit's solution will lead to problems that Gina's will not, while still appreciating both solutions as correct within the given task. They might then recognise the discrepancy between the task's purpose of leading up to a specific type of coordinate system and the much greater set of correct solutions it allows, and question whether the task is even suited for its purpose.

The scaffolding is designed to guide PSTs towards these considerations and directly draw on content from the geometry course, specifically about bases and coordinate systems in affine spaces [1]. Section (a) serves to establish a connection between the textbook task and the mathematics from the lecture. Prompt (a.1) asks PSTs to solve the textbook task themselves and describe a positioning system, and prompt (a.2) then asks for a justification in university-mathematical terms why the positioning system is an affine coordinate system. Section (b) prompts a detailed mathematical analysis of the two students' solutions: In (b.1), PSTs determine whether they fulfil the textbook task. Then the scaffolding asks for specific arguments, using university mathematics, that Gina's solution does correspond to an affine coordinate system (b.2) while Hamit's does not (b.3) [2]. Section (c) directs the PSTs' attention back to the textbook task, prompting them to reflect upon whether it is suited to fulfil its purpose, and which learning difficulties a teacher should anticipate. The further continuation of the scaffolding, where PSTs craft a response to the students, will not be the focus here.

A central goal of our research is to gain detailed insights into how PSTs engage with interweaving problems, especially how they connect specific ideas from university mathematics with realistic teaching situations. In this paper, we argue that a discursive analysis of PTSs' written solutions allows for meaningful insight into their learning processes. For this, we will draw on ideas from commognition theory.

THEORETICAL CONSIDERATIONS

The theory of Commognition is a framework that interprets cognition as well as communication as discourse, and learning, especially of mathematics, as learning to participate in a new discourse (Sfard, 2008). Different discourses can be distinguished by four characteristics: *vocabulary* and *visual mediators*, which characterise the typical language and sign systems of a discourse, *routines*, meaning recurring patterns within the discourse, and *endorsed narratives*. These are utterances about the objects of a discourse, their relations and properties, which are accepted by the participants in the discourse. Narratives can be on the *object level*, pertaining to the objects of the discourse, or on the *meta level*, representing rules by which participants in the discourse may act. Specifically concerning the meta-narratives of university mathematics

discourse, Viirman (2021) found that lecturers do not always communicate them explicitly, but that university students are sometimes implicitly expected to recognise them on their own. In contexts where several discourses are involved simultaneously, researchers have pointed out the importance of *interdiscursivity*. This can include discursants flexibly switching between discourses to combine them into one (Mosvold, 2015), or supporting productive engagement in one discourse by carrying elements from another discourse into it (Sfard, 2008, p. 220).

Teachers' ability to analyse teaching situations based on their professional knowledge and experience has been discursively conceptualised as proficiency in *discourse about teaching* mathematics (cf. Cooper, 2014). Mosvold (2015) observed that this type of discourse is highly interdiscursive per se, as elements from mathematical, didactical, pedagogical and other discourses come together. Due to this interdiscursivity, it is very challenging for inexperienced teachers to participate in discourse about teaching, and especially to include insights about the mathematical subject matter in it (ibid.). Considering the discourse by which such insights are generated highlights another interdiscursive challenge, which closely corresponds to Klein's (1908) observations: While the general consensus is that mathematics teachers need to know university mathematics for a better mathematical understanding of teaching situations, it has not been explored in much detail how exactly elements of *university mathematics discourse* can be beneficially integrated into teachers' *discourse of mathematical meaning-making about a teaching situation.* Research on PSTs' beliefs about the relevance of university mathematics for the teaching profession (Eichler & Isaev, 2023) suggests that they perceive this as a particular challenge.

The aim of the interweaving problems from our project is to support PSTs' engagement in *discourse about teaching* with a specific focus on the mathematical subject matter. The scaffolding is designed to guide PSTs through the two interdiscursive challenges outlined above. First, PSTs are prompted to engage in *discourse of mathematical meaning-making about a teaching situation* and to directly draw on elements from *university mathematics discourse*. Then, the scaffolding asks them to engage in *discourse about teaching*, but explicitly asks for a focus on the mathematical subject matter so that elements of *discourse of mathematical meaning-making about a teaching situation* will be incorporated. One goal of our research is to gain detailed insight into the discursive mechanisms of this. In this paper, we will present a discursive analysis of data and argue why such an approach is promising. We aim to answer the following research questions, in accordance with the two challenges identified above: 1) How do PSTs' solutions to interweaving problems use discursive elements from university mathematics to support mathematical meaning-making regarding a teaching situation? 2) How does mathematical meaning-making in PSTs' solutions compare discursively to their taking the perspective of a teacher?

METHODS

The data set analysed for this paper consists of the written solutions to the interweaving problem presented in Figure 1, handed in by the participants of the geometry course

during the 2025 summer semester. 28 PSTs participated in ten small groups of two or three, all of which gave consent for their solutions to be analysed. With regard to the first research question, solutions to prompts (a.2), (b.2) and (b.3) were analysed, as they ask to draw on university mathematics to gain a greater understanding of the presented situation. A comparison between section (c) and previous sections yields the most insight for the second research question, as (c) asks PSTs to take the perspective of a teacher again after some more mathematical prompts.

To identify discursive engagement, we analysed for the four characteristics mentioned above: *vocabulary*, *visual mediators*, *routines* and *narratives.* As the data was all in written form, the analysis for vocabulary and visual mediators was concerned with the words and representations of mathematical objects in the solutions. Due to great variation in how routines and narratives were structured and presented, each solution was additionally paraphrased by the first author. These paraphrases were then analysed for the routines and narratives that could be recognised in them. The analysis for interdiscursivity requires familiarity with a number of discourses, mainly the specific mathematical discourse from this geometry course and teachers' discourses about teaching. The first author of this paper has been involved in teaching the geometry course for three semesters, and can therefore be considered familiar enough with its discourse to conduct the analysis. Since the team of authors has many years of experience as teachers, teacher educators, and teacher education researchers, an analysis for elements of typical discourses about teaching was also possible.

FINDINGS

In the following, we present aggregated findings across groups alongside a few especially salient examples. As expected considering the scaffolding prompts, the vocabulary and visual mediators used in sections (a.2), (b.2) and (b.3) mostly align with university mathematics discourse. Expressions like 'affine basis' and 'translation vectors' find a lot of use, which can clearly be attributed to the vocabulary of the geometry lecture rather than school geometry. All groups denote positions in vector notation instead of with verbal or graphical descriptions.

Prompts (a.2) and (b.2) both formulate the same task – to argue that a set of points is an affine basis. Out of 20 total solutions to this task (two from each of the ten groups), our analysis will focus on eight solutions which use a strategy we will refer to as the *affinity routine*: finding an affine bijection that maps the set of points onto the affine standard basis. The affinity routine is established in the lecture as a universally valid way to solve this task. Of the eight solutions that apply it, seven adhere closely to typical realisations from university mathematics: providing an affinity mapping onto the affine standard basis, and using matrix-vector notation or mentioning the identity function. There is one central difference between the situation presented by this vignette and the typical contexts in university mathematics where this routine is realised: Whereas in university mathematics, it can always be pre-supposed that the affine standard basis exists and is known, it has not been established here. This means that these seven realisations of the affinity routine can only be considered valid if they

are preceded by an argument of where the affine standard basis is located and why it is in fact a basis. The scaffolding explicitly mentions this necessity in both prompts (a.2) and (b.2). Still, though most of the seven solutions make some attempt at characterising the standard basis, none fully succeed at creating the formal prerequisites for their realisation of the affinity routine to be considered valid by the standards of the university mathematics discourse. In contrast, one solution approaches the routine entirely differently: instead of mapping onto the affine standard basis, they map onto an affine basis established in an earlier section of the group's own solution (see Table 1). Rather than giving an affinity in formal notation, they verbally explain which visual transformation would need to be performed for the two bases to become identical.

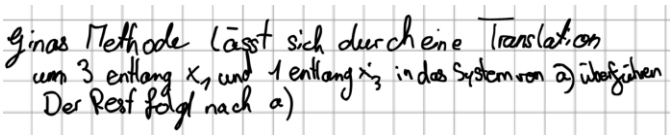 Ginas Methode lässt sich durch eine Translation um 3 entlang x_1 und 1 entlang x_3 in das System von a) überführen. Der Rest folgt nach a)	Gina's method can be transformed into the system from (a) by a translation of 3 along x_1 and 1 along x_3. The rest follows with (a).

Table 1: Excerpt from a solution and English translation by first author

Prompt (b.3) sets the task to argue that a positioning system is not an affine coordinate system, with detailed instruction on how to structure the argument. Of the ten groups, seven give mathematically correct solutions, all employing the same routines from university mathematics with only slight variation in order. Interestingly, a few groups employ additional strategies to connect the formal mathematical argument to the visual situation in the vignette. Four groups explicitly refer to the visual situation when determining numerical coordinates. Two groups precede the formal mathematical argument with a visual explanation of what the formal property means: distances between neighbouring beads are not constant. One group even explicates the discursive shift that is happening (see Table 2). After the visual explanation, they preface the formal argument by stating that it is another way to observe the same property.

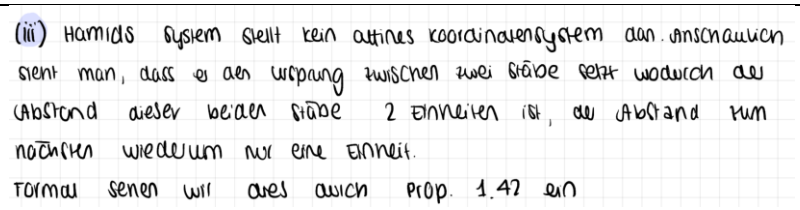 (iii) Hamids System stellt kein affines Koordinatensystem dar. Anschaulich sieht man, dass es den Ursprung zwischen zwei Stäbe setzt wodurch der Abstand dieser beiden Stäbe 2 Einheiten ist, der Abstand zum nächsten wiederum nur eine Einheit. Formal sehen wir dies auch Prop. 1.42 ein
Hamit's system does not represent an affine coordinate system. We can see visually that he places the origin between two sticks so that the distance between those two sticks is 2 units, whereas the distance to the next one is only one unit. Formally, we can also see this with Theorem 1.42.

Table 2: Excerpt from a solution and English translation by first author

In prompt (c), a shift in the vocabulary and visual mediators used immediately becomes apparent. Where written text, numerical-symbolic notation, and graphical realisations were all used with some frequency before, now all solutions exclusively use written text. Keywords that do not appear in prior sections are now mentioned by several

groups: *Spatial awareness* (six mentions), the *realistic context as a motivating factor* (three mentions), *inquiry-based learning* and *creative thinking* (two mentions each). Despite the shift in vocabulary and visual mediators, some narratives are carried over from previous sections. The previous prompts are designed to reflect upon the variety of coordinate systems that would be acceptable solutions to the textbook task, and narratives about this are mentioned frequently here: Seven groups mention the fact that non-affine coordinate systems are allowed as solutions, and eight groups state that several affine coordinate systems with different origins and basis vectors can be found.

Even when several groups state the same narrative about the vignette, however, there is some variation in their further discursive engagement with it. For instance, eight groups mention the fact that the textbook uses a visual model to introduce a more abstract mathematical idea. Three groups connect this to a typically endorsed narrative from discourses about general pedagogy: using contextualised real-life examples is motivating and therefore supports learning. Five groups, however, engage with the *mathematical* properties of the model. Two of them state that the visualisation is specifically suited to facilitate the transition from a two-dimensional Cartesian coordinate system to a three-dimensional one, consolidating this claim by explaining how a two-dimensional coordinate system can easily be recognised in the model, and how the emergence of a third dimension can be understood visually (see Table 3).

[The students] already know 2D coordinates. Looking at the bead model directly from the front, the 2D coordinate system emerges with the directions ‘to the right’ and ‘up’. The students are familiar with this. From this point, it is not too hard anymore to accept the direction ‘towards the back’ as the third coordinate. Thus, students can familiarise themselves visually with coordinates in three-dimensional space.

Table 3: Excerpt from a solution and English translation by first author

DISCUSSION

Having analysed the data from the PSTs’ solutions, we will now interpret the analysis to answer the research questions. Regarding the first research question, the instance of the affinity routine is of particular interest, as it illustrates an obstacle we presume to be common when trying to relate university mathematics to school practice: a routine that in university mathematics is universally applicable to solve the task *identifying an affine basis* cannot be applied in the school context without modification. Creating the formal prerequisites for a typical realisation of the routine to be applicable is evidently very challenging to the PSTs, as none out of seven solutions is fully successful. We interpret this as a glimpse into the details of why Klein’s second discontinuity is such a big problem. Even though the PSTs generally seem to recognise the connections

between the school situation and concepts from university mathematics, they struggle greatly with relating them to each other explicitly enough for the university routine to become applicable in the school context. It is easy to see how this can diminish the perceived benefit of knowing university mathematics as a school teacher.

The same instance also provides an example of how some PSTs forego this difficulty by treating the university mathematics routine more flexibly: Rather than giving an affinity **in vector notation** that maps onto the **affine standard basis,** one group **visually describes** an affinity that maps onto **any previously established basis.** Recognising this as an equivalent possibility, though, requires a deep understanding of the underlying logic of the routine, as well as the conscious breaking of meta-narratives from the university mathematics discourse, as this realisation of the affinity routine would not be considered valid there. This is one example of a type of interdiscursive practice we tentatively label 'interweaving routines'; drawing on university mathematics discourse when engaging in discourse of mathematical meaning-making about a teaching situation, but breaking with its conventions in productive ways. A second, rather more intuitive, example of these interweaving routines can be found in section (b.3). Two groups go beyond merely completing the formal mathematical argument and explicitly interpret its meaning visually. This indicates an interdiscursive awareness of how one can benefit from including elements from university mathematics discourse here. These groups do not treat the formal argument itself as the goal, but use it to support their mathematical understanding of the situation presented in the vignette.

This analysis gives an impression of how PSTs' *discourse of mathematical meaning-making about a school situation* can directly benefit from integrating elements of *university mathematics discourse*, but that these might need to be applied in unfamiliar ways. The two discourses pursue fundamentally different aims and thus follow different meta-narratives, but this is rarely explicitly addressed to PSTs and therefore presents a challenge to them. Moreover, the meta-narratives of *university mathematics discourse* itself are not always explicitly communicated by lecturers (Viirman, 2021) and may therefore not be entirely clear to PSTs, adding to the difficulty of this situation.

To answer the second research question, an analysis of the vocabulary and visual mediators used in the solutions shows a clear shift between sections (a) and (b), focused on mathematical meaning-making, and section (c), which evaluates the textbook excerpt as learning material. While the former draw heavily on elements from university mathematics discourse, the latter barely use those anymore, and keywords from discourses about pedagogy become more frequent. It needs to be noted here that the scaffolding in section (c) explicitly asks PSTs to focus on mathematical subject matter in their evaluation. This indicates that PSTs perceive *discourse about teaching*, even when explicitly asked to focus mathematical subject matter, not as a fundamentally mathematical discourse, but perhaps more as a pedagogical one. That is, of course, a valid perspective and should not be judged negatively, but it does highlight the challenge of incorporating insights from discourses of mathematical

meaning-making into this discourse, which has been observed by several researchers (e.g., Mosvold, 2015; Cooper, 2014). Nevertheless, the fact that most groups mention narratives from the former sections when engaging with the latter sections suggests that these discourses are not perceived as entirely disjoint by our PSTs.

A closer look at how different solutions engage with the vignette's use of a visual model reveals some variation. Even though the PSTs were explicitly prompted to focus on mathematical subject matter, only about half the groups do this, while the others draw solely on pedagogical discourses. This aligns with Mosvold's (2015) observation that authentic discourse about teaching is highly interdiscursive and that inexperienced participants struggle with navigating this interdiscursivity. Considering this, the variety in our data is not too surprising and can be appreciated as an approximation of the complex, interdiscursive reality. It may, however, pose a difficulty to the interweaving problems as a learning opportunity, and to the specific development of discourse about teaching with a focus on mathematical subject matter.

In this paper, we have presented a discursive analysis of PSTs' solutions to the interweaving problems and argued why this approach may be fruitful. Although data from only one interweaving problem has been analysed, and analysis of the entire data set is still ongoing, the analysis allows a focus on the specific challenges that PSTs face when trying to use university mathematics beneficially in the school context: utilising elements from university mathematics discourse to further mathematical meaning-making, but also incorporating mathematical subject matter into discourse about teaching. Beyond that, it has the potential to reveal in detail the ways in which PSTs fail to productively use university mathematics in mathematical meaning-making, as well as interdiscursive practices that help them succeed. This can deepen our understanding of why Klein's second discontinuity poses such an immense challenge from PSTs' perspective, and inform the design of future learning opportunities.

ACKNOWLEDGEMENT

The interweaving problems are designed and implemented in cooperation with Hendrik Kasten and Denis Vogel from the Mathematical Institute at Heidelberg University.

NOTES

1. An epistemological difficulty we face for the design is that most school geometry is Euclidean, but the university lecture only arrives at Euclidean spaces quite late in the semester. Therefore, the interweaving problems sometimes cannot draw on the Euclidean geometry that would be natural in a situation because the lecture has not yet dealt with it. Despite this, the problems have been carefully constructed to allow for the most possible insight from the university mathematics available when they are presented to PSTs. Furthermore, the aim of the interweaving problems is to establish broader strategies to connect university mathematics with teaching practice, with each vignette serving as an emblematic example, rather than achieving a complete and in-depth mathematical deconstruction of each individual vignette.

2. The most obvious flaw in Hamit's system is that distances cannot be consistently defined. As the lecture will only introduce distances later in the semester, prompt (b.3) instead points towards the closest equivalent available: ratios between collinear vectors are not constant when basic transformations are applied, such as a shift to the right by one bead.

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A Reference Epistemological Model for Swedish high school Calculus

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Keywords: Calculus, Anthropological Theory of the Didactic, Reference Epistemological Model, textbook analysis, Klein's double discontinuity.

BACKGROUND AND RATIONALE

Calculus is one of the most important areas of mathematics, both at the upper secondary and university levels, and it is well known that it is a difficult topic for students (e.g., Bressoud et al., 2016). In spite of this, there is little research on exactly what knowledge upper secondary mathematics teachers (USMTs) need in order to teach calculus and how teacher education can support prospective teachers in developing it (exceptions include Yan et al., 2020). One way of understanding the demands of USMT education is through the so-called *double discontinuity* (Klein, 1908), referring to two abrupt transitions in the experience of USMT students; the first when they leave school and begin their university studies, the second when they leave university and enter the workplace as teachers. In both cases the mathematics they encounter and their own role in relation to it change radically. As part of a larger project addressing the double discontinuity in the context of Calculus/Analysis, and focusing in particular on how to prepare prospective USMTs for teaching Calculus, we aim to give a characterization of Swedish upper secondary Calculus, and how it differs from the university Calculus that prospective USMTs study. Since there is evidence to suggest that textbooks are a major influence on the teaching in Swedish mathematics classrooms (e.g., Bergqvist et al., 2010), it makes sense to focus such a study on how upper secondary Calculus is presented in textbooks. In this poster proposal, I will describe the analytical tool we have developed for this study, the so-called Reference Epistemological Model (REM).

DEVELOPING A REFERENCE EPISTEMOLOGICAL MODEL

Central to the Anthropological Theory of the Didactic (ATD) is the question of how knowledge is constructed within and transformed between institutions. In the ATD, a model of a knowledge organisation is called a *praxeology*, and is divided into a praxis block, consisting of the *types of tasks* (T) and their corresponding *techniques* (τ) and thus centred on the doing; and a logos or theoretical block, describing the discourse in terms of *technology* (θ) (directly justifying the technique) and *theory* (Θ) which justifies and structures the technologies (Barbé et al., 2005). Praxeologies can be centred around a particular type of task T, outlining the set of techniques available for solving this task, and their associated technologies. In this case, we talk about a *point praxeology*. A *local praxeology*, on the other hand, is centred around a specific technology θ, and a number of local praxeologies sharing the same theory Θ are grouped together as a *regional praxeology* (Barbé et al., 2005).

The aim of the proposed textbook study is to present a praxeological characterisation of Swedish upper secondary Calculus. To this end, a reference model of the intended

mathematical knowledge is needed (Barbé et al., 2005). In our case, this Reference Epistemological Model (REM) will consist of a set of local praxeologies jointly making up a regional praxeology of Calculus (Table 1). In the analysis, the REM can be used,

TYPE OF TASKS (T)		TECHNIQUE (τ)	TECHNOLOGY (θ)	
T_2: Differentiate functions	$T_{2.1}$: differentiate power functions (polynomials)	$\tau_{2.1}$: Algebraic methods $\tau_{2.2}$: Digital methods $\tau_{2.3}$: Graphical methods		θ_2: Derivative (including rate of change and difference quotient) $\theta_{2.1}$: Conditions for differentiability θ_5: Function
	$T_{2.2}$: differentiate exponential functions		$\theta_{5.1}$: The number *e*	
	$T_{2.3}$: differentiate logarithmic functions		$\theta_{5.2}$: The natural logarithm	
	$T_{2.4}$: differentiate trigonometric functions		$\theta_{5.3}$: Trigonometric functions	
	$T_{2.5}$: differentiate composite functions		$\theta_{2.2}$: The chain rule $\theta_{5.4}$: Composite functions	
	$T_{2.6}$: differentiate products of functions		$\theta_{2.3}$: The product rule	
	$T_{2.7}$: differentiate quotients of functions		$\theta_{2.4}$: The quotient rule	
	$T_{2.8}$: Compute the derivative of a function in a specific point			

Table 1: Part of the REM – local praxeology of derivative.

for instance, to determine whether some praxeological elements are missing or have been added. To build the REM I drew on Swedish national curriculum documents, but since these are very short on detail, I also made use of Swedish standardized national tests and university level Calculus textbooks. I began by identifying the relevant technologies, mainly drawing on the national curriculum, but adding 'real numbers' and 'continuity', not mentioned there but crucial to the development of Calculus. Then I added related types of tasks, drawing on curriculum documents and national tests. As for techniques, the curriculum documents are quite laconic, merely speaking of digital and graphical methods, but, for instance, the national tests make it clear that students are also expected to use algebraic methods, so this was added. In the actual poster I will present the full REM, and exemplify its use in the analysis.

ACKNOWLEDGMENT

This research was funded by the Swedish Research Council, Grant No. 2024-04698,

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TWG4: Digital technologies in the teaching and learning of university mathematics, including artificial intelligence

Mathematical Discourse of Generative AI Tools on Improper Integrals: The Case of ChatGPT

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Drawing on the commognitive framework, this study discusses ChatGPT's evaluations of a set of improper integrals, focusing on the extent to which the tool fosters ritualistic and exploratory participation in integral discourse. The analysed data consist of 24 evaluations by ChatGPT of eight improper integrals obtained across three trials. The findings indicate that ChatGPT was able to recognize the integrals as improper in most cases (20 out of 24 evaluations), but it did not consistently promote explorative mathematical discourse. For example, its routine often lacks explicit reference to the improperness of the integrals, including endorsed narratives about the definition and the use of routines for their evaluation. Furthermore, in most evaluations, ChatGPT did not invite users to engage with graphical realizations of improper integrals.

Keywords: calculus, commognition, rituals and explorations, case study, generative AI tools.

INTRODUCTION

The rapid rise of generative artificial intelligence (GenAI) is reshaping education, presenting both new possibilities and complex challenges. Recent studies highlight its growing influence on teaching (including assessment) and learning (e.g., Celik et al., 2022). Central to this shift, tools such as ChatGPT are gaining attention across disciplines, including mathematics education (e.g., Pepin et al., 2025), offering personalized, real-time support and assisting teachers in diverse educational tasks such as writing lesson plans and posing word problems (Trust et al., 2023). Additionally, GenAI tools may expand access to information for students and teachers by often producing coherent, human-like responses in a way that may be more efficient than traditional search engines (Farrokhnia et al., 2024). Policymakers increasingly call for research to guide their responsible use and to promote AI fluency in education. As these tools grow more powerful and accessible, understanding their capabilities and limitations is critical. Reported challenges include occasional misinterpretation of concepts, off-topic outputs, and risks of overreliance, which may undermine creativity and critical thinking (Farrokhnia et al., 2024; Oh et al., 2024; Trust et al., 2023).

In mathematics education, GenAI tools have been explored for various purposes, including lesson planning, task design, problem solving, and proof generation (for a review, see Pepin et al., 2025). These tools aid in solving a wide range of mathematical tasks, not only generating solutions but also providing detailed justifications, including step-by-step procedures or explanations that may support students' mathematical learning (Walkington, 2025). However, reliability issues persist, particularly at

advanced levels (Yoon et al., 2024), and outputs often lean toward teacher-centred approaches with limited responsiveness to student needs (Gurl et al., 2024). Despite evidence that university students increasingly use GenAI tools (e.g., Kock et al., 2025), few studies have examined the mathematical accuracy or quality of their explanations (e.g., Yoon et al., 2024). In this paper, we therefore investigate the extent to which solutions produced by a GenAI tool may support or hinder students' learning of *improper integrals*—a key topic in calculus and analysis that has been less investigated in mathematics education research compared to definite integrals. Accordingly, we pose the following research question (RQ): *"To what extent does ChatGPT foster mere computations or reasoning when evaluating improper integrals?"*

THE COMMOGNITIVE FRAMEWORK

The commognitive framework is a socio-cultural, discursive theory developed by Sfard (2008). Within this theory, mathematics is perceived as a historically formed discourse delineated by four interrelated components: unique *keywords* (e.g., improper integral), *visual mediators* (e.g., $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$), endorsed narratives (e.g., the fundamental theorem of calculus (FTC)), and routines (Sfard, 2008). A routine is a "set of metarules defining a discursive pattern that repeats itself in certain types of situations" (Sfard, 2008, p. 301). Metarules describe regularities in participants' discursive actions and are categorized into two components: the *how*, which outlines the procedural steps of the routine, and the *when*, which defines the contextual conditions under which the routine is considered appropriate by leaders of a discourse (Lavie et al., 2019).

Rituals and *explorations* (Lavie et al., 2019; Sfard, 2008) form two ends of a spectrum of routines. Rituals are process-oriented where the primary aim is to carry out a specific task, often motivated by social expectations. In such routines, the focus is on procedural execution where the discursant is typically concerned with questions like *"How do I proceed?"* or *"How can I carry out a particular procedure?"* (Nachlieli & Tabach, 2019, p. 255). Explorations are product-oriented aimed at "constructing and endorsing new narratives about mathematical objects" (Lavie et al., 2019, p. 166). For instance, this may include the use of endorsed definitions of a mathematical object such as an improper integral to create or endorse new narratives about it. For instance, creating a fully-fledged narrative rather than merely computing a number, choosing among options, or endorsing a narrative, points towards explorative routines. Commognitive researchers argue that mathematical routines can only become truly beneficial when they "evolve into full-fledged explorations" (Sfard, 2008, p. 167). The transition from ritual to exploration ("deritualization"; Lavie et al., 2019) has been shown to be often gradual and slow, with many students never fully making this shift during their schooling (Sfard, 2017), but this transition is essential at university level.

Three key aspects may be used to examine the spectrum between ritual and exploration (Sfard, 2017): (1) the nature of the objects discussed—explorative discourse focuses on abstract mathematical objects, whereas ritualized discourse centres on unobjectified signifiers; (2) the source and justification of mathematical narratives—explorative

discourse builds on logically derived and previously endorsed narratives, while ritualized discourse relies on memory or authority; and (3) the implied goals of the activity and roles of participants—explorative discourse frames mathematics as an inquiry-based practice where learners act as independent thinkers and problem-solvers, whereas students participating ritualistically appear "with no agency in the discourse" (Sfard, 2017, p. 55) and who merely recall what they expect to be institutionally warranted. Hence, the RQ is conceptualised as *"To what extent does ChatGPT foster ritualistic or explorative routines when evaluating improper integrals?"*

IMPROPER INTEGRALS

Improper integrals play a vital role in calculus and analysis and across multiple STEM disciplines, including physics and economics, where the validity of several theorems and various models depend on the evaluation and analytical properties of improper integrals. We follow Stewart et al. (2021) distinguishing two main cases (provided that the occurring limits exist and with the usual arithmetic conventions for $\pm\infty$):

- *Type 1* (unbounded intervals): The improper integral of $f\colon [a,\infty) \to \mathbb{R}$ is $\int_a^\infty f(x)\,\mathrm{d}x \coloneqq \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$ (similarly for $f\colon (-\infty, a] \to \mathbb{R}$) and the improper integral of $f\colon (-\infty,\infty)$ is $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$ for any $a \in (-\infty,\infty)$.
- *Type 2* (half open intervals): The improper integrals of $f\colon [a,b) \to \mathbb{R}$ is $\int_a^b f(x)\,\mathrm{d}x \coloneqq \lim_{s\to b} \int_a^s f(x)\,\mathrm{d}x$ (similarly for $f\colon (a,b] \to \mathbb{R}$) and the improper integral of $f\colon [a,b] \to \mathbb{R}$ with exactly one discontinuity/singularity at $c \in [a,b]$ is $\int_a^b f(x)\,\mathrm{d}x \coloneqq \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

The notion of improper integral may be considered an extension of the (proper/definite) Riemann integral in that the integrals in the limits must exist. Depending on the authors, discourses on improper integrals may focus on specific cases of functions. For example, Roussos (2004, p. 8, emph. and line breaks omitted) refers to the three cases "1. The integrated function is unbounded over the interval of integration. 2. The interval of integration is not closed. 3. The interval of integration is unbounded" for "a piecewise continuous real function" and Stewart et al. (2021, p. 542) refer to "an infinite discontinuity in $[a,b]$" for Type 2.

Empirical studies investigating students' discourse on improper integrals have reported that many students participate ritually, focusing on procedural routines and experiencing difficulties in coordinating algebraic and graphical realizations of the improper integral (González-Martín & Camacho, 2004; Radmehr & Drake, 2021). Furthermore, many students avoid engaging with its graphical realizations and rely solely on algebraic realizations. For example, Radmehr and Drake (2021) reported on an evaluation task in which upper secondary and first-year university students checked whether the area enclosed between the curve $y = x^{-2}$ and the x-axis from $x = -1$ to

$x = 1$ had been computed correctly in terms of $\int_{-1}^{1} x^{-2}\,\mathrm{d}x = \left[\frac{x^{-1}}{-1}\right]_{-1}^{1} = -2$. Several students simply checked the calculations supposedly relying on the FTC and accepted -2 as the area. Only three of the 17 interviewed students referred to the narrative that the integrand must be continuous to apply the FTC. Past studies have also shown that students face challenges with improper integral, often linked to their discourse on other mathematical objects, including limits, convergence, sequences, and definite integrals. For instance, students in González-Martín and Camacho's (2004) study appeared to "conceive of the definite integral ALWAYS as an area" (p. 85). Focusing more on the graphical realizations of improper integrals and reinforcing students' mathematical discourse on related objects could help them overcome the challenges reported in previous studies (Camacho & González-Martín, 2006). Furthermore, careful selections of tasks in which the functions have similar-looking graphs but enclose different areas (e.g., $\int_0^{\infty} e^{-x}\,\mathrm{d}x = 1$ and $\int_1^{\infty} x^{-\frac{1}{3}}\,\mathrm{d}x = \infty$) may help students realize that similar-looking graphs can correspond to very different areas. This also emphasizes that, while the graphical realizations can aid in predicting divergence, they have limitations for predicting convergence, thereby underscoring the importance of using more formal routines—such as using the comparison theorem—when evaluating the convergence of improper integrals (González-Martín & Correia de Sá, 2008).

METHODOLOGY

The present study is conceptualized as an intrinsic case study, involving one of the most widely used GenAI tools—ChatGPT-4o—as the chosen case. The tools were asked to evaluate eight improper integrals (T1.1 $\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x$, T1.2 $\int_{-\infty}^{-1} \frac{x^2+x}{x^3}\mathrm{d}x$, T1.3 $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x$, and T1.4 $\int_{-\infty}^{\infty} \frac{x^2}{x^2+1}\mathrm{d}x$ for Type 1, and T2.1 $\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$, T2.2 $\int_0^{\frac{\pi}{2}} \sec(x)\,\mathrm{d}x$, T2.3 $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\mathrm{d}x$, and T2.4 $\int_0^{\pi} \tan(x)\,\mathrm{d}x$ for Type 2; see Table 1). Six integrals were selected from worked examples and exercises from Stewart et al. (2021) and the remaining two (T1.4. and T2.4.) were added to ensure a wider range of task types and such that two integrals of each type converge and diverge. For Type 1, the tasks represent different scenarios involving infinite bounds: at the upper bound (T1.1), at the lower bound (T1.2), and at both (T1.3 and T1.4). For Type 2, the tasks include singularities at the lower bound (T2.1), the upper bound (T2.2), and at a point within the interval requiring the integral to be split (T2.3 and T2.4). All functions in T1.1 to T2.4 have antiderivatives, since these suit the introduction of improper integrals in a calculus course (e.g., Stewart et al., 2021).

Data were collected on 17–18 July 2025. Each integral was pasted as an image into the chat interface of ChatGPT without any additional prompt. To be able to analyse its discursive actions and to enhance the credibility of the findings, each integral was submitted three times to capture potential variations in the outputs. Before each trial, we used the 'new chat' function to ensure a fresh response independent of previous

interactions. Although additional prompting might have led to a greater variety in responses, we restricted ourselves to a minimalist prompt for three reasons. First, students with limited prompting skills may use only basic queries, such as submitting an integral for evaluation (e.g., when completing homework assignments). Second, we exclude effects from more specific prompting. Third, ChatGPT did not only answer in terms of numerical values, but often already added reasoning or occasionally organized its answers into steps, already accounting for the presence of a rich discourse.

Table 1. Values of the selected improper integrals and their associated primitives

	T1.1	**T1.2**	**T1.3**	**T1.4**	**T2.1**	**T2.2**	**T2.3**	**T2.4**
Result	1	does not converge	π	does not converge	$2\sqrt{3}$	does not converge	$\frac{9}{2}$	does not converge
Primitive	$-\frac{1}{x}$	$\log(x)-\frac{1}{x}$	$\tan^{-1}(x)$	$x-\tan^{-1}(x)$	$2\sqrt{x-2}$	$\log(\tan(x)+\sec(x))$	$\frac{3}{2}\sqrt[3]{(x-1)^2}$	$-\log(\cos(x))$

RESULTS

Across the 24 evaluations, ChatGPT explicitly identified the integrals 20 times as improper. In the remaining cases, one trial for T1.3, described it as a "definite integral," but the evaluation relied on a routine typically associated with improper integrals, as discussed below. In all three trials for T2.1, ChatGPT referred to the improper integral simply as an "integral" or "definite integral" and then used a substitution routine. For T2.2, ChatGPT invited the user to consider a graphical substantiation: "If you want, I can also explain this [why the integral is improper] graphically […]". However, in all other cases, ChatGPT did not encourage the user to further engage with the discourse on improper integrals; in all evaluations, it concluded its response after providing the evaluation, without offering additional explanations. The detailed findings are presented separately for Type 1 and Type 2 improper integrals.

Explorations and rituals for Type 1 improper integrals

Three main routines were identified in ChatGPT's evaluations across the trials, with *R1* as central, while *R2* and *R3* are dependent on the nature of the given integrals:

- *R1: Recall the definition of improper integrals of Type 1, evaluate the integral in the limit, and compute the limit.*
- *R2: Simplify the integrand before evaluating the integral.*
- *R3: Use symmetry to transform the integration over the entire real line* $(-\infty,\infty)$ *into twice the integral over the half-line* $[0,\infty)$.

In the following, we discuss how these routines are employed across the four Type 1 improper integrals. ChatGPT used *R1* in several variants when evaluating the four given Type 1 integrals. In all twelve trials conducted for Type 1, it showed both, rather routine driven computation as well as explorations on the convergence/divergence of the integrals and further substantiations.

In essence, ChatGPT's answers to all Type 1 integrals contain, even if not always made fully explicit, the narratives "$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$" and "$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}[F(x)]_a^b$" for an antiderivative F of f. We illustrate two variants of *R1* with the integral $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$: In one trial, ChatGPT used the definition of Type 1 integrals, "$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\mathrm{d}x$", followed by recalling "the antiderivative of $\frac{1}{x^2}$", "$\int \frac{1}{x^2}\mathrm{d}x = -\frac{1}{x} + C$", and then computing the limit "$\lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{1}\right) = 0 + 1 = 1$". In another trial, similar steps were mentioned but carried out differently: First, an antiderivative was given, "$\int \frac{1}{x^2}\mathrm{d}x = \int x^{-2}\mathrm{d}x = x^{-1} = -\frac{1}{x}$", and then, the narrative "$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1}\right) = 0 + 1 = 1$" was used to conclude that "$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = 1$". Notably the definition of Type 1 improper integrals was not recalled explicitly and was only used implicitly, which suggests a ritualistic computation. For example, one variant of the narrative in *R1* is $\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty}[F(x)]_a^b$ for $F' = f$ which relies on the existence of an antiderivative; this happened in the third trial for T1.1 as well. However, ChatGPT introduced the evaluation here with the narrative "This is an **improper integral** because the upper limit is infinity" and concludes its answer with the narrative "This integral **converges** and its value is **1**" (all emph. orig.).

For T1.2, $\int_{-\infty}^{-1} \frac{x^2+x}{x^3}\mathrm{d}x$, ChatGPT, across the three trials employed R2 (i.e., ($\frac{x^2+x}{x^3} = \frac{1}{x} + \frac{1}{x^2}$) first before utilising *R1*. Here, we also find a substantiation about the divergence of the integral that is not merely computation driven. ChatGPT recognised that $\int_{-\infty}^{-1}\frac{1}{x}\mathrm{d}x$ diverges: "Even though $\int_{-\infty}^{-1}\frac{1}{x^2}\mathrm{d}x$ does converge (its antiderivative is $-1/x$, which tends to 0 as $x \to -\infty$), the **whole integral diverges** because one term diverges". This is an important substantiation, recognizing that a single divergent integral suffices to make the sum of two integrals divergent, hence a fully explorative answer.

For the integral T1.3 $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x$, we observed inconsistencies with scholarly mathematical discourse. In each trial, *R1* is initiated, however, only partially endorsable: In one trial, we find the narrative "$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x = \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{1+x^2}\mathrm{d}x = \lim_{a\to\infty}[\tan^{-1}(x)]_{-a}^{a} = \cdots = \pi$" (1), and in another trial "$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x = \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b\frac{1}{1+x^2}\mathrm{d}x = \lim_{a\to-\infty}\lim_{b\to\infty}[\tan^{-1}(x)]_a^b = \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \cdots = \pi$" (2). A blend of (1) and (2) appeared in the third trial. In (1), the narrative and result are correct but there is no justification. Further, "$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x = \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{1+x^2}\mathrm{d}x$" is not the narrative from the definition of Type 1 improper integrals, but rather for Cauchy principle values, it however happens to be valid in this case. This hints towards an overgeneralisation of the narrative in the definition of one-sided improper integrals.

Moreover, no keywords and narratives deal with convergence or divergence, hinting at a mere ritualistic computation of the integral. In (2), the evaluation is also mathematically correct, even though it does not align with standard textbook definitions as reported earlier. Additionally, $\tan^{-1}(\pm\infty)$ are not actual function values.

Regarding T1.4, ChatGPT utilised *R2* and *R3* before evaluating the integral, although not both routines were used in every trial. For instance, in one trial, it recognised the integrand as an "even function […] because replacing x and $-x$ leaves the expression unchanged", and deduced "$\int_{-\infty}^{\infty} \frac{x^2}{x^2+1}\,\mathrm{d}x = 2\int_{0}^{\infty} \frac{x^2}{x^2+1}\,\mathrm{d}x$", then evaluated "$\int_{0}^{\infty} \frac{x^2}{x^2+1}\,\mathrm{d}x = [\ldots] = \int_{0}^{\infty} 1\,\mathrm{d}x - \int_{0}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x$" after applying R2. Across the three trials, no substantiation was provided as to why $\int_{0}^{\infty} 1\,\mathrm{d}x$ and $\int_{0}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x$ diverge, nor was R1 used. Yet, it was concluded that $\int_{-\infty}^{\infty} \frac{x^2}{x^2+1}\,\mathrm{d}x$ diverges. The same was observed in the trial when R3 was not employed.

Explorations and rituals for Type 2 improper integrals

For Type 2 improper integrals, four main routines were identified:

- *R4: "Recall the definition of improper integrals of Type 2, such as* $\int_a^b f(x)\,dx = \lim_{s\to b}\int_a^s f(x)\,dx$*, evaluate the integral in the limit, and compute the limit*".
- *R5: "Split* $\int_a^b f(x)\,dx \coloneqq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *and evaluate the two improper integrals in that sum when* c *is a singularity of* f".
- *R6: Use the substitution routine for definite integrals* (which may have appeared for the Type 1 integrals as well for differently chosen examples).
- *R7: The use of the not scholarly endorsable object-level rule: "If an integrand in an improper integral is unbounded at a point over the interval of integration, then the improper integral diverges".*

For, T2.1 $\int_{2}^{5} \frac{1}{\sqrt{x-2}}\,\mathrm{d}x$, we found no exploratory routines, no reference to *R4*, nor keywords related to improper integral discourse such as "improper" or "convergence/divergence". Instead, ChatGPT presented an application of *R6* across the three trials. For example, in one trial the substitution "$u = x - 2 \Rightarrow du = dx$" was merely followed by the grammatically incomplete narratives such as "When $x = 2, u = 0$", leading to "$\int_0^3 \frac{1}{\sqrt{u}}du = \int_0^3 u^{-1/2}\,du = \left[\frac{u^{1/2}}{1/2}\right]_0^3 = \left[2\sqrt{u}\right]_0^3 = 2\sqrt{3} - 2\sqrt{0} = 2\sqrt{3}$". For the other integral involving a root, T2.3 $\int_{0}^{9} \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x$, *R5* was utilised and ChatGPT interpreted the cube root as a real-valued function and evaluated it correctly. The tool recognized that the values of the root change from negative to positive at $x = 1$, that the integral is improper, split it into $\int_{0}^{1} \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x + \int_{1}^{9} \frac{1}{\sqrt[3]{x-1}}\,\mathrm{d}x$ and computed it correctly with the substitution $u = x - 1$ and *R4*. For the integrals involving

trigonometric functions, *R7* alone was employed for T2.2 in one trial and for two trials for T2.4. For instance, in one trial for T2.2, ChatGPT wrote that the "integral **does not converge** because the function $\sec(x) = \frac{1}{\cos(x)}$ becomes unbounded as $x \to \frac{\pi}{2}^{-}$", which was further substantiated with "$\cos(x) \to 0$ as $x \to \frac{\pi}{2}^{-}$", concluding that $\int_0^{\frac{\pi}{2}} \sec(x)\,\mathrm{d}x$ is "**divergent**". In the other trials, however, the integral is recognized as "**improper** because the function $\sec(x) = \frac{1}{\cos(x)}$ has a **vertical asymptote at** $\boldsymbol{x = \frac{\pi}{2}}$ (since $\cos\left(\frac{\pi}{2}\right) = 0$)" or "because the function $\sec(x) = \frac{1}{\cos(x)}$ becomes unbounded as $x \to \frac{\pi}{2}^{-}$". Following these endorsable narratives, ChatGPT utilised *R4*. For the remaining trial of T2.4, after utilizing *R7*, substantiation is briefly provided using *R5* without any further substantiation why each $\int_0^{\frac{\pi}{2}^{-}} \tan(x)\,\mathrm{d}x$ and $\int_{\frac{\pi}{2}^{+}}^{\pi} \tan(x)\,\mathrm{d}x$ diverges: "The improper integral $\int_0^{\pi} \tan(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}^{-}} \tan(x)\,\mathrm{d}x + \int_{\frac{\pi}{2}^{+}}^{\pi} \tan(x)\,\mathrm{d}x$ diverge because both parts tend toward ∞ and $-\infty$, respectively".

DISCUSSION AND CONCLUSIONS

Our analysis of ChatGPT's evaluations of improper integrals points to both opportunities and limitations in its role as a learning tool. The obtained results in terms of numerical values or divergence were correct. We observed that in some trials, it invited users only to ritualistic participation in the discourse on improper integrals and, in a few cases, even relied on not scholarly endorsable object-level rules (i.e., R7) or did not recognize an integral as improper (T2.1). It is therefore important that students develop AI literacy in the sense of approaching its routines and narratives critically, rather than accepting them passively. The findings also indicate that if students rely heavily on GenAI tools to learn mathematics—even if these tools eventually produce correct results consistently—students may be primarily invited into ritualistic routines, thereby undermining their engagement in exploratory mathematical discourse. As Sfard (2017) noted, "what learners are offered is what you get from them in return" (p. 41). Hence, if such tools are to play a meaningful role in supporting mathematics teaching and learning, they must be trained to foster exploration, rather than merely ritualistic participation in mathematical discourse. In addition, given that the final answers were fully correct, it is increasingly difficult for students to tell at which points ChatGPT could have made a mistake. While the use of AI as a learning resource may not be actively encouraged by mathematics lecturers, students may still engage with such tools independently. Therefore, the findings highlight the critical role of the lecturer in guiding students to critically evaluate AI-generated discourse, particularly in identifying non-endorsable narratives and routines, as well as responses that may appear mathematically valid but lack appropriate justification. This may include guiding students to question the validity of AI-generated narratives, to justify the

routines used, and to examine whether the discourse aligns with canonical mathematical discourse.

A further limitation concerns ChatGPT's lack of using different realizations in its responses. Although connecting different realizations of mathematical objects is crucial in learning mathematics (Sfard, 2008), ChatGPT seldom prompted users to consider graphical realizations—doing so only once across all 24 evaluations—even when addressing relevant keywords such as "vertical asymptote". Previous studies (e.g., Camacho & González-Martín, 2006), however, have shown that the engagement with graphical realizations is essential for learning convergence and divergence of improper integrals. Finally, our minimalist prompting—entering integrals as images without additional instructions—likely reflects how students would use such tools when working on assignments. While this design enhanced ecological validity, it also revealed how easily interactions may remain ritualistic.

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Touching the Imaginary: A Multi-Touch Interface for Complex Numbers

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Keywords: Complex numbers, Embodied learning, Geometry-Algebra integration

INTRODUCTION

An understanding of complex number arithmetic, including the ability to transition between algebraic and geometric representations, is a prerequisite for undergraduate studies in complex analysis. Research has highlighted students' difficulties with geometric interpretations of complex numbers (Nemirovsky et al., 2012) and points to a lack of research-informed educational resources. In the Indian educational context, complex numbers are introduced in the secondary school-level (Grade 11) and form a foundational component of undergraduate studies in engineering and science. Teachers at this level are therefore expected to have profound subject matter and pedagogical content knowledge to bridge the gap between procedural fluency and conceptual understanding (Karakok et al., 2015). In this paper, we discuss the genesis of a touch-based interface designed for complex number visualization and arithmetic, explore teachers' experiences with it, and its affordances for instruction. Our broad research questions are:

1) How does a gesture-based embodied interface influence students' conceptual understanding of complex arithmetic and associated geometric aspects?
2) How do teachers interact with and perceive the utility of a gesture-based, embodied interface for the instruction of complex numbers?

THEORETICAL FRAMEWORK OF EMBODIED LEARNING

Embodied learning theory (Stolz, 2015) posits that cognitive processes are deeply rooted in the body's interactions with the world. Supporting this, recent research indicates that physical engagement with mathematical content can reduce cognitive load and ground abstract reasoning in perceptual and motor experiences (Rajan et al., 2025). Consequently, pedagogical frameworks like embodied design leverage interactive technologies to translate this theory into practice, encouraging students to use physical actions as a means of embodying mathematical concepts (Trninic & Abrahamson, 2012). Utilizing these insights, we developed the Touchy Feely Complex Numbers (TFCN) system.

GENESIS OF TFCN SYSTEM

Prior to development, a diagnostic study with 94 Indian undergraduates pinpointed specific challenges with complex numbers (Rajan & Chari, 2026). Key difficulties

included integrating algebraic procedures with geometric explanations for addition, conceptually representing conjugates, and comprehending multiplication as a geometric transformation. These findings directly informed the TFCN's design.

The TFCN is a customised touch-based interface - developed to create, manipulate, and perform various complex number arithmetic operations such as addition, subtraction, multiplication, division, and conjugation. Unlike traditional interfaces like GeoGebra, Desmos, or Wolfram Alpha, which produce automated results, in TFCN, each operation requires the user to perform a series of embodied gestures. This pedagogical framework utilizes complex numbers constrained to the unit circle. It features two distinct operational modes: a decimal system and an integer system. The decimal mode permits the full suite of complex arithmetic on all numbers, while the integer mode is restricted to complex numbers with integer real and imaginary components. The primary rationale for the integer system is to simplify user interaction and to provide a more tangible, embodied representation of complex multiplication. A deliberate constraint of this system is the exclusion of the division operation.

The TFCN platform was tested with a group of five teachers from diverse undergraduate education systems in India. With some training of the TFCN system, we gathered their experience about the system and its potential use during complex numbers instruction. The insights gathered from interviews informed enhancements to the system's design, instructional structure, navigational layout, and data-collection methods. The evolved design is used for the comprehensive study with undergraduate students.

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The effect of using CalcPlot3D to teach partial derivatives in different educational settings

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We study the results of using the digital application CalcPlot3D on students' understanding of partial derivatives. To do so, we compare the constructions of the students of three instructors teaching multivariable calculus at three different types of institutions. Instructor P taught at a two-year community college, S at a highly competitive undergraduate college, and T at a comprehensive state university. All three used different instructional approaches. We used APOS to classify students' mental structures for partial derivatives based on their written work on an instrument. The results indicate that using a digital resource for teaching partial derivatives is insufficient to ensure students' understanding. Institutional and pedagogical factors significantly contribute to determining students' learning outcomes.

Keywords: multivariable calculus, digital resources, partial derivatives, APOS.

INTRODUCTION AND LITERATURE REVIEW

Phenomena in nature often involve quantities that depend on multiple variables and measures. Therefore, studying multivariable functions and analyzing their changes with the tools of multivariable calculus (MVC) is vital for modelling these phenomena. A core concept in understanding change is the idea of partial derivatives, which describe how a quantity varies when only one component changes while the others remain constant. As a result, finding the most effective ways to support the teaching and learning of partial derivatives is crucial.

In the simplest case, two-variable functions are usually represented as surfaces in three-dimensional (3D) space. This change in dimensions creates a new challenge for students, who must now generalize from the 2D context of their earlier studies to the 3D context required by two-variable functions (Trigueros & Martínez-Planell, 2010). There is a growing body of research dealing with the teaching and learning of different topics of differential MVC, including partial derivatives (e.g., Borji et al., 2024). There is also work discussing students' construction of the schema of differential MVC (Trigueros et al., 2025). Further, the interaction between instructors, students, and resources has become a subject of investigation (Moore-Russo et al., 2024; Martínez-Planell et al., 2025). These studies include instructors' adoption and use of resources to support visualization and student learning of multivariable calculus and the relationships between instructional resources, instructors' pedagogy, and MVC content. However, we are not aware of research that explores the role of context (e.g.,

type of institution, class size, pedagogical practices) in students' learning of differential MVC that is facilitated by the use of digital resources.

The present study investigates how instructors' pedagogical practices interact with two types of resources (digital and 3D-printed surfaces) as well as the classroom context, and how all combine to impact students' understanding of partial derivatives. More specifically, there were two research questions that drove the study.

1. What types of conceptions of partial derivatives were students able to construct?
2. What relations are observed between students' performance on partial derivatives and the type of institution and instructors' pedagogical practices when using digital or physical resources?

THEORETICAL FRAMEWORK

We use Action-Process-Object-Schema theory (Arnon et al., 2014) since it has been used in previous studies of partial derivatives (Borji, et al., 2024) as a helpful tool for analyzing students' mental constructions. In APOS, an Action is a transformation of a previously constructed mathematical object that an individual perceives as external. It could be applying a memorized or explicitly available procedure step by step. When an Action is repeated and reflected upon, it might be interiorized into a Process. A Process is perceived as internal and allows an individual to omit and anticipate steps, typically generating dynamic imagery. The internal perception of a Process means that significant connections to other mathematical knowledge have been constructed, allowing an individual to discuss the transformation in general terms and perceive it as independent of representation. Different Processes can be coordinated to form new Processes. When a Process is perceived as an entity in itself, and an individual can perform Actions on this entity, then the Process has been encapsulated into an Object. When determining the type of structure a student has developed, it is important to consider the overall student tendency in different problem situations. This study did not explore the relation between partial derivatives and directional derivatives, the total differential, or tangent planes. Focusing on a single aspect rather than multiple interrelated components allowed us to model the construction of partial derivatives using only Actions, Processes, and Objects, without needing Schemas, which are typically used to model more complex interrelated ideas.

A genetic decomposition (GD) is a model of how a student might construct a specific mathematical notion. It is expressed in terms of the structures (i.e., Actions, Processes, Objects) and the mechanisms (e.g., interiorization, encapsulation, coordination) of APOS theory. A GD does not claim to be the best or only way for a student to learn a specific notion. It is merely an initial hypothesis that requires experimental support. A GD can be used to design data-gathering instruments for experimentation and to analyze data. We adapt the GD for partial derivatives, in Figure 1, as suggested by Borji et al. (2024) to analyze students' construction of the GD components (i.e., GD1, GD2, GD3, GD4) in order to determine their APOS conception of partial derivative.

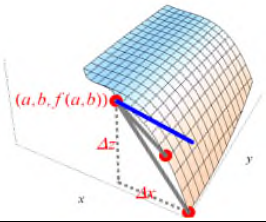
	Graphical path	Symbolic path
		
GD1: Action on slope (graphical or symbolic)	Slope of secant line in x direction $\frac{\Delta z}{\Delta x}$	Average rate of change as difference quotient $\frac{f(a+\Delta x, b)-f(a,b)}{\Delta x}$
GD2: Process of varying slopes to determine tangent	Secant lines progressively getting closer to the tangent line; the slope of secant lines approaching the slope of tangent line	Horizontal change Δx gets "closer and closer" to zero
GD3: Coordination of graphical & symbolic Processes for the partial at a point	Visualization of tangent line and its slope $f_x(a,b)$= slope of tangent line $= \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x}$	Instantaneous rate of change $\lim_{\Delta x \to 0} \frac{f(a+\Delta x, b)-f(a,b)}{\Delta x}$
	Inter-representational coordination of processes to consider the partial at a point	
GD4: Partial at a point leads to partial as a function	Actions of point evaluations on a Process of partial at a point allow encapsulation into an Object of partial at a point. This is de-encapsulated and coordinated with a Process of function of two variables to construct a Process of partial derivative as a function. Actions on a Process of partial derivative as a function enable its encapsulation into an Object.	

Figure 1: Summary of genetic decomposition of partial derivative (with respect to x)

METHODOLOGY

Instructors and students

We investigate how students of three instructors (P, S, and T) understand partial derivatives. All three instructors worked at different types of institutions and used CalcPlot3D and/or 3D-printed surfaces to teach partial derivatives. Table 1 summarizes the different contexts for instructors P, S, and T.

Instructor	Type of Institution	Participants, Pedagogical Practices	Use of Resources
P	Community College	14 students; primarily lecture with a few collaborative activities	Extensive use of CalcPlot3D for instructor demonstrations; some use of 3D-printed surfaces for collaborative activities
S	Selective Undergrad College	30 students; lecture punctuated by many collaborative activities	Extensive use of CalcPlot3D for instructor demonstrations; use of CalcPlot3D allowed on all assessments; concentrated effort to connect embodied experiences, 3D-printed surfaces, CalcPlot3D, and paper-and-pencil exercises
T	Research University	98 students in a large hall; lecture without collaborative activities	Extensive use of CalcPlot3D for instructor demonstrations; encouraged use of CalcPlot3D (through instructor-created videos) on student homework

Table 1. Summary of the different contexts for instructors P, S, and T.

Besides instructor demonstrations, P also used CalcPlot3D for some individual activities, including the use of individually manipulable figures in a digital textbook, and the use of digital figures in WeBWorK for scaffolded exercises. P used a collaborative activity for partial derivatives, the second of three such activities. The three collaborative activities, all in differential calculus, involved 3D-printed surfaces. After covering partial derivatives as a skill and using CalcPlot3D to discuss the geometric meaning of both the first and second partial derivatives in each direction, P's students formed groups to work with a 3D-printed mountainous surface. Students verbally explained what they expect to see on a surface with various combinations of positive, negative, or zero partial derivatives and second partial derivatives. They located a point on the surface that corresponded to each and labelled these points on the surface.

Instructor S used CalcPlot3D mostly for instructor demonstrations. However, she had a substantial number of collaborative activities (six in differential calculus) that involved a concentrated effort to tie the digital renderings in CalcPlot3D to the 3D-printed surfaces and 2D-drawn figures. S's first partial derivative activity involved a field trip out to a hill with an instructor-led walk around and discussion where the concepts behind partial derivatives were discussed, including ideas of gradients, directional derivatives, and extrema. S later tied this to having a "bug" walk on a 3D-printed surface. Small groups were given a mountainous surface and a whiteboard marker. Following a scaffolded worksheet, students identified points on the surface with various derivative properties and described the features at a given point using everyday terms (such as "uphill" in the positive x direction). Instructor S also used a pre-saved CalcPlot3D URL demonstration in class, to visualize trace lines for the partial derivatives in the x and y directions and to discuss both first-order and second-order partial derivatives on a graph of a function. Instructor S had her students use CalcPlot3D during class, for homework outside of class, and during quizzes and exams.

Instructor T used CalcPlot3D and Desmos for instructor demonstrations. In previous semesters, T used a variety of physical resources in collaborative student activities, but he explained that he was constrained by the classroom environment that semester. T encouraged students to use CalcPlot3D on their homework through a series of short videos explaining how to employ CalcPlot3D for various introductory topics. T's students uploaded screenshots of CalcPlot3D for digitally submitted homework assignments. T used CalcPlot3D to show the traces and partial derivatives as slopes in class, demonstrating how partial derivatives are represented geometrically.

Data collection

A six-question, task-based, paper-and-pencil instrument was designed to help determine which GD structures, described in the Figure 1, students evidenced. Data were collected at the end of the semester. For Instructor P, this was a for-credit, individual, take-home assignment. For Instructor S, students worked collaboratively and submitted the assignment individually, but not for credit. For Instructor T, it was an extra-credit, individual homework assignment. Table 2 provides information about

each task; it relates the possible GD information that could have been noted in students' responses to each task. However, a student with a Process or Object structure might have only provided a response that was consistent with an Action. We now present the instrument as Figure 2.

1. Suppose the graph of $z = f(x, y)$ is as follows. State the sign (positive, negative, zero) of f_y (3.5, 0) and of f_x (3.5, 0). Explain your reasoning.

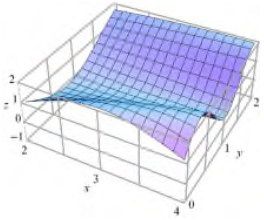


Figure for task 1

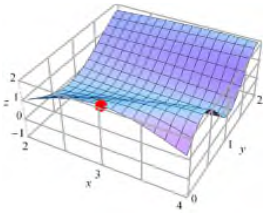


Figure for tasks 3 and 4

2. If $h(x) = f_x(x, 0)$, draw how the graph of h may look for $2 \leq x \leq 4$. Explain your reasoning.

3. State the sign (positive, negative, zero) of $f_{xx}(3, 0)$. Explain your reasoning. Can you also explain this in a second way?

4. State the sign (positive, negative, zero) of $f_{yx}(3,0)$. Explain your reasoning.

5a. The following is a table of values of a differentiable function f of two variables. Approximate the value of $f_y(1,2)$ the best you can. Show all your work.

	$y = 1$	$y = 2$	$y = 3$
$x = 0$	5	7	11
$x = 1$	6	10	18
$x = 2$	8	14	16

5b. The following is a table of the function as in 2a above. Approximate the value of $f_y(1,2)$ the best you can. Be sure to show all your work.

	$y = 1$	$y = 2$	$y = 2.01$
$x = 0$	5	7	7.04
$x = 1$	6	10	10.06
$x = 2$	8	14	14.10

6. The following figure shows the contour diagram of the tangent plane of a function $z = f(x, y)$ at the point $(0,3)$. Find $f_x(0,3)$. Show all your work.

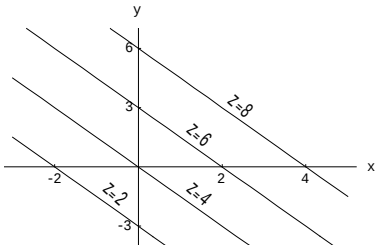


Figure 2. Data collection instrument.

Tasks were designed to allow students the opportunity to show constructions from GD1 to GD4. The relation between each task and GD is detailed in Table 2 on the next page.

Data analysis

Codes GD1, GD2, GD3, and GD4 were used when a student's argument was consistent with the corresponding GD component. Other data-grounded codes were introduced as needed. For example: di (dynamic imagery), J (justification provided), c (argument based on concavity), and so on. The coding was done by consensus in weekly meetings.

The general tendency of the student was used to determine the final accorded APOS structure. For an Object structure of partial derivative as a function, the student would need to have constructed GD1 through GD4.

Task	Relation to GD and its Components
1	Can provide evidence of GD3, as students may explain the sign by referring to the tangent line at the point in the chosen direction (*Action*); can give information about the graphical understanding of partial derivatives, which is significant because perceiving partial derivatives as independent of representation is consistent with a *Process* of partial derivative at a point
2	Can provide evidence of GD3 since the partial derivative at a point might be constructed before performing the *Action* of graphing; can give information about student's use of dynamic imagery for partial derivatives, which is consistent with a *Process* of partial derivative at a point
3	Can provide evidence of GD3 by seeing if the student can generate dynamic imagery when describing the partial derivative of f_x at the given point (*Process*); can provide evidence of GD4 if the student describes the behavior of f_x on an interval which is consistent with thinking of partial derivative as a function (*Object*); however, it is also possible to answer this task by invoking a memorized property of concavity (*Action*)
4	Can provide evidence of GD4 if student imagines how f_y changes in an interval of *x* values, consistent with thinking of partial derivative as a function (*Object*)
5a	Can provide evidence about GD1, as the task requires forming a difference quotient or referring to the slope of a secant line (*Action*)
5b	Can provide evidence about GD2, as the task requires thinking of a partial derivative as a limit and thus choosing the closest point (*Process*); as with 5a, use of difference quotient can give evidence of GD1 (*Action*)
6	Can provide evidence of GD1 as it requires forming a difference quotient (*Action*); when considered together with the other tasks, it can give evidence of a student's perception of partial derivative at a point as independent of representation, which is consistent with a *Process* conception (GD3)

Table 2. Relation of task items to GD.

RESULTS

We first present Table 3, comparing the APOS structure of a partial derivative constructed by students. We then give paradigmatic examples of students with Action, Process, and Object structures. Table 3 shows there is a marked difference in the types of structures constructed by the students of the three instructors.

Instructor	Action	Process	Object
P	71% (10/14)	14% (2/14)	14% (2/14)
S	33% (10/30)	37% (11/30)	30% (9/30)
T	94% (92/98)	5% (5/98)	1% (1/98)

Table 3. Students showing Action, Process, or Object structures by instructor.

Action conception

Most students who demonstrated an Action conception showed a few unjustified computations and failed to answer some tasks correctly. In tasks 5a and 5b (using tables) and in task 6 (using a contour diagram), T10 (representative of instructor T's students) performed actions to compute difference quotients to approximate f_x and f_y

at the given points (Figure 3). Further, as shown in Figure 3, T10 showed awareness that these are slopes by writing $\frac{\Delta f}{\Delta y}$ and $\frac{\Delta f}{\Delta x}$. So, T10 seemed to have constructed GD1.

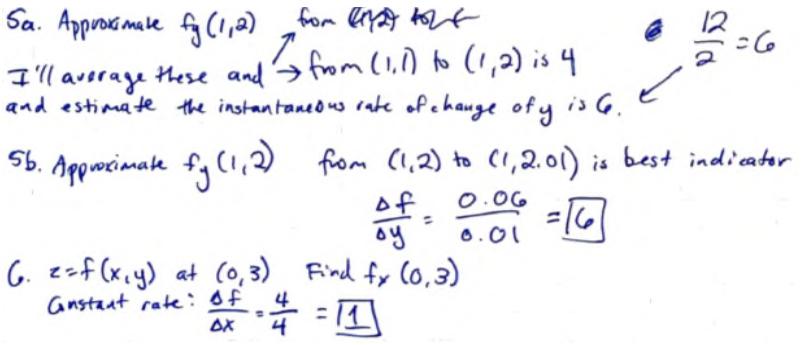


Figure 3. T10's work on tasks 5a, 5b, and 6.

T10 might have even constructed GD2, as there is a brief, vague justification ("is best indicator") for task 5b. T10's answer to task 3 was based on concavity, which could be another memorized property. In task 4, T10 assumed that $f_x(3,0) = f_y(3,0) = 0$ implied that the mixed partial $f_{yx}(3,0) = 0$. There was no evidence in tasks 1 and 2 that T10 constructed a Process of partial derivative at a point or as a function. The general tendency of T10 was to perform numeric or algebraic computations, or to apply memorized facts, all of which are examples of Actions.

Process conception

Most students who demonstrated a Process conception went beyond using memorized procedures but were not consistent in solving tasks often missing at least one component of the GD. For example, P3 (one of instructor P's students constructing more than an Action) did not show the construction of GD2 on task 5b, when given a table of function values and asked to approximate $f_y(1,2)$, P3 acknowledged that it was "asymmetric" but only performed the calculation "$\frac{f(1,2.01)-f(1,1)}{2.01-1}=\frac{10.06-6}{1.01}=\frac{4.06}{1.01}\approx 4.0198$." P3 did not allow for the fact that the point to the right $(1,2.01)$ was much closer to $(1,2)$ than the point to the left $(1,1)$. This can be interpreted to mean that P3 was not showing awareness that the partial derivative is a limit of difference quotients. On the other hand, P3's responses to other tasks were consistent with a Process conception. For example, in task 4, when asked about the sign of f_{yx} at $(3,0)$, P3 wrote, "positive, as you move in the positive *x* direction, the slope in the positive *y* direction gets steeper uphill," and accompanied this with the drawing in Figure 4. P3 demonstrated the ability to generate dynamic imagery when writing "… as you move … the slope … gets…", consistent with a Process conception. P3 (in Figure 4) drew three arrows in the *y* direction that could have been used to imagine the change in slopes as one moves in the *x* direction; this gives evidence of P3 thinking of f_y as a function defined on an interval (GD4). We omit discussing other responses of P3, since they are similar to those of student T10 above (Action) or student S28 (Object) below.

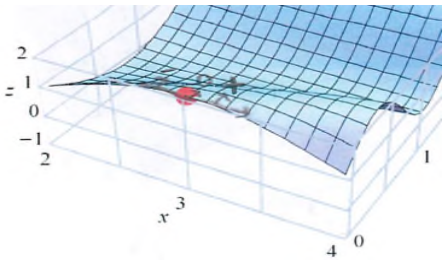


Figure 4. P3's work on task 4.

Object conception

Students who demonstrated an Object structure would have consistently shown evidence of having constructed GD1 through GD4. Most students of instructor S constructed either a Process or, as her Student S28 did, an Object of partial derivatives. Task 5a included a table of function values and asked students to approximate the value of $f_y(1,2)$. S28 averaged the difference quotients to the right and left (Figure 5), giving evidence of thinking of these as slopes by using the symbols Δz/Δy, and also giving evidence consistent with being aware that the computation will result in the slope of a particular secant line (GD1), see Figure 5. Being able to think of a mathematical notion as independent of representation is consistent with a Process conception. In this task, S28 related symbolic, numeric, and graphical representations.

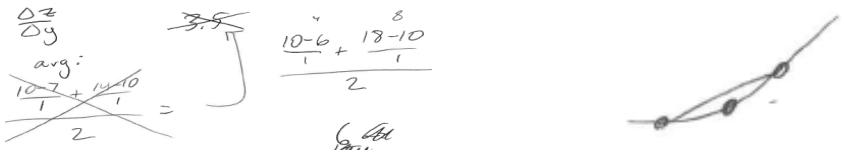


Figure 5. Students' S28 answer to task 5a.

On task 5b, the students were given a table and asked to approximate$f_y(1,2)$. One of the points on the table was very close to the base point (1, 2). S28 consciously chose to employ (Figure 6) a different approach than in task 5, suggesting that S28's answer was more than just the repetition of a memorized procedure (an Action). The student shows some awareness of the partial as a limit by writing "don't use the far point" (Figure 6). This was consistent with Process GD2.

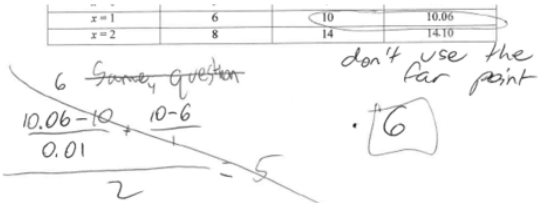


Figure 6. Students' S28 answer to task 5b.

Task 2 asked for the graph of $f_x(x,0)$ for the same function as that in task 1. S28's answer is in Figure 7. The student interpreted partial derivatives as slopes leading to tangent lines at varying points (GD3) and also showed evidence of dynamic imagery ("going from … to … start … and get") across an interval, consistent with a Process

conception of partial derivative as a function (GD4). Then in task 3, S28 wrote that the "slope of f_x is negative $\therefore f_{xx} < 0$" showing an action on f_x which is consistent with an Object conception of partial derivative as a function.

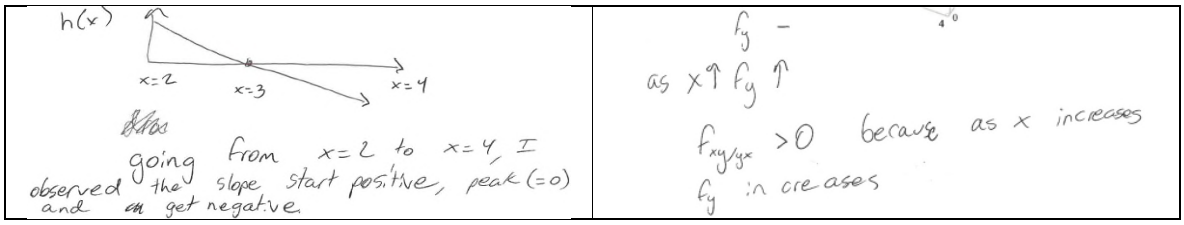


Figure 7. Student S28's answers to task 2 (left) and task 4 (right).

In task 4 (Figure 7, right), S28 was asked for the sign of $f_{yx}(3,0)$. The student recognized that mixed partials were equal, which could be memorized, but also showed dynamic imagery ("as x increases f_y increases"), which is consistent with a Process conception of a partial derivative as a function (GD4). The student seemed to have imagined f_y on an x-interval and so seemed to think of f_y as a function on which the Action of finding its partial derivative with respect to$x$ can be performed.

To summarize, S28 showed the construction of GD1, GD2, GD3, and GD4, and so seemed to have constructed a Process of partial derivative as a function. Further, in tasks 3 and 4, S28 performed actions on f_x and f_y, respectively, suggesting these had been encapsulated into an Object of partial derivative as a function.

DISCUSSION AND CONCLUSIONS

The results in Table 3 show that students' conceptions of partial derivatives can vary greatly and depend on more than the digital resources used. For the most part, students were able to show the construction of GD1. That is, they could compute and relate difference quotients and slopes of tangent or secant lines. This was shown in tasks 5a and 5b, which gave a tabular representation of a function, and task 6, which gave a contour diagram of the tangent plane at a point. Students could answer these by performing standard numerical computations that might have been memorized, showing Action structures. Students showed difficulty constructing GD2, which required treating the derivative as a limit and, thus, tested for a Process structure. Task 5b gave information about this. The construction of GD3 (Process structure of partial derivative at a point, likely to be noted on tasks 1 through 3) and GD4 (Object structure of partial derivative as a function, likely to be noted on tasks 2 through 4) challenged students. The use of digital resources did not guarantee that students were able to perform these constructions. Rather, the instructors' pedagogical practices when using resources, and how student-oriented those practices were, seemed to have played a more important role than the type of institution.

The increased use of 3D-printed models and collaborative activities corresponded with better student results as shown in the case of S, who made a concentrated effort to have students connect all resources, followed by P, who incorporated some collaboration

and use of 3D-printed models. Many more of S's and a few more of P's students showed evidence of having constructed a Process or Object of partial derivative than T's students. We also found that the type of institution and its competitiveness in accepting students did not determine students' results. The institution of T is more competitive than that of P, but the students of P performed better than those of T. These results can be explained by considering that in the institution of T, calculus is taught in large lecture sections. This teaching arrangement constrained the use of collaborative learning opportunities and class discussions. Instructor T had to change his usual teaching practice to focus more on lecturing and less on the use of 3D-printed models and collaborative activities.

Our study has some important limitations. All the data were obtained from students' written work. This did not allow for a deeper inquiry into students' understanding. Further, students frequently did not justify their work, and so we could not make inferences regarding their understanding.

DISCLAIMER

The views expressed in this article do not necessarily reflect those of the U.S. Air Force Academy, Air Force, Department of Defense, or Government.

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Insights into Mathematics by In-Depth Studies on Mathematicians with the Digital Interactive Mathematical Map for Calculus

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Keywords: University mathematics education, Digital technologies, History, Calculus.

RESEARCH TOPIC

Digital Interactive Mathematical Maps (DIMM) were developed in an iterative design-based research process and evaluated in different contexts primarily addressing perceived usefulness and ease of use (e.g., Brandl & Hackstein, 2024; Brandl & Vinerean, 2023; Vinerean et al., 2023). The DIMM are freely accessible at https://math-map.fim.uni-passau.de and consist of 3-d knowledge nets and a 2-d timeline for the areas of Algebra, Calculus, Geometry and Stochastics. Essentially, it offers "information on selected mathematical results and concepts with links to suitable internet content such as texts and YouTube videos, and sometimes suitable problems from mathematical competitions." (Brandl & Hackstein, 2024, p. 701)

RESEARCH QUESTION

A central current research question is: To what extent and in which way does the use of the Digital Interactive Mathematical Map for Calculus help to get insight into mathematics (conceptual understanding and historical development of content knowledge) by in-depth studies on mathematicians?

RESEARCH PROCESS AND RESULTS

Developed by Design-Based-Research (Bikner-Ahsbahs et al., 2015; Brandl et al., 2024) the DIMM' technical features have already been evaluated by the Technology Acceptance Model of Davis (1985) and using Qualitative Content Analysis (Vinerean et al., 2023). Following the TPACK framework specialized to mathematics education (Buchholz et al., 2023), a study in a Calculus Course at the University of Karlstad (Sweden) was conducted together with the professorship for the didactics of mathematics of the University of Passau (Germany). The main concern was to look at the potential of collaborative activities of students in order to get significant insight into mathematics by in-depth studies on mathematicians using the DIMM for Calculus. To seven groups of students the mathematicians Cauchy, Euler, Gauss, Leibniz, Newton, Riemann, Taylor, and Weierstrass were randomly assigned. Subsequently, they were given tasks in order to search for information on the mathematician, to investigate the mathematical contributions, to explore the context and contemporaries (by the Horizontal Cut function of the DIMM), to see interactions and influences of different results (by the Vertical Cut function), and to describe the historical context. The task result was expected as a written report (essay format) containing biographical information, description of mathematical results, references – and the completing of a

corresponding quiz, with questions on the potential of the DIMM whether to support the essay task (open answer format).

Results of the corresponding quiz showed that the DIMM mainly helped the students to get insight into mathematics and retrieve information about the mathematicians they had to investigate; especially the connections between the different, but related, topics contributed to an easier accessibility of the knowledge structure. Critical statements addressed the missing of a Swedish version and the possibly main suitability for university mathematics in higher levels than for younger students or high school level.

The poster presents in a structured way the DIMM, the conducted study with connection to TPACK, and the data analysis with examples from students' answers.

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Integration of generative AI tools in linear algebra learning: Effects on the Personal Mathematical Working Space

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Keywords: linear algebra, generative artificial intelligence, mathematical working spaces, paradigms, university teaching.

Introduction

Teaching linear algebra at university often leads to understanding difficulties among students and can reduce their motivation (Díaz et al., 2024). Recent research shows that ChatGPT has significant limitations in linear algebra. Bagno et al. (2024) warn that this tool can generate responses that appear accurate and well-structured but contain contradictions and inaccuracies that may mislead students with little experience. More generally, the recent emergence of generative artificial intelligence (GAI) tools raises the issue of integrating a new digital artefact into teaching, thus raising didactic questions about learning. These tools can give students the impression that they are interacting with a human interlocutor, which requires an analysis of their advantages and limitations in learning processes (Bagno et al., 2024).

In this regard, our research question is as follows: How does the integration of GAI tools influence the personal mathematical work of future teachers when solving a linear algebra problem?

Theoretical framework and methodology

The Theory of Mathematical Working Spaces (ThMW) (Kuzniak et al., 2022) allows mathematical work to be analysed by linking the epistemological (representations, artefacts, theoretical framework) and cognitive (visualisation, construction, proof) planes, articulated by three geneses (semiotic, instrumental and discursive).

Vandebrouck et al. (2024) distinguish three paradigms structuring work in linear algebra: LA1 related to algorithmic work on linear systems and geometry in dimensions 2 and 3, with direct visual representation; LA2 related to linear applications and matrix algebra; and LA3 relating to abstract vector spaces with proofs and validation rely on the axioms and properties of the algebraic structure, independently of geometric or numerical representations.

Our sample consists of 21 students in initial training to become mathematics teachers at the Universidad de Concepción (Chile). A group activity was carried out at different times on a task from the ECOS-ANID C22H03 project, with semi-structured interviews (2 students). The data consists of screenshots of interactions with the GAI, written work, and video recordings of the interviews.**Activity:** In groups (8), with free access to GAI tools and GeoGebra. Problem 1: find the matrix that transforms $\triangle ABC$ into $\triangle PQR$; Problem 2 (generalization of 1): is there a matrix that transforms any triangle into any other triangle?

Data analysis followed Kuzniak et al. (2022): activated geneses (semiotic, instrumental, discursive) and vertical planes. The LA1, LA2 and LA3 paradigms were identified based on the mathematical objects mobilized, the techniques used and the level of formalization. A priori analyses anticipated a possible LA2 mobilization for this activity when generalizing.

Preliminary results

The analysis reveals three contrasting types of use of GAI, each with different impacts on the personal MWS. Type A (dependent-delegative), where students transcribe GAI responses without validation; Type B (critical-strategic), where students actively reformulate and validate GAI responses; and Type C (delegative-uncritical), where students follow GAI reasoning without questioning it. The activity revealed the limitations of GAI, particularly in generating contradictory or imprecise answers (Bagno et al., 2024), which proved to be difficult for students to interpret. This situation illustrates the importance of a structured personal MWS for evaluating and validating GAI responses. Problems 1 and 2 seem to favour the LA2 paradigm in GAI responses, while students predominantly work in LA1. The LA3 paradigm was not observed, as the proposed activities did not require work in abstract vector spaces beyond $\mathbb{R}^n$.

This research shows that GAI does not replace conceptual mastery. Delegated use hinders the development of conceptual understanding and prevents the development of cognitive autonomy. Conversely, critical appropriation could promote paradigmatic flexibility.

A semi-structured interview (E013, Type B group) revealed: (1) triangulation between GAI tools reflected an implicit validation criterion ("the answer was clearer in MathosAI"), evidencing emergent instrumental genesis; (2) critical reflection on problem 2 only emerged via a researcher-introduced counterexample, showing a human actor can trigger the critical posture absent in the student–GAI interaction; (3) E013 recognised that "if I don't understand the concepts, AI doesn't help me," confirming GAI cannot substitute the theoretical reference frame.

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Supporting Appropriate and Identifying Inappropriate Uses of AI in Mathematics Teacher Seminar Papers

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This study investigates how preservice mathematics teachers define and enact appropriate use of generative AI in seminar papers. Framed by the TPACK model, it examines the alignment between students' stated principles and their actual practices, as well as their ability to recognize AI-generated texts. A qualitative-dominant design across two academic years compares conditions with and without structured guidance. Findings reveal a gap between beliefs and practices, while guidance supports more critical and transparent use. Practical indicators for identifying inappropriate AI use are proposed, with implications for teacher education.

Keywords: mathematics teacher education, artificial intelligence, academic integrity, TPACK, qualitative research.

INTRODUCTION, THEORETICAL FRAMEWORK AND AIMS

The increasing availability of generative artificial intelligence (AI) tools raises new questions about academic integrity and pedagogical practice in higher education. In mathematics teacher education, these tools offer both opportunities for learning and risks related to uncritical or inappropriate use. A central challenge lies in defining what constitutes appropriate use of AI and supporting students to act accordingly (Jelson et al., 2025).

This study is framed by the Technological Pedagogical Content Knowledge (TPACK) framework (Mishra & Koehler, 2006), which emphasizes the integration of technological, pedagogical, and content knowledge. Appropriate AI use thus requires not only technical skills but also critical evaluation and pedagogically meaningful application. The study also draws on academic integrity perspectives emphasizing understanding and responsible knowledge construction (Bretag, 2013).

The study addresses three questions: how students define appropriate AI use, to what extent their practices align with these definitions, and how AI-generated or inappropriately used content can be recognized.

METHODOLOGY

The study was conducted with third-year preservice mathematics teachers at an Israeli teacher education college. A qualitative-dominant design compared two cohorts: Year 1 (29 students, minimal guidance) and Year 2 (40 students, structured guidance including discussions, reflection tasks, and a text-classification activity). The comparison is exploratory.

Data included seminar papers, student reflections and reported AI prompts, classroom discussions (Year 2), and a task requiring students to distinguish between AI-

generated, peer-written, and encyclopaedia texts. Data were analyzed thematically to identify patterns in students' definitions, reasoning, and practices. Seminar papers were examined for alignment with stated principles, and descriptive statistics were used for the classification task.

RESULTS AND DISCUSSION

Across cohorts, students consistently stated that AI use is appropriate only when the user understands and critically evaluates the content. However, this principle was often not operationalized in practice. A gap emerged between stated beliefs and actual use: while some students demonstrated reflective engagement, others used AI uncritically or inconsistently. In Year 1, about one third of students - mainly non-native Hebrew speakers - used AI extensively without reporting it, whereas structured guidance in Year 2 was associated with increased awareness and more consistent reporting.

Students showed moderate ability to distinguish between AI-generated, peer-written, and encyclopaedia texts, relying mainly on stylistic cues rather than explicit criteria. Instructors identified several recurring indicators of inappropriate use, including mismatches between writing style and student ability, limited use of mathematical representations (e.g., diagrams or examples), superficial understanding during oral defenses, inconsistencies in references, and generic phrasing.

From a TPACK perspective, these findings suggest that students develop partial integration of technological and content knowledge, while pedagogical judgment regarding AI use remains underdeveloped. Structured guidance appears to support more critical and transparent use, yet inappropriate practices persist, indicating that ethical AI use must be explicitly addressed in teacher education. Detection of misuse currently relies largely on tacit professional judgment, highlighting the need for more systematic frameworks.

POSTER CONTENT AND CONCLUSION

The poster will present a conceptual model linking AI use to TPACK and academic integrity, a comparison of teaching approaches, and key findings on students' conceptions and practices. It will also include results from the text-classification task and a practical set of indicators for identifying inappropriate AI use. Short anonymized examples will be used to illustrate findings and support discussion.

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Didactic transposition through generative AI: Exploring the structure, functioning, and utility of the mean value theorem

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Generative AI tools like ChatGPT are widely used in scientific research, including mathematics education. This multiple case study examines how ChatGPT models knowledge about the mean value theorem (MVT) praxeologically, and how it conceptualises the theoretical notions of structure, functioning, and utility (SFU) recently introduced within the Anthropological Theory of the Didactic. The findings indicate that ChatGPT reproduced key elements of the MVT and articulated praxeological components relevant to SFU, yet its organisation of these elements varied across trials and diverged from our conceptualisations. We conclude that ChatGPT may support constructing epistemological reference models, but it currently lacks reliability in conceptualising theoretical notions with coherence and precision.

Keywords: Calculus, use of generative AI, ATD, case study, mean value theorem.

INTRODUCTION

Mathematics educators and students increasingly use generative artificial intelligence (GenAI) tools like ChatGPT in various contexts such as generating explanations, problem-solving, or lesson planning (Pepin et al., 2025; Zhu, 2025). Similarly, GenAI tools are increasingly incorporated into scientific research: Researchers use them for reviewing and summarising literature, quantitative and qualitative data analysis, generating data, formulating hypotheses, or drafting and refining texts (Ray, 2023; Lehr et al., 2024).

Researchers discussing the use of GenAI tools in scientific research—including in mathematics and mathematics education—seem to agree that while their powerful and constantly evolving capabilities depict clear benefits, they cannot replace human researchers: GenAI tool have to be prompted carefully and the given context influences their capabilities in qualitative data analysis (Chatterji et al., 2025; DeCoster et al., 2025), their outputs have to be checked and refined in terms of accuracy (e.g., as a result of hallucination of facts or references), and research ethics concerning accountability, justifiability, or authorship must be respected (Lund et al., 2025). In this paper, we investigate the use of ChatGPT in mathematics education *research* working with the *Anthropological Theory of the Didactic* (ATD) (Chevallard & Bosch, 2020) addressing the line of inquiry described by Luzano (2025, p. 790):

> As mathematics education continues to evolve in increasingly diverse environments, it is crucial to examine whether AI-assisted analysis can accommodate various academic discourses and research traditions […]. Addressing these gaps will provide a clearer picture of ChatGPT's applicability in mathematics education research.

Recently, Chevallard (2024) included the notions of *structure*, *functioning*, and *utility* (SFU) into ATD, which have already been taken up by Strømskag and Chevallard (2024) and Topphol (2023) when developing so called *(epistemological) reference models (RM)* for mathematical topics such as concavity/convexity and the fundamental theorem of calculus. However, the three notions are not yet fully conceptualised or worked out for multiple mathematical topics. Addressing it in research is thus not immediate. Moreover, current versions of ChatGPT could not yet be trained much with related data – circumstances under which its performance is considered weaker than in its other strengths (Lehr et al., 2024). Accordingly, we address how ChatGPT could contribute to the genuine scientific task of conceptualisation of theoretical notions and their application in research practice. We do so by examining RMs created by ChatGPT about the mean value theorem (MVT), in which we ask it to draw on the notions of SFU. Our research question is:

> *How do different versions of ChatGPT (at a given point of time) present the notions of structure, functioning, and utility in reference models of the mean value theorem that they create when prompted with current literature on these notions?*

ANTROPOLOGICAL THEORY OF THE DIDACTIC

The Anthropological Theory of the Didactic (ATD) is one of the major theoretical frameworks in mathematics education (Chevallard & Bosch, 2020). Within this framework, knowledge is modelled through praxeologies. A praxeology is a unit of knowledge consisting of a quadruplet $[T/\tau/\theta/\Theta]$, where T is a type of tasks, τ is the technique(s) used to carry out those tasks, θ is a technology, a discourse that explains and justifies the techniques, and Θ is a theory that represents a broader discourse justifying θ. Praxeologies are usually the product of institutional activity, whether located in a single institution or shared across multiple institutions (Strømskag & Chevallard, 2024). The sub-theory of didactic transposition (DT) addresses how institutionalised knowledge and its associated praxeologies are transformed from their original contexts of production into forms intended for teaching, and ultimately into realisations in classroom practice (Figure 1).

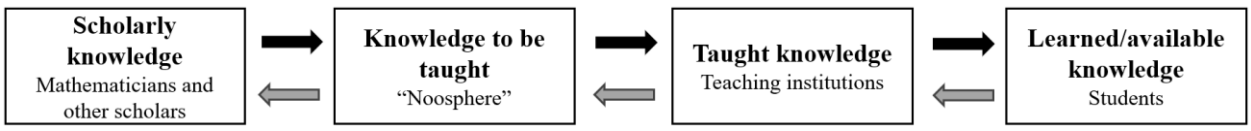

Figure 1: The DT process (adapted from Strømskag & Chevallard, 2024, p. 120).

More specifically, ATD posits that knowledge taught in schools and universities originates from *scholarly knowledge*—that is, the knowledge developed by scholars (e.g., mathematicians) within universities or other academic institutions. When transposing a body of knowledge from its originating institution, an appropriate milieu must be constructed and supported by activities that make it "teachable, meaningful, and useful" (Chevallard & Bosch, 2020, p. 214). This process involves several actors, collectively referred to as the *noosphere*—those who "think" about teaching—such as mathematicians, teachers, lecturers, and curriculum designers (Chevallard & Bosch,

2020, p. 214). Their responsibility is to maintain the integrity of scholarly knowledge mediating between its academic origins and the expectations imposed by society on the educational system (Chevallard & Bosch, 2020).

We focus in our study on the notions of *structure*, *functioning*, and *utility* as recently introduced to ATD by Chevallard (2024, p. 9):

> In a very general way, with regard to any possible object *o*, three major questions arise [...]: the question of its structure (what is the object *o* made of?), the question of its functioning (how does it 'work'?), and finally the question of its utility (what can it be used for?).

To the best of our knowledge, these notions have not yet been transposed in detail into *knowledge to be taught* within mathematics education textbooks or papers. However, it can now be accessed and learned as *taught knowledge* through GenAI tools such as ChatGPT. Therefore, in this study, we consider GenAI tools as *teaching institutions* through which mathematics education researchers may engage in learning *scholarly knowledge* about SFU in ATD as produced by ChatGPT. Consequently, we conceptualise the DT process of knowledge related to SFU when mathematics educators use GenAI tools to learn about mathematics education theories as shown in Figure 2. Such knowledge is at the boundary of scholarly knowledge and knowledge to be taught because the researchers are also research communicators.

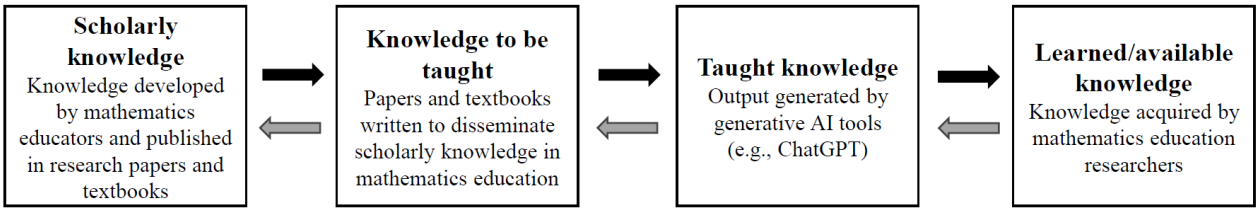

Figure 2: DT through GenAI for learning about mathematics education theories.

In our study, we consider the product (i.e., the presentation and use of the notions of SFU in reference models of the MVT generated by ChatGPT) from the perspective of a DT process. This means that we adopt the methodological approach commonly used in ATD to analyse such processes. Analysing how a body of knowledge is transposed primarily requires a basis for comparison, that is, an (epistemological) reference model (RM) (Bosch & Gascón, 2014; Gascón & Nicolás, 2022). An RM models the scholarly knowledge prior to DT. To construct such a model, the researcher develops an independent description of the knowledge (Bosch & Gascón, 2014). An RM should draw on multiple academic sources to ensure a robust representation of scholarly knowledge (Topphol, 2023). However, it inevitably reflects the researcher's own perspective and therefore, "should constantly be questioned and submitted to empirical confrontation" (Chevallard & Bosch, 2020, p. 216).

METHODOLOGY

This study is conceptualised as a multiple case study comprising two cases, each representing a trial in which ChatGPT-5 was used to generate an RM for the MVT. The data for the two trials (T1) and (T2) were collected in November 2025 using the *Deep*

Research mode of the latest version of ChatGPT-5 with a new free account. We deleted prior chats and disabled customisation. The prompts for the trials are shown in Figure 3. For the design of the prompts, we were guided by Chatterji et al. (2025), most of whose authors are affiliated to OpenAI, who used ChatGPT for qualitative data analysis of texts. For (T1), we uploaded Chevallard's chapter (2024) via the Chat interface. Then, we entered the first part of the prompt followed with a short answer to a question by ChatGPT to clarify the context of the RM. After having received the first RM, we uploaded Topphol's paper (2023) and asked ChatGPT to refine the RM. For (T2), we did not upload any paper and provided the quote from Chevallard (2024, p. 9) cited above instead.

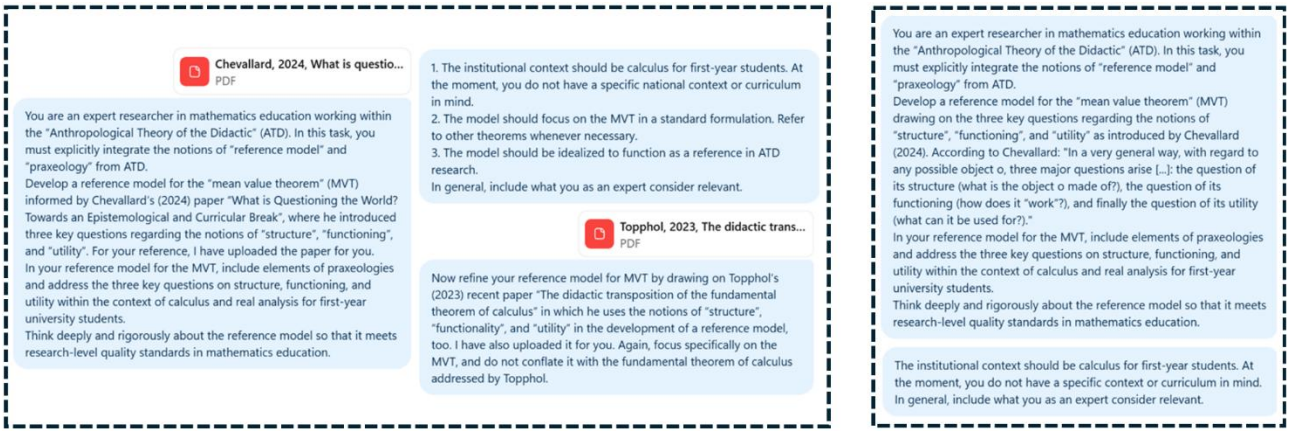

Figure 3: Prompts for trial (T1) (left) and trial (T2) (right).

Our own RM of the MVT delineates scholarly knowledge about this theorem in the context of *calculus in one variable*. We initially collected tasks related to the MVT. We further backed our RM with tasks identified in textbooks such as Apostol (1981), Forster (2016), and Stewart et al. (2021). Then, we abstracted the tasks into types and organised them according to SFU, being aware that this organization is contingent. Our understanding of SFU is as follows: The *structure* of a theorem is its statement, its proof, and special cases or generalisations. *Functioning* concerns those praxeological elements that are directly involved in the practical operation with the theorem and its proof. Thus, functioning addresses how the theorem operates as a tool for reasoning and problem-solving. The *utility* of the MVT consists of both its inner-mathematical applications—such as for proofs and computations—and its extra-mathematical applications in real-world contexts based on the interpretation of derivatives and rates of change (e.g., velocities and growth rates). Accordingly, we distinguish between intra-mathematical (IMU) and extra-mathematical utility (EMU). Then, we analysed deductively which of the types of tasks in our RM appear in ChatGPT's two RMs. Moreover, we analysed how ChatGPT described to have conceptualised the notions of SFU.

OUR REFERENCE MODEL ON THE MEAN VALUE THEOREM

Techniques and technologies centre in the theories Θ_C calculus, Θ_A real analysis, and Θ_T topology of real numbers, but they cannot be presented here in more detail.

Structure: Mean value theorem (MVT): Let $a < b$ be real numbers. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b). Then, there is a $c \in (a,b)$ (an "MVT point") such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ (S1). An immediate consequence is that function values can be estimated via the derivative: $f(b) - f(a)$ is bounded by $(b-a) \cdot \inf f'_{|(a,b)}$ and $(b-a) \cdot \sup f'_{|(a,b)}$ (S2). The MVT is a generalisation of Rolle's theorem (RT) (S3) and a special case of Cauchy's mean value theorem (S4) (Stewart et al., 2021). The MVT as well as the other theorems are existence theorems. In neither of them, the point c must be necessarily unique (S5). Graphically, the MVT guarantees that there is a tangent to the graph of the function f whose slope is identical to the slope of the secant through the points $A\big(a; f(a)\big)$ and $B\big(b; f(b)\big)$ (see Figure 4), while Rolle's theorem guarantees a horizontal tangent to the graph of f (S6). In applications, the derivative of f may be considered as an instantaneous rate of change of a quantity represented by f: Accordingly, the MVT states that there is a point at which the instantaneous rate of change is equal to the average rate of change of the quantity between a and b (S7).

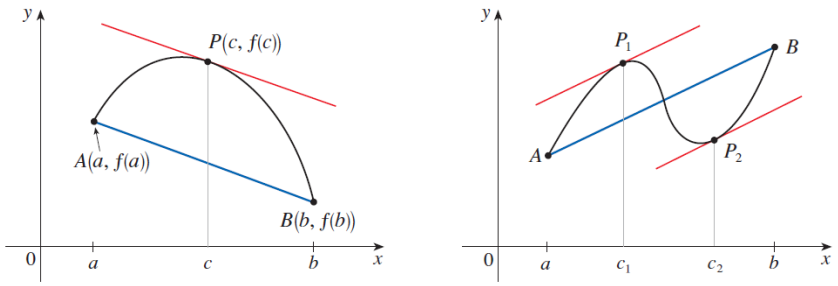

Figure 4: Graphical representations of the MVT (from Stewart et al., 2021, p. 292).

Rolle's theorem appeared first in a publication by Rolle in 1691 and the MVT was first stated by Lagrange in the 18th century (S8) (Stewart et al., 2021). Besenyei (2012) describes that the proof (S9) of the MVT commonly find in calculus textbooks (e.g., Stewart et al., 2021) is due to Dini and Bonnet (S10). Figures 5 and 6 show tasks and/or task types related to F and U. Belonging to the intra-mathematical utility task type T_{I1}: *Prove theorems with the MVT*, Figure 7 shows our list of theorems in a calculus context that are usually proven with the MVT in calculus literature.

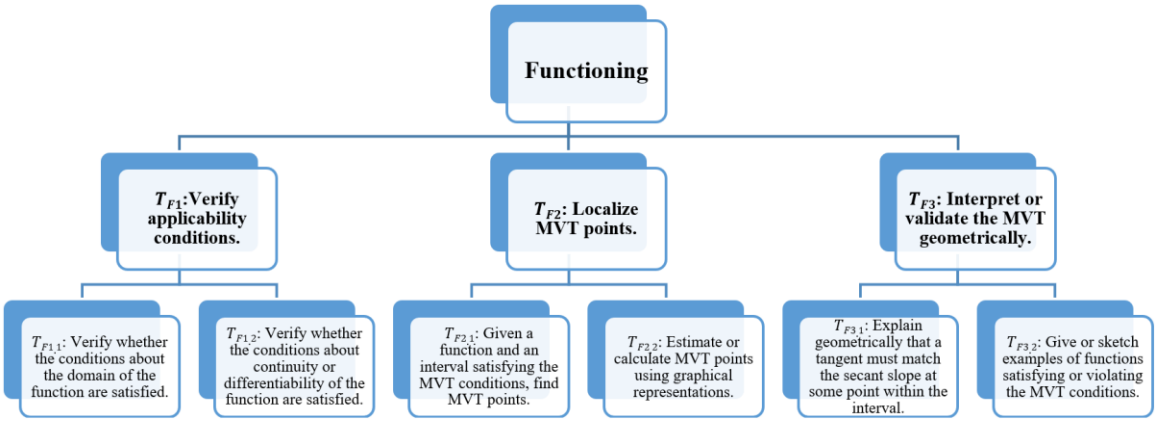

Figure 5: Tasks / Task types in our RM related to *functioning*.

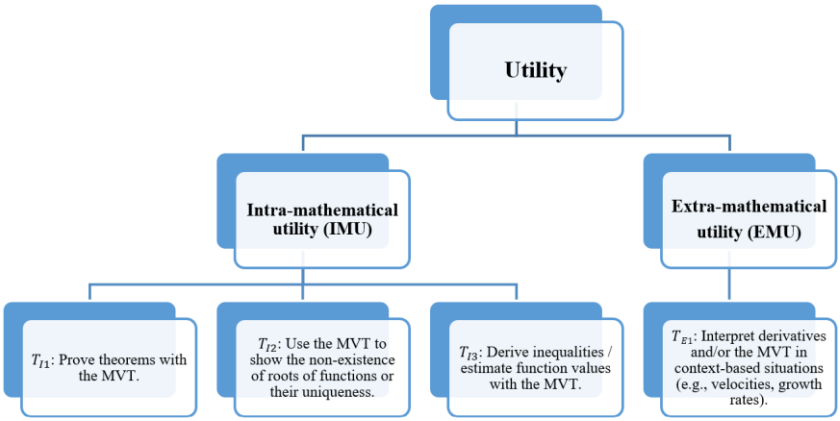

Figure 6: Tasks / Task types in our RM related to *intra-* and *extra-mathematical utility*.

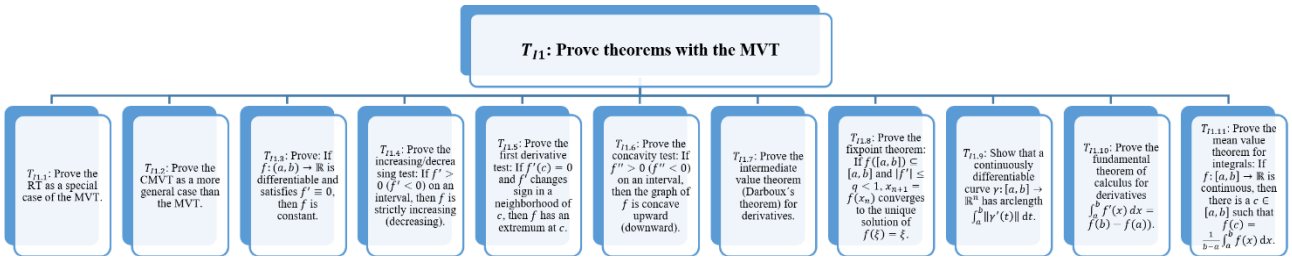

Figure 7: Tasks in our RM related to the task type T_{I1}: *Prove theorems with the MVT*.

RESULTS

ChatGPT's conceptualisation of SFU

Almost all elements we found in the two RMs by ChatGPT could be subsumed under our own RM. However, the three RMs do not align with respect to how they group these elements into SFU. In (T1), structure is described as "what the theorem consists of (its statement, conditions, logical formulation)" and "underlying definitions", functioning as "how it operates or is used in practice" and "operates as a mathematical tool", and utility as "its applications (intra- and extra-mathematical uses)" and in terms of "long-term consequences". More specifically, and in alignment with our RM, ChatGPT describes IMU as "its use in function analysis and proofs" and "corollaries," and EMU "refers to real-world contexts." However, in (T2), ChatGPT conceptualised structure as "statement and components" and "form and content of the theorem", not necessarily underlying definitions or proofs. Functioning is described here as "how it works/proofs" and "proof method and logical mechanisms." In fact, the whole section on functioning ChatGPT produced in (T2) may be considered as a praxeological analysis of the proof of the MVT, addressing as well several of the elements we listed under structure. ChatGPT seem to operationalise "how it works" with "how the proof is carried out." Utility is not further divided in (T2).

Results from the deductive analysis

Table 1 shows the results of our deductive analysis of the elements in our RM compared to ChatGPT's RMs. In this table, we also highlight whether ChatGPT situated an element as we did, differently, or at multiple points.

		Trial (T1)	Trial (T2)			Trial (T1)	Trial (T2)
Structure (S)	S1	X	X (also in F)	**Intra-mathematical utility (IMU)**	$T_{I1.1}$		
	S2	X (implicit in F, & without ref. to SFU)	X (in U)		$T_{I1.2}$		
	S3	X (partly without ref. to SFU)	X (also in F)		$T_{I1.3}$	X (also in F and IMU)	X (also in U)
	S4		X (also in U as a technique)		$T_{I1.4}$	X (in F and without ref. to SFU)	X
	S5				$T_{I1.5}$	X	
	S6	X (also in F)	X (also in F)		$T_{I1.6}$	X (also without ref. to SFU)	X
	S7	X (in F & EMU)	X (in U)		$T_{I1.7}$	X (as theory for $T_{I3.1}$ & without ref. to SFU)	X (without ref. to SFU)
	S8				$T_{I1.8}$		
	S9	X (also without ref. to SFU)	X (predominantly in F)		$T_{I1.9}$		
	S10				$T_{I1.10}$		
Functioning (F)	$T_{F1.1}$	X ($T_{F1.1\&2}$ seen together in S, and without ref. to SFU)	X ($T_{F1.1\&2}$ seen together in S)		$T_{I1.11}$		
	$T_{F1.2}$	X	X		T_{I2}	X	
	$T_{F2.1}$	X (in F & without ref. to SFU)	X (in S)		T_{I3}		
	$T_{F2.2}$			**EMU**	T_{E1}	X (also in F & without ref. to SFU)	X
	$T_{F3.1}$						
	$T_{F3.2}$	X (in S)	X (in S)				

Table 1: Deductive analysis of ChatGPT's RMs in trials (T1) and (T2).

In both trials, ChatGPT mentioned several of the elements without explicitly situating them to S, F, or U. At the level of mathematical content, its responses largely fit within the elements of our RM, except for its additional link between the MVT and Taylor's theorem on remainders in power series developments. Although, we limited our scope to calculus in one variable, the IMU we addressed is broader, encompassing areas beyond differential calculus (e.g., integrals). Historical remarks, the non-uniqueness of MVT points, and graphical representations were absent in ChatGPT's RMs as well (like other missing elements, they might have been produced in response to different prompts though). By contrast, ChatGPT articulated praxeologies in greater detail than we did here in terms of (types of) tasks, techniques, technologies, and theories. These differences underline that RMs are situated within (implicit) institutional constraints, and that they are researchers' products, and in this case also researcher-GenAI joint work. Leaving open the institutional context (e.g., "calculus" as a broad unspecified whole), leads to different, potentially reasonably structured RMs.

Institutional embedding and the notion of theory

In (T2), ChatGPT devoted a section on a "Praxeological Model" and "Institutional Constraints and Didactic Transposition" each. While the presentation of tasks, techniques, and technologies, seemed to align with what we consider their use in ATD, the notion of theory was inconsistent. At some points, ChatGPT considered theory as "fundamental properties of continuous and differentiable functions […] ensuring the hypothesis of the MVT" or certain theorems, which we may interpret as "Θ_C calculus." Yet at other points, ChatGPT considered theory to consist of the "interpretation of the derivative as an instantaneous rate" or of rather unrelated singular facts such as "the fundamental theorem of calculus," or "the formal MVT statement" itself. ChatGPT also addressed the issue of DT and hypothesised scenarios how "the scholarly praxeology (with full proofs and rigorous conditions) [is transformed] into 'mathematical knowledge to be taught'". Consistent with literature on ChatGPT's hypotheses-generation abilities, some of its hypotheses on DT seemed reasonable (e.g., a potential under-emphasis of the auxiliary function in the proof of the MVT as a corollary to RT), while others underlined ChatGPT's inability to grasp mathematics (e.g., hypothesising an under-emphasis on the need for differentiability, which is rather absurd since then $f'(c)$ could not even be stated). Finally, we spotted that ChatGPT mentioned references not completely aligning with the text, but also that ChatGPT referred to research related to SFU such as the notion of "archeorganisation" (Strømskag & Chevallard, 2024), which did not appear in our prompts.

DISCUSSION AND CONCLUSIONS

Our findings suggest that ChatGPT may support mathematics education researchers in developing RMs by providing an initial overview of (types of) tasks and other elements in the quadruplets of praxeologies related to the SFU of a mathematical theorem. In both trials, ChatGPT reproduced only a subset of the SFU elements that we considered. In both trials, ChatGPT stated explicit conceptualisations of the three notions of

structuring, functioning, and utility. These conceptualisations and their use differed within the trials and from our RM. Additionally, researchers constructing an RM are faced with the necessity to remain neutral with respect to a specific institution so that the RM can be used as a comparative foil. Yet, they produce their RMs from the perspective of a (position in an) institution, which may be left implicit or may be difficult to make explicit. The same happens with ChatGPT. It co-authored RMs based on our prompts and its underlying large language model, producing ad hoc groupings of elements of praxeologies and without specifying the context (e.g., whether calculus is considered restricted to differential calculus). Overall, our study shows that ChatGPT has potentials as a tool for constructing RMs and for conceptualising theoretical notions within a theoretical framework, even if the theoretical notions are itself sensitizing. Yet, the models it produced contained notable gaps. Our aim is not to advocate for or against the use of GenAI, but to offer an initial exploration of how such tools might be incorporated into mathematics education research for conceptualizing theoretical notions. Further data and more systematic investigations are needed to identify how prompts may influence ChatGPT's production of reasonable descriptions of theoretical notions within a framework and in interaction with mathematics education researchers.

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Designing targeted exercises and a scaffolded learning environment for learning to work with the theorem prover Lean

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Keywords: proof, digital tools, scaffolding, design-based-research.

INTRODUCTION

Theorem provers have recently been shown to support the development of students' proof competencies but are not intuitive to use. We follow a Design-Based-Research approach (Bakker, 2018) to develop design principles for learning environments for the theorem prover Lean. Building on our work presented at CERME 14 (Garnelo & Liebendörfer, 2025), we present types of tasks that illustrate how we implement scaffolding according to different kinds of knowledge needed to master Lean.

THEORETICAL FRAMEWORK

Theorem provers are digital tools for processing proofs in a formalized language that enable automated verification of statements and proofs. Although originally developed for research, their interactive feedback and explicit representation of mathematical concepts also make them promising tools for teaching (Thoma & Iannone, 2022).

However, working with Lean requires complex knowledge. We draw on computer science education, where Bayman and Mayer (1988) structured the knowledge needed to master a programming language in three consecutive levels: syntactic, conceptual and strategic. Addressing all levels is crucial for students' progression in learning the principles and strategies of the language (McGill & Volet, 1997). Our earlier work confirmed that this structure supports describing students' knowledge and tracking their progression with Lean (Garnelo & Liebendörfer, 2025). There, we also proposed the design principle of designing a scaffold for the learning environment based on the levels (syntactic, conceptual, and strategic) at which students may struggle. Scaffolding is a common design principle in DBR for students faced with new tools (e.g. Azimi et al., 2023). It refers to the provision of temporary, targeted support to help students learn new, challenging concepts that is gradually removed as students gain proficiency.

DESIGN AIM AND METHOD

We report results from the second design-cycle that started by re-specifying our learning goals based on students' progression we found in our prior data analysis. We now advanced to the development stage of this cycle where we focus on the design principle of scaffolding. We pursue the question of how tasks and task sequences can be planned and designed in a way in which they integrate the components of the levels of knowledge in a way which supports students journey to mastering Lean. To this end we designed tasks that target one or more components of knowledge that may belong to different types of knowledge.

RESULTS

One example for a type of task is to give students a full Lean proof with a syntactical error. As students are interacting with a complete proof, they are not required to have strategic knowledge and the required conceptual knowledge is strongly reduced, in particular regarding the types of proof steps involved. In solving such a task, students practice making sense of Lean code, as well as making small adjustments (targeting the syntactical knowledge in general) and working with the interactive proof state and error messages of Lean (targeting this specific aspect of the conceptual knowledge).

Our poster will present the different types of exercises we have developed and show the intended scaffolding. This includes an exemplary structure for arranging these exercises in a learning environment to support a smooth progression of learning Lean.

DISCUSSION

The design principle of scaffolding, combined with the knowledge types model, has proven its worth by supporting the design of tasks that form a sequence of reasonable steps for working with Lean. However, it was sometimes difficult to coordinate the sequencing of exercise types with the underlying mathematical topics. Practical problems further include managing the information load that students need to simultaneously work with. In the near future, the evaluation of this design cycle's implementation will show in how far the proposed progression proves effective.

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Integrating programming and mathematical proof perspectives: towards a framework for conceptual knowledge in Lean

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Conceptual knowledge (CK) has been explored in programming and in mathematics, but its role in using interactive theorem provers (ITPs) is still not well understood. In this paper, we propose a framework for CK for proving in Lean that integrates insights from computer science education (programming) and mathematics education (proof). Drawing on a narrative literature review, we identify five CK dimensions relevant to tactic-style proofs in Lean: objects, constructs and concepts; relationships and connections; principles and rules; language and representation; structure and organisation. In the discussion, we outline related opportunities and challenges, and suggest directions for future research on students' proving and the potential transfer between Lean and pen-and-paper proofs.

Keywords: Conceptual knowledge, Mathematical proof, Programming, Lean, Framework development.

INTRODUCTION

Students' difficulties with proof are well documented, including challenges in recognising what counts as a proof (Inglis & Alcock, 2012), understanding the role of definitions, theorems and examples (Fukawa-Connelly, 2012), and grasping the language and structure of proofs (Selden & Selden, 2013). These issues underline the importance of conceptual knowledge (CK) e.g., knowledge of objects, meanings, relationships, and their role in proving (Crooks & Alibali, 2014). Existing accounts of CK emphasise concepts and connections in mathematics (Rittle-Johnson et al., 2001) and, in the context of proofs, CK includes objects, reasoning strategies, and proof structures (Stylianides, 2007).

Among digital tools that may address these difficulties, interactive theorem provers (ITPs) have been gaining traction in undergraduate mathematics (Yan & Hanna, 2023). Across several countries, ITPs have been introduced in university modules for two main purposes: to learn how to prove and to formalise mathematical results. A widely used ITP is Lean [1]. This ITP is both a functional programming language [2] and a proof assistant for formalising mathematical proofs. Hence, it encompasses programming-like features, such as syntax and constructs (e.g., Qian & Lehman, 2017) and proof-related features, like mathematical definitions, theorems and proving strategies (Thoma & Iannone, 2022). Lean supports two proof styles: tactic and term. In tactic-style proofs, users write commands, including tactics, on the left side of the screen to give instructions to the software to proceed in the proof. Lean checks every command and provides feedback on the right side of the screen, the Infoview, which

displays hypotheses, goals and sub-goals, and shows how tactics change them (Yan & Hanna, 2023). Term-style proofs are written as complete expressions rather than as step-by-step tactic commands. Here we focus on tactic-style proofs because these are more flexible and typically easier to write rather than term-style proofs; commands are checked step by step and the proof status is constantly updated in the Infoview (Yan & Hanna, 2023). Accordingly, tactic-style proofs feature in Introduction-to-Proof university modules and other educational resources. Figures 1 and 2 present an example of a tactic-style proof and show some features of the Lean interface.

The Lean interface makes some proving aspects, such as goal/sub-goal decomposition and justification at each step, more visible. However, students do not always connect the use of Lean with traditional pen-and-paper proving (Iannone & Thoma, 2024). This issue, together with the growing use of ITPs in undergraduate mathematics (e.g., Avigad, 2019), suggests the need to characterise CK for proving with this ITP in an innovative way that draws on both computer science education (programming) and mathematics education (proof) perspectives.

In this paper, we propose a framework for CK which addresses two research questions: What dimensions describe CK for proving in Lean? What features characterise these dimensions? In the following sections, we outline our methods, review the relevant literature and present the proposed dimensions that highlight key aspects of CK for proving in Lean. We then describe how the two perspectives are integrated in the framework, discuss opportunities and challenges associated with Lean across the dimensions, and suggest how the framework could contribute to inform the design of teaching resources and future research.

METHODS

To develop the framework, we followed the guidelines for conceptual papers by Kallia and Cutts (2023), and conducted a narrative literature review (Demiris et al., 2019) to synthesise theoretical and empirical CK insights relevant when proving in Lean. Due to the limited space, here we report only the most relevant sources from a larger set which also reflect our socio-cultural approach to learning. For research related to CK in computer science education, we started from the review by Qian and Lehman (2017) which highlights the need for syntactic, conceptual and strategic knowledge for programming. We then identified two seminal articles (Bayman & Mayer, 1988; McGill & Volet, 1997) that conceptualise CK for programming. Similarly, to investigate CK in mathematics education, we started from Crooks and Alibali's (2014) review of CK definitions and then focused on proof-specific work (e.g., Stylianides, 2007; Mejia-Ramos et al., 2012; Fukawa-Connelly, 2012) to find aspects related to proof that account for CK. We chose Stylianides' (2007) definition of proof because it explicitly foregrounds central elements of CK for proving such as definitions, theorems, rules and representation. To start devising the framework, we identified key aspects from both literatures by highlighting keywords used to define CK. We grouped similar aspects (i.e., same words used or similar concepts), chose representative labels (usually the most used across different studies) for each group and then synthesised

them into dimensions integrating the two perspectives. The process was iterative: alternatives were discussed and a final set agreed by the authors. No dimensions were discarded, but some (e.g., language and representation) were merged. We then reviewed recent educational research that examines aspects of CK when engaging with an ITP (and specifically with Lean), although CK itself has not been the explicit focus (e.g., Avigad, 2019; Iannone & Thoma, 2024). Finally, we mapped the emerging Lean features (e.g., tactics, goals, syntax, Infoview, correctness) onto the dimensions identified to characterise the CK framework for proving in Lean. Because these features mapped coherently, we considered the dimensions suitable. However, we acknowledge that these dimensions are not mutually exclusive; where overlap occurs (e.g., type theory aspects), we located elements according to our conceptualisation. We will refine these choices in future empirical work.

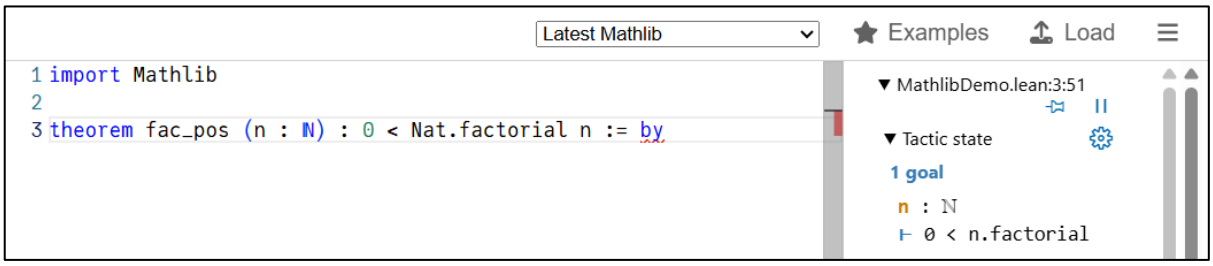

Figure 1: Screenshot of the beginning of a proof in the online version of Lean 4. Line 3 on the left represents the statement "for every natural number n, $n!$ is greater than zero". The hypotheses and goals are visible on the right.

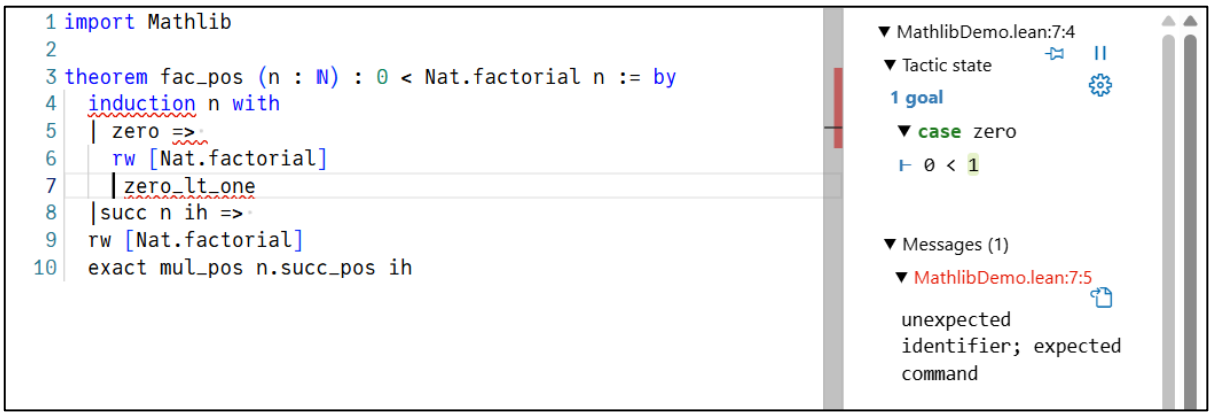

Figure 2: Screenshot of the proof by induction constructed in Lean for the statement presented in Figure 1. On the right, error messages appear because some commands (underlined in red on the left) are wrong.

LITERATURE REVIEW

Conceptual knowledge in computer science education for programming

In computer science education, CK related to programming is described as an understanding of programming objects, constructs and how they behave during execution (Bayman & Mayer, 1988; McGill & Volet, 1997). McGill and Volet (1997) describe CK in programming as the understanding of constructs and principles, supported by mental models of how a system works and of the semantics of

programming actions. CK enables learners to combine language features within a program and to design solutions. Qian and Lehman (2017) further distinguish between syntactic, conceptual, and strategic knowledge in their review of misconceptions in programming. They define CK as knowing how programming constructs and principles work and what happens inside the computer; for example, how loops operate, what occurs during an assignment, or why certain syntax (such as semicolons) is required. Qian and Lehman (2017) argue that such knowledge underpins the use of variables, conditionals, and functions, and supports debugging as a process of making sense of program behaviour rather than relying on trial-and-error.

Conceptual knowledge in mathematics education with a focus on proof

In mathematics education, CK has been described both in narrow and broad terms. The literature review by Crooks and Alibali (2014) collects different definitions of CK in mathematics and classifies them. The two most common types of definitions, identified in their work, emphasise knowledge of concepts and principles, and understanding of relationships and connections between the objects within a domain. Consistent with the focus on these types of definitions, Rittle-Johnson et al. (2001) stress the role of principles and interrelations between units of knowledge, portraying CK as flexible, generalisable knowledge that supports meaningful application across contexts. Skemp's (1976) distinction between relational and instrumental understanding further clarifies that CK involves not only knowing how to perform procedures but also understanding why they work and how they relate to broader structures. This emphasis extends to proof, where CK encompasses knowledge of objects (definitions, axioms, theorems that must be accepted by the local community), valid forms of reasoning, and structural features of arguments (Stylianides, 2007). Other findings in the literature, related to proof comprehension and writing, shed light on other relevant aspects for CK in mathematical proof such as high-level ideas, that summarise an argument's structure, detailed sub-arguments, appropriate language and notation (Mejia-Ramos et al., 2012; Fukawa-Connelly, 2012).

TOWARDS A FRAMEWORK FOR CONCEPTUAL KNOWLEDGE IN LEAN

Our synthesis of the narrative literature review from computer science education and mathematics education extends the limited literature about Lean and details both programming elements and proof features related to Lean. Research in computer science education for programming stresses constructs, operations, principles and system behaviour (Bayman & Mayer, 1988; McGill & Volet, 1997; Qian & Lehman, 2017) and mathematics education concerns knowledge of concepts and, more importantly, the connections and principles that link them (e.g., Crooks & Alibali, 2014). Proof research foregrounds definitions and statements, valid forms of reasoning, representations, language and proof structures (e.g., Stylianides, 2007; Mejia-Ramos et al., 2012). Recent research about Lean describes key features of this software. For example, Avigad (2019) considers Lean as a language for constructing proofs that are immediately checked by the computer, foregrounding issues of rigour and correctness. Iannone and Thoma (2024) explore students' perceptions of the usage of Lean

considering also CK as knowledge of how Lean works, including its constructs and operations. Considering all these aspects, we eventually identify the dimensions that constitute the framework presented in this paper: objects, constructs and concepts; relationships and connections; principles and rules; language and representation; structure and organisation. A summary of these dimensions and their key elements is shown in Table 1. Below, we describe each of the dimensions.

Objects, constructs and concepts

In computer science and mathematics education, developing CK involves knowledge of objects, constructs and concepts. In computer science, programming objects that can be acted upon (e.g., variables and functions [3]) and constructs (e.g., loops) are crucial elements of CK (e.g., McGill & Volet, 1997). Similarly, in mathematics, knowledge of concepts is equally important. Focusing on proof, key concepts include definitions, axioms, theorems, and examples (e.g., Stylianides, 2007; Fukawa-Connelly, 2012).

In Lean, these two perspectives intersect. Mathematical definitions and theorems are implemented as programming objects, while tactics act like programming functions: they operate on other objects and give Lean the instructions to complete proof steps (Yan & Hanna, 2023). For example, tactics enable users to apply a hypothesis, rewrite a statement and implement a specific strategy such as proving by induction (see tactic *induction* in line 4, in Figure 2). Through Lean's Infoview, the roles and statuses of objects, such as hypotheses, assumptions and goals (Iannone & Thoma, 2024), are foregrounded. In Figure 1, the tactic state in the Infoview shows the hypothesis that n is a natural number and the goal that the factorial of n is always positive.

Relationships and connections

In programming and in mathematical proving, it is not enough to know the relevant objects, constructs and concepts. CK encompasses a full understanding of the ways in which objects and constructs can be combined (McGill & Volet, 1997) and an understanding of the usage of mathematical concepts, their logical status and relationships between them (e.g., Fukawa-Connelly, 2012).

Beyond the interplay between distinct objects as part of a mathematical proof, Lean highlights connections between these objects by checking coherence between hypotheses, steps and conclusion (Thoma & Iannone, 2022). The Infoview makes visible these connections by showing how tactics dynamically change the goals, and so the status of the proof, at each step (Yan & Hanna, 2023). In Figure 2, it is visible how, in line 7, the goal changed after applying the tactic *induction* and rewriting with the definition of factorial for natural numbers (*rw [Nat.factorial]*).

Principles and rules

CK involves understanding the principles that constrain actions and rules that justify each move. Programming principles and rules provide guidelines for writing code and indicate how to construct a logically correct program (McGill & Volet, 1997). Logic and systematic rules of inference must be applied for constructing a mathematical proof

and recognised to understand a given one (Stylianides, 2007). As one mathematical principle, every proof step should be justified (e.g., Mejia-Ramos et al., 2012) by referring to some mathematical definitions, previous results or principles.

In Lean, logical requirements for a compiling program and a correct proof are both necessary and often interconnected. The system, before compiling, checks whether the command is typed correctly and verifies that the requirements for applying the chosen tactic are met (Thoma & Iannone, 2022; Yan & Hanna, 2023). This syntactic and mathematical correctness is guaranteed by Lean's mathematical library (Thoma & Iannone, 2022) and, if there is an error in a command, error messages appear in the Infoview (see Figure 2).

Language and representation

Programs and proofs both require precise technical language and appropriate representations of their objects. Programming languages have features that frame the syntax and representation of objects and constructs, while proofs rely on formal language, the use of symbols and linking words (e.g., Fukawa-Connelly, 2012; Thoma & Iannone, 2022). CK encompasses understanding of the semantics of programming features to understand, modify and create programs (McGill & Volet, 1997) and the use of appropriate and known modes of representation to articulate reasoning and present concepts in proofs (Stylianides, 2007).

As a programming language, Lean requires precise syntax which closely aligns with mathematics notation (Thoma & Iannone, 2022), although proving in Lean is not a direct translation of pen-and-paper proofs (Avigad, 2019). In terms of representation, Lean is based on a version of dependent type theory [5], so that every object has a type that defines it (Yan & Hanna, 2023). In Figure 1, $\mathbb{N}$ (natural numbers) is the type for n. In Lean, the type of objects is fixed, but their role and status in hypotheses or goals can change; the dynamic representation of these elements is shown in the Infoview. The representation of the whole tactic-style proof is a sequence of all tactics applied to prove the statement, visible on the left side of the screen. However, the proof is never displayed entirely. The sequence of tactics (e.g., lines 4-10 in Figure 2), once written, is always visible; but the proof state changes at each step and, in the Infoview, users can only see hypotheses, goals and error messages related to the specific line where the cursor is (Yan & Hanna, 2023). In Figure 1, the initial hypothesis and goal are shown in the Infoview; in Figure 2, only the information related to line 7 is visible.

Structure and organisation

Structuring and organising correctly both programs and proofs is essential. In programming, modularity refers to breaking a program down into smaller, independent and reusable modules. Although not emphasised in the CK literature we reviewed, modularity influences how programs are structured. In mathematics, the structure of a proof reflects strategic choices and should inform both high-level ideas and sub-proofs. Recognising these high-level ideas and how they fit together is key to developing CK (Fukawa-Connelly, 2012).

The basic structure of a tactic-style Lean proof is a sequence of commands which includes tactics and gives Lean instructions on how to build a proof. The choice of tactics (e.g., *induction* in Figure 2) reflects the proving strategy (proving by induction in Figure 2), whereas Lean's dynamic step-by-step approach (Yan & Hanna, 2023) mirrors how programs are usually constructed. This approach allows users to focus on each step at a time and thus decomposes the proof into small steps that represent sub-goals (Iannone & Thoma, 2024). In our example, in line 7 the *tactic state* shows only the sub-goal related to the base case (*case zero*) of the proof by induction; the user has to complete this goal or move to other lines with the cursor to see the other goals.

Dimensions	Programming	Proof	Lean
Objects, constructs & concepts	Objects (e.g., variables, functions) and constructs (e.g., loops)	Definitions, axioms, theorems, examples	Definitions, theorems, tactics; assumptions, goals, hypotheses
Relationships & connections	Objects and constructs interaction during the execution of the program	Logical status of and connections between objects; coherence between hypotheses, steps, and conclusions	Interplay of tactics, theorems, and proof status; explicit goal/ sub-goal dynamics visible in the interface
Principles & rules	Programming principles; correctness	Logical rules; norms of justification	Logical coherence required at every step; rigour and correctness
Language & Representation	Syntax and semantics of programming languages; how computer works	Proof language; accepted representational modes	Functional programming language; precise syntax; tactic-style proof; Type theory; dynamic representation
Structure & organisation	Program modularity and design	Proof structures; goal/sub-goal decomposition	Tactic sequence; dynamic-incremental approach

Table 1: Summary of the dimensions that characterise the framework for CK and their key elements across programming, proof and Lean

DISCUSSION

Separating programming and mathematical aspects hinders a full characterisation of CK in Lean, therefore we integrate both perspectives across every dimension of our framework. At the level of objects, constructs, and concepts, definitions and theorems in Lean are both mathematical and programming objects. Thus, when constructing or reading a proof, the user needs a clear grasp of the mathematical concepts, their role/status, and their implementation in Lean. Objects, constructs, and concepts are the building blocks, and understanding the relationships among them, as both mathematical and programming objects, lead to develop CK. Interplay between objects

can change roles/statuses, enabling progress toward the goal of completing the proof. Principles and rules regulate this interplay, guiding proof construction and enforcing stepwise coherence and correctness (Thoma & Iannone, 2022). Mathematical correctness requires that the conditions for applying a specific tactic hold. At the programming level, commands must be typed correctly. This connects to the language and representation dimension. Users must learn Lean's functional-programming-based and precise syntax, which mirrors mathematical language and notation. In representation, functional programming and dependent type theory strongly shape how every object is expressed (Yan & Hanna, 2023). Moreover, tactic-style proofs are sequences of tactics, while the proof state (hypotheses and goals) is dynamic and shown in the Infoview. Understanding Lean's structure and organisation, and how these reflect the steps of a mathematical proof, is essential for CK of proving in Lean.

This integration of computer science and mathematics education perspectives helps users develop CK that connects Lean with pen-and-paper proving because it integrates key elements from both fields. Moreover, the discussion of opportunities and challenges related to the framework's CK elements could guide researchers in exploring current issues: the potentially beneficial use of Lean for learning how to prove and the transferability of aspects of CK from Lean to traditional proving. We highlight here some opportunities in using Lean. Ready access to descriptions of definitions and theorems, thanks to Lean's mathematical library (Thoma & Iannone, 2022), can support students in recognising and using these objects. This information, together with knowledge of the logical status of statements and their links (Fukawa-Connelly, 2012), is also supported by Lean's Infoview, which makes the status of objects visible and shows how tactics modify goals (Iannone & Thoma, 2024). Beyond the Infoview, other aspects of Lean's environment (e.g., step-by-step representation and clear structure) can be leveraged to scaffold understanding. In addition, the guaranteed correctness (syntactic and mathematical) allows users to check their work immediately and offers instant gratification (Avigad, 2019). However, using Lean also presents challenges. It requires becoming familiar with the specific syntax and can lead to trial-and-error approaches rather than planned reasoning and to over-reliance on the system's feedback. Users could become inclined to follow step-by-step feedback and focus mostly on the current goal rather than comprehending the high-level ideas of the proof (Iannone & Thoma, 2024).

CONCLUSION

In this paper, we propose a framework for CK required for proving with Lean. Reflecting on both programming and mathematical proof features of Lean, the framework draws on literature from computer science education (programming) and mathematics education (proof). The integration of these two perspectives is considered necessary because Lean is both a functional programming language and a proof assistant. Moreover, keeping the features of both domains would help students see the connection between Lean and traditional pen-and-paper proving (Iannone & Thoma, 2024). Our synthesis of the literature suggests that CK for proving in Lean implies

understanding system behaviour, core constructs and principles, recognising mathematical concepts and objects and their interconnections, being aware of the language and acknowledging proof structure and organisation. These aspects correspond to the devised dimensions of the presented framework. This framework may not be exhaustive and would not be a definitive account of CK for proving in Lean. Instead, it offers a theoretically grounded basis to inform future work. Further empirical research is needed to refine, operationalise and examine these dimensions. Future studies could analyse students' CK during proving activities in Lean, investigate how learners coordinate these dimensions, and explore which aspects of CK may transfer between Lean and pen-and-paper proofs. The framework can also guide researchers in identifying relevant CK elements for educational resources to overcome proving difficulties and limitations in terms of Lean usability.

NOTES

1. For more information access the Lean community website: https://leanprover-community.github.io/index.html
2. A brief insight: functional programming is a programming paradigm where programs are constructed by applying and composing functions. It is a declarative programming paradigm in which function definitions are trees of expressions that map values to other values, rather than a sequence of imperative statements which update the running state of the program.
3. In functional programming languages, programming functions are considered as objects because they can be treated as any other data type.
4. Italic font highlights (parts of) commands written in Lean or elements of the Infoview present in Figures 1 and 2
5. A brief insight: type theory is a formal logical framework in mathematics and computer science that classifies objects into "types". It is an alternative to set theory.

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Transitions Within and Across Mathematical Discourses in a Digital Dynamic Task on the Archimedean Property

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This pilot design study explores how students' interaction with a digital dynamic task on the Archimedean Property initiates their transition within and across mathematical discourses. Drawing on commognition, we use the Realisation Tree as a visual and analytical tool to map how first-year university students connect different realisations of the Archimedean Property, each with roots in different discourses, while interacting with the task in GeoGebra. Analysis of two students' interactions shows three emergent learning opportunities: from strategic gameplay to spatial reasoning, from spatial reasoning to formal definition, and from reasoning with examples to reasoning with quantified relationships. These transitions illustrate how dynamic digital environments can support students' secondary–tertiary transition.

Keywords: analysis, transition from school to university mathematics, commognition, design research, dynamic digital environments.

INTRODUCTION

The transition from secondary to tertiary mathematics (STT) requires students to shift from familiar, often empirical or visual ways of working toward formal, rigorous, and abstract ways of reasoning (Sfard, 2014). Central to this shift is learning to participate in a mathematical discourse that relies on explicit definitions, logical deduction, and quantification features that are generally absent or implicit in secondary school mathematics. Instead of working through examples, diagrams, or heuristic narratives as is typically the case in school mathematical discourses, students are now expected to reason by articulating relationships between mathematical objects through statements organised around universal and existential quantifiers (Schüler-Meyer, 2022). Therefore, this shift involves changes within and across discourses, such as from algebraic manipulation to logical deduction, or from visual to symbolic reasoning, requiring students to be able to talk, produce and endorse narratives about mathematical objects of the relevant mathematical discourses (Wallach et al., 2022). Supporting such shifts entails task designs that expose students to the different mathematical discourses and facilitate a variety of ways of reasoning across them. Tasks that invite students to coordinate these different discourses, such as numerical, graphical, algebraic, and logical, can therefore provide fertile ground for participation in- and for connecting the multiple mathematical discourses (Wallach et al., 2022; Weingarden et al., 2024).

Dynamic digital environments can support these shifts by offering visual and immediate feedback, enabling students to explore mathematical relations in ways that

foreground underlying structure (Weingarden et al., 2024). As students manipulate variables and observe the effects of these changes, their early reasoning may remain grounded in the discourse of the dynamic digital environment. Yet, these interactions can become a productive starting point for transforming experiential narratives into more formal ones. For example, students begin to reinterpret actions as mathematical relations, assign symbolic names to quantities, and coordinate these relations across different representations (Schüler-Meyer, 2022). Such coordination is foundational for entering formal discourse, where quantified statements like "for every $\varepsilon > 0$ there exists an n…" are no longer experienced as unfamiliar formalities but as logical reorganisations of previously encountered relationships.

The present study investigates these shifts in the context of a dynamic environment designed to foreground the structure of quantified reasoning. We examine how students' narratives evolve as they move from describing actions and perceptual features toward articulating logical relations between quantities. In particular, we analyse how students construct and connect different realisations of the target concept and how their discourse gradually indicates shifts from experiential or algebraic descriptions to logically organised, quantified formulations.

THEORETICAL CONSIDERATIONS

Realisation Tree

Our study situates the design and interpretation of tasks in dynamic digital environments within commognition (Sfard, 2008). In commognition, learning mathematics is conceptualised as a change in a discourse, aiming to become a participant in a certain mathematical community. A mathematical discourse is set, among others, by its mathematical objects (Sfard, 2008). Mathematical discourses are autopoietic, with mathematical objects built upon previously endorsed ones.

Realisation Trees (RT) (Sfard, 2008) represent the network of possible ways a mathematical object can be realised and the relations among them, and can be used as a visual and analytical tool. Each branch of the tree shows different realisations of a mathematical object, and can be seen as belonging to a discourse, such as geometric, numerical, algebraic, and logical. RTs visualise the potential connections that can be made across and within the realisations in the branches (Weingarden et al., 2024). These connections are defined as saming narratives (SNs) and refer to the mapping of (parts of) a mathematical object to its equivalent in the different discourses. The students' SNs reveal how they move across or within different discourses. For instance, the Archimedean Property does not pre-exist in the discourse but emerges as students talk about and act upon their various realisations. Therefore, engaging with mathematical discourses involves becoming familiar with different realisations of a mathematical object.

Learning mathematics, thus, requires not only encountering different realisations of an object, but also recognising their coherence through *saming*. Saming refers to the process of mapping hitherto separate realisations of an object, possibly in different

discourses such as geometric and algebraic, into one (Sfard, 2008). Through saming, students move from context-dependent interpretations toward more unified and abstract conceptions, a shift that is central to participation in mathematical discourse. In STT, saming is particularly important because it marks students' shifts from empirical realisations of narratives about mathematical objects toward logical ones that rely on definitions, logical deduction, and quantification. RTs map these shifts by visualising students' navigation of the branches. Also, they can highlight the saming narratives through which students align branches of intuitive realisations to logical ones. This can be seen as horizontal movement between the branches of the RT. By examining how students construct saming narratives across the RT, we can trace how their reasoning begins to shift toward the logical branch of the RT, indicating early steps in entering the logical discourse that is central to tertiary mathematics.

When implemented in dynamic digital environments such as GeoGebra, these connections can be enacted and observed directly. For instance, dragging sliders or manipulating shapes makes visible how students shift between discourses, moving from visual comparison in the dynamic digital environment to algebraic formulation or from empirical observation to logical justification. The RT thus provides a tool for designing and analysing digital tasks: it helps predict which realisations are made available in the environment, what SNs between them can occur, and how SNs may develop as students coordinate these realisations in real time, navigating different discourses.

Task Design

In recent years, dynamic digital environments have been integrated into university mathematics teaching and learning to foster precisely the moves across and within discourses. The integration of dynamic digital environments in university mathematics shows promising results, as they support students in interpreting and later proving mathematical objects (Zengin, 2022; Yorganci, 2023). For example, Zengin (2022) focused on the proof of the Fundamental Theorem of Calculus using dynamic digital environments for third-year university students. The students engaged with GeoGebra to visualise the accumulation of area under a curve, using the software's dynamic feedback to explore the dependence between differentiation and integration and to reconstruct the formal proof of the theorem through these visual–symbolic connections. However, while such studies highlight the affordances of digital dynamic environments for bridging empirical and formal reasoning, the emphasis on how these environments may support students in navigating connections between discourses during STT, especially in introductory university courses, is limited.

In the present study, we designed a task around the Archimedean Property: S(ε,ν): $\forall \varepsilon > 0, \exists\, \nu \in \mathbb{N}$ such that $\nu\varepsilon > 1$, in GeoGebra, aiming to provide first-year students with opportunities to explore different discourses and construct saming narratives within and across these discourses. Two students, the falsifier and the verifier, interacted with sliders representing ε and ν, in a game-based format. The falsifier's

goal was to choose an ε for which no ν could make the inequality $\nu\varepsilon > 1$ true, while the verifier aimed to find an ν large enough to satisfy it. Visually, the condition was modelled by two areas: a blue square of area 1 and a green rectangle of area $\nu\varepsilon$. The statement $S(\varepsilon, \nu)$ held whenever the green rectangle became larger than the blue square, meaning that the verifier successfully countered the falsifier's move (Figure 1).

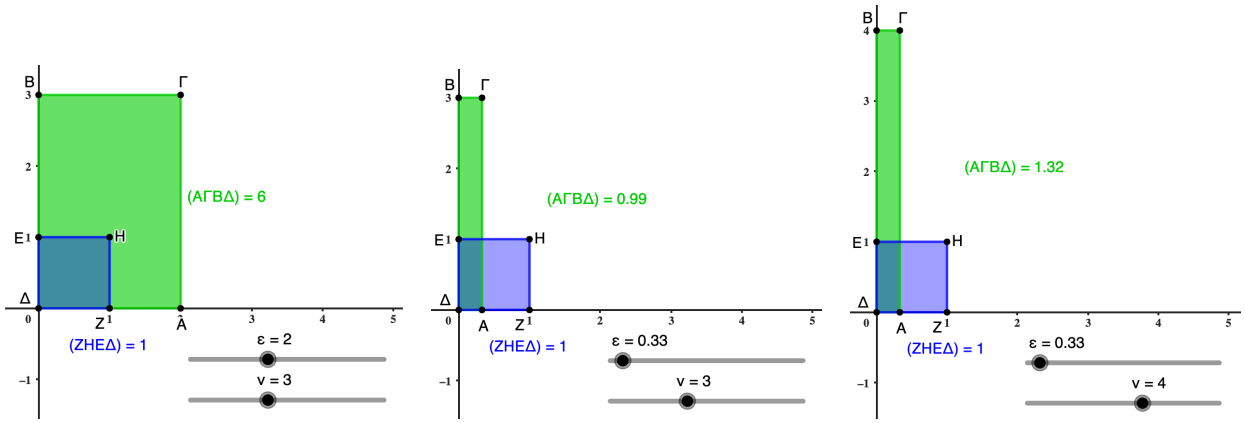

Figure 1: Hypothetical rounds of the task.

This dynamic environment provided immediate visual feedback, allowing students to experience the dependency $\nu = \nu(\varepsilon)$ and to reason while interacting with the environment. In this way, the task situates the Archimedean Property within multiple realisations across interconnected discourses: the informal discourse of a game and beating your opponent, the dynamic digital environment discourse of the task that gives visual feedback, the geometric discourse of magnitudes and areas, the numerical discourse of repeated addition, the algebraic discourse of inequalities, and the logical discourse of quantification. By engaging with these discourses simultaneously, the aim is to support students in connecting intuitive and formal reasoning by constructing saming narratives.

Research Question

Drawing on this theoretical and design perspective, the present study investigates how first-year university students engage with and connect multiple realisations of the Archimedean Property in a dynamic digital environment. We ask: *How does the interaction with a digital dynamic task on the Archimedean Property afford saming narratives that support students' transition within and across mathematical discourses, from empirical to the intended logical discourse that centers quantification?*

METHOD

This study is part of a design study exploring how tasks in dynamic digital environments can support students' STT in mathematics. The present study is a pilot implementation, aiming to examine how two students, S1 and S2, engage with and connect different realisations of a mathematical object through interaction with the task. The study took place in Greece, where mathematics curricula in upper secondary

education place considerable emphasis on geometry and calculus. At the time of data collection, the students were first-year mathematics students who had recently begun engaging with first-year, university-level analysis courses.

We collected data from the interaction of the two students who were involved with the task centred on the Archimedean Property. The first author was the facilitator during the data collection, introducing students to the task, setting up the audio recording, and capturing the students' interactions with GeoGebra alongside their spoken dialogue. The duration of the implementation was 45 minutes. All students participated voluntarily, and gave written informed consent. The audio-recorded data were transcribed and translated from Greek to English, while snapshots from the interactions with GeoGebra were added to the transcript.

The goal of the analysis was to identify the learning opportunities towards logical reasoning by analysing the saming narratives within and across discourses using commognition. We examined the students' narratives, the series of utterances about mathematical objects (Sfard, 2008), as they interacted with the task, focusing on how their narratives reflected realisations in different discourses. For instance, when the students discussed the interdependence between the variables ε and ν, their utterances were characterised as interdependence narratives within the algebraic realisation, indicating engagement in the algebraic–logical discourse. Words such as "for every," "there exists," and "always" signalled their movement into the logical discourse, where they reasoned about quantifiers and the truth conditions of the statement. At the same time, we focused on how these realisations were connected through saming narratives. For example, the statement *"the product of ν and ε is the area of rectangle $AB\Gamma\Delta$"* (Greek for $ABCD$) connected the algebraic–logical and geometric discourses, illustrating a transition between them. Tracing such connections allowed us to visualise how students navigated the branches of the Realisation Tree and to identify the learning opportunities afforded by the dynamic environment for moving towards logical and quantified reasoning while investigating the Archimedean Property.

FINDINGS

Realisation tree for Archimedean Property as first design outcome

First, we exemplify how the Realisation Tree (RT) for the task in GeoGebra is constructed. The task's goal is to support first-year students to move to formal discourse through endorsing saming narratives that move them from informal to quantified realisations. The students were asked to explore "how to choose the number ν so that the area of the square ΑΒΓΔ (ABCD) is greater than 1, the area of the square ΕΗΖΔ (EHZD)," as appears in Figure 1. Solving the task involves exploring the relation between ε and ν in the context of the Archimedean Property (AP). Thus, the AP serves as the central node of the RT, from which several branches extend, each representing a distinct way in which the property can be realised and experienced in the task.

The RT illustrates how students engage with different discourses as they explore the task. Each branch represents a distinct way of realising the AP through the digital

environment, such as manipulating dynamic sliders, describing numerical relations, or formulating formal logic and algebraic expressions. These branches capture the diversity of visual mediators and narratives that students might produce as they move between experiential and formal ways of talking about AP. The dynamic affordances of GeoGebra invite students to test, adjust, and compare different realisations while receiving immediate feedback. Thus, the RT provides a structured overview of the potential learning opportunities embedded in the task, showing how the designed task can support transitions within and between discourses and the construction of saming narratives that link visual, numerical, and symbolic aspects of the AP.

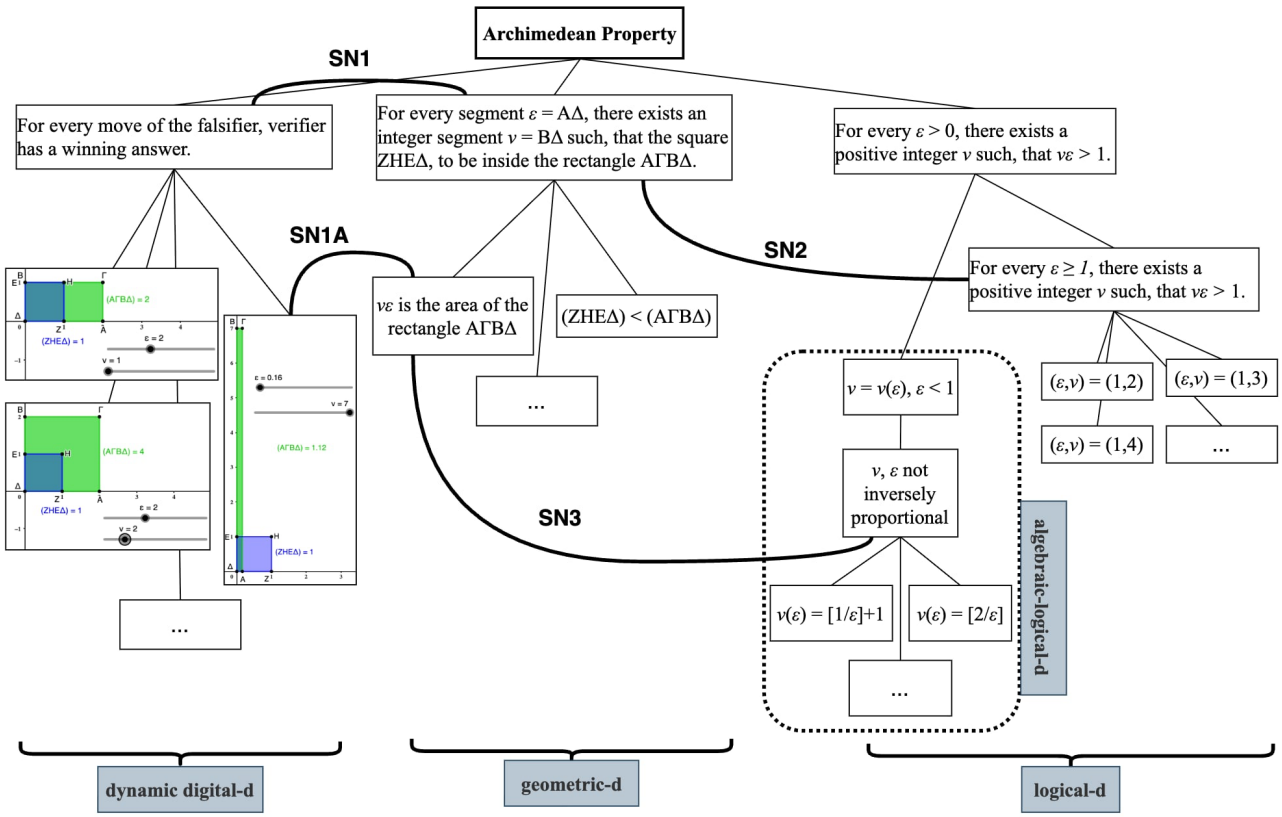

Figure 2: The Realisation Tree for the AP.

The first branch draws on the dynamic digital environment discourse, where students use the sliders to beat their opponents' chosen ε. They explore winning strategies and observe the verifier's systematic success. The second branch extends this experience to a geometric discourse, where the same relationships are expressed through measurable magnitudes, height (ν) and length (ε), and their product, the rectangle's area $\nu\varepsilon$. The third branch leads students toward the logical discourse of quantification, where they are encouraged to encapsulate the empirical patterns discovered earlier into the quantified statement $\forall\varepsilon > 0, \exists\, \nu \in \mathbb{N}$ such that $\nu\varepsilon > 1$. In this way, the branches of the RT represent the designed pathway from intuitive to formal reasoning, illustrating how different realisations of the AP emerge in the interconnected discourses [Figure 2].

Saming narratives in the analysis of the task

The two students have already interacted with the task for about 20′ and they have observed that the verifier always wins. At the start, they acted as opponents, and their aim was just to win against each other, but then they decided that they would collaborate and work together to find a way to win the task.

1 S2 ν is the height, ε is the length.

2 S1 Correct.

…

30 S2 Is it the famous ε? For every ε, there is ν... [silently] If we move ε to the left, say 0.24, only then will it become smaller.

31 S1 Right. Put ε on 0.22 where you found it before.

32 S2 Ok. Let's see for what values of ν, it's done each time. That is, if ν is 4, then ε must be 0.24 to make it smaller.

33 S1 Right. Let's see. Let's reduce it to 3.

34 S2 If ν is 3 then ε should go somewhere between 0.33 and 0.34.

…

49 S1 It doesn't make sense in general. The verifier always wins.

50 S2 No matter what ε we take, you can always find an ν that does the job. If we put ε on 2, you can see then, that for all ν, the green [area] is bigger. If we put it, let's say, at 0.66 there you can find some ν that prevents it from getting bigger.

51 S1 In general, the smaller the values for ε, the better.

52 S2 If ε is smaller than 1 then you can find an ν that doesn't make it bigger. So, its limit value is probably at 1. So, ε must be below 1, so that we can play the game.

The excerpt illustrates a sequence of saming narratives across branches of the Realisation Tree, from the dynamic digital environment to the geometric [SN1, SN1A] and from geometric toward the formal discourse [SN2, SN3]. Initially, the students' narratives are situated within the rules of the game, where "winning" is interpreted procedurally as the verifier's repeated success (dynamic digital discourse). As they begin collaborating and assigning measurable attributes to the game elements ("ν is the height, ε is the length"), their narratives transition from describing actions to defining geometric quantities, reinterpreting the game dynamics as relationships between magnitudes (geometric discourse). Later, the students engage in quantifier sense-making through reasoning across two ε-cases, connecting with the emerging definition form of the AP ($\forall \varepsilon > 0, \exists\, \nu \in \mathbb{N}$ such that $\nu\varepsilon > 1$) (logical discourse). The saming narratives mark the move from the game's procedural logic to a formal realisation of the Archimedean Property, where variation in ε and ν becomes central.

Each saming narrative opens up possible learning opportunities for the students to shift to distinct realisations within the discourses. SN1 enables students to move *from strategic gameplay to geometric reasoning*: the identification of ν and ε as measurable magnitudes, "ν is the height, ε is the length" (line 1), allows them to reconceptualise winning strategies not as moves but as proportional relationships. SN1A extends this opportunity by stabilising these geometric quantities as they explore how adjusting ε affects outcomes: for instance, S2's comment, "If we move ε to the left, say 0.24, only then will [it] become smaller" (line 30), shows the shift toward interpreting the game's dynamics as comparisons of magnitudes rather than isolated moves.

SN2 indicates a learning opportunity to link geometric quantities to variable co-variation, moving *from spatial reasoning to a formal definition*. As students test specific (ε, ν) pairs, "If ν is 4 then ε must be 0.24 to make it smaller" (line 32); "If ν is 3 then ε should go somewhere between 0.33 and 0.34" (line 34), they begin to treat coordinated change as a structural feature of the situation, moving from spatial reasoning to a first move towards formal understanding of dependency. Finally, SN3 supports the transition *from reasoning with examples to reasoning with quantified relationships*. Statements such as "No matter what ε we take, you can always find a ν that does the job" (line 50) and "If ε is smaller than 1 then you can find a ν that doesn't make it bigger" (line 52) indicate that the students are beginning to articulate general statements, showing the development of quantifier sense-making and the emerging objectification of the ε-ν relation in the algebraic-logical discourse.

Together, these saming narratives show how interactions within the task give rise to learning opportunities that progressively allow students' entry to different discourses, ultimately to an early formal understanding of the Archimedean Property.

DISCUSSION

This study explores how the interactions with a designed dynamic digital task can afford students' saming narratives for the transition within and across discourses. The RT functioned as both a visual and an analytical tool, making visible how students' participation evolved from engagement in the task to logical discourse that centres quantification. Within the dynamic digital environment, the task served as a mediational bridge between intuitive and formal reasoning, enabling students to gradually navigate across and within discourses.

In the excerpt, students' the four identified saming narratives revealed learning opportunities that were not explicitly built into the task but developed through their interaction. For example, the students produced saming narratives that helped them shift *from strategic gameplay to spatial reasoning,* allowing students to reconceptualise winning strategies as proportional relationships between magnitudes. Similarly, another saming narrative allowed the move *from spatial reasoning to formal definition,* suggesting the transition from the geometrical to formal discourse. Last, another saming narrative focuses on the move *from reasoning about examples to*

reasoning with quantified relationships that appeared within a formal discourse and showed the development of quantifier sense-making.

These findings resonate with recent work on the potential of dynamic digital environments to promote continuity between empirical and formal reasoning (Weingarden et al., 2024; Wallach et al., 2022). However, the present analysis extends this by showing how such continuity can be captured and visualised through the RTs. The RT made it possible to trace how students' saming narratives evolved from focusing on game procedures to articulate quantification. In commognitive terms, these transitions reflect learning as a change in discourse, with the saming narratives allowing the exploration of discourses that emerge for the students.

Furthermore, the findings offer initial implications for task design in the context of the secondary–tertiary transition. By embedding multiple discourses, digital dynamic environment, geometric, and logical, within a single task, the task supported flexible movement across realisations of the same mathematical object. Similar to Weigarden et al. (2024), the digital dynamic environment thus served not merely as a representational medium but as a discursive mediator that may facilitate students to experience different realisations of the mathematical object.

Nevertheless, this study is a pilot attempt, and it is limited to a small sample and focus on a single task and mathematical object. The analysis, while providing in-depth insight into saming narratives, cannot account for broader variations in students' participation across contexts, topics, or institutional settings outside Greece. Future research could therefore expand on these findings by exploring how saming narratives unfold across different mathematical domains, by examining how teachers' interventions can facilitate students' construction of saming narratives, or by longitudinally tracing how such realisations contribute to students' sustained participation in university mathematics.

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Cultivating undergraduate students' agentive participation in mathematical discourse when ChatGPT is involved

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Generative Artificial Intelligence is becoming a transformative factor in university mathematical experiences. In this paper, I endorse a commognitive lens to discuss undergraduate students' agentive participation in mathematical discourse when ChatGPT is involved. I present an activity that involves problem-solving and reflection on a teaching/learning situation in which a fictional student interacts with ChatGPT to solve the same problem. Based on the responses of two students, I propose manifestations of agentive participation that we would like our students to cultivate.

Keywords: Commognition, agentive participation, GenAI, ChatGPT, problem-solving

INTRODUCTION

Digital resources and especially Generative Artificial Intelligence (GenAI [1]) are becoming an essential part, and a transformative factor, of university mathematical experiences (Geraniou et al., 2024). Especially, since the release of ChatGPT [2] in 2022, and its rabid integration to institutional and individual practices, both researchers and policy makers were mobilised to investigate the impact, in terms of limitations and potentialities (e.g. for the impact of GenAI in higher education, see the review of Bond, et al., 2024 and for the impact of ChatGPT in mathematics education, see the review of Pepin et al., 2025). So far, studies highlight the importance of students - GenAI interactions and what students expect from the GenAI to do (or not do) for them (e.g., (Zhuang, 2025) and how they interact with GenAI (e.g., Yoon, 2024). Emerging studies observe discursive moves in the interaction with GenAI, that see GenAI as a conversational partner (Gíslason, 2025). Overall, studies report an urgent need of guidance for undergraduate students towards a critical engagement with GenAI tools, rather than passively accepting their outputs (Yoon, 2024; Zhuang, 2025). In this work students' critical engagement in the context to the mathematical learning is seen from a *commognitive perspective* (Sfard, 2008) through students' *agentive participation* in mathematical discourse when digital resources are involved. This paper aims to contribute to the discussion around the cultivation of students' agentive participation when GenAI (here ChatGPT) is involved in mathematical problem-solving activities.

CONCEPTUALISING PARTICIPATION IN MATHEMATICAL DISCOURSE WHEN DIGITAL RESOURCES ARE INVOLVED

The theoretical perspective of this work is discursive and draws on the commognitive framework (Sfard, 2008), according to which mathematical practices, including teaching and learning, are seen within a community and its discursive activity. Mathematical discourse is characterised by *used words*, *endorsed narrative*, *routines*, and *visual mediators* (Sfard, 2008). However, discourses of other disciplines, e.g.

science or engineering, as well as pedagogical discourses, e.g. teaching of mathematics, are also involved and interact with mathematical experiences. In the context of this study learning is the process through which students gradually become capable of employing mathematical routines they encounter in their university studies through their *agentive participation* (Nachlieli & Tabach, 2022) in university mathematical discourse – namely, "the freedom and initiative performers exercise in deciding what should be said and done" (p. 4). In previous works I/ argued that the availability of digital resources may alter the engagement with mathematical tasks (Biza, 2024; Biza & Nardi, 2023b). In such situations, students' *agentive participation* in mathematical discourse is seen "in the initiative students exercise as they decide what, when and how they use a digital (or not) resources in their mathematical activity" (Biza, 2024, p. 733). With the spread and the availability of GenAI, now it becomes even more vital to open the discussion around how such agentive participation can be cultivated.

In previous works, and in order to explore how online search routines are integrated into undergraduate students' mathematical activity, we invited students "to solve a problem and then reflect on said solutions in the light of how others (other undergraduates or school children) have solved the problem" (Biza & Nardi, 2023b, p. 2284). In this paper, I draw on a similar approach to trigger undergraduate students' reflections on the impact ChatGPT has on problem-solving activities. The approach follows the design and principles of the MathTASK programme. In MathTASK, we design, and we engage, participants with fictional but realistic classroom situations which we call "*mathtasks*" (Biza & Nardi, 2023a,b). *Mathtasks* are presented as short teaching/learning situation where a teacher and student(s) deal with a mathematical problem. Participants are invited to solve the mathematical problem, and then, to interpret and respond to the teaching/learning situation. In this paper, the situation includes interaction (and explicit input) from ChatGPT. Undergraduate students are invited not only to solve the problem, but also to reflect on the response from a fictional student, Peter, who articulates his response through his exchanges with ChatGPT. I analyse participants' responses to the mathematical problem also in the light of the issues they identify in the fictional student's response as well as how they interpret and address those issues to investigate the research question: *To what extent, and in what forms, is undergraduate students' agentive participation in mathematical discourse manifested in their responses when ChatGPT is involved?*

METHODS, CONTEXT AND PARTICIPANTS

The study was conducted in the context of a Mathematics Education course (*The Learning and Teaching of Mathematics*) for finalist (Year 3) students of Bachelor of Science programmes (BSc in Mathematics and BSc in Physics). The course introduces students to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum (Biza & Nardi, 2023a). The course is assessed through a *Portfolio of Learning Outcomes* that involves, amongst others, the use of mathematics education research literature to reflect on teaching/learning situations (*mathtasks*): identifying issues in those situations, interpret those issues and

responding to student needs. Such activities were used also in the weekly sessions of course. In this study, I draw on the responses to ONE mathtask (Figure 1) from two students, Leighton and Nataliia both finalist of a BSc in Mathematics course, out of the nine who consented their work to be used for research purposes.

Methodologically I approach the study as a case of multiple components consisting of the students of the course; the course they attended; and, the mathtask they were invited to engage with. I am interested in the routines students engage and the narratives they endorse in the context of the *task situation* of the mathtask. Namely, I endorse Lavie et al.'s (2019) contextualization of a routine within task situations, namely "any setting in which a person considers herself bound to act—to do something" (Lavie et al.'s, 2019, p. 159) and to a particular person who performs this routine. In the next sections, first, I present the mathtask with attention to the routines and endorsed narratives in ChatGPT's and Peter's interactions. Then, I present Leighton's and Nataliia's discursive activity in the light of their engagement with the mathtask.

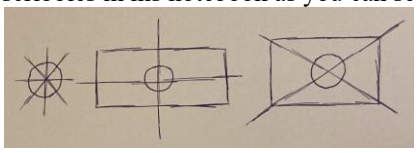
MATHTASK: PETER'S WORK WITH CHATGPT ASSISTANCE

The context of the learning/teaching situation in the mathtask (Figure 1) is a Year 11 (students aged 15-16) lesson on symmetry. The context was kept modestly open to allow participants flexibility with the Year 11 curriculum. The fictional student (Peter) engages with some exploration shown in the scribbles on his notebook and then posts the problem on ChatGPT. ChatGPT's response and the follow up exchanges were video-recorded, saved on a shared folder and provided to the students through a weblink (see Appendix). In comparison to previous *mathtasks,* here GenAI (in our case ChatGPT) acts as a legitimate interlocutor, a conversational partner (Gíslason, 2025) that interacts with Peter and influences his response. ChatGPT first suggests how "you can conceptualize the solution" in an accurate notation and a clear description of the shapes (and their areas) that are involved. Then, in the "Practical Solution", ChatGPT recommends that the "center of mass (centroid) of the paper-hole system" should be found. The use of "centroid" shifts the solution to routines that are more common in Physics and not yet seen at the Year 11 curriculum. Geometrical narratives are also part of ChatGPT recommendations: "Draw a line through the center of the mass that divides the rectangular area into two equal halves" and "[e]nsure this line also divides the circular hole's area symmetrically". ChatGPT's response, although well phrased, combines interrelated but not same routines of "halving" the area and identifying "symmetry". When Peter asks for a diagram, ChatGPT provides a specific case of the problem; when the centre of the circle is at the centre of the rectangle. When Peter asks for "any other positions of the circular hole", ChatGPT distinguishes cases and concludes that the line must "[s]plit the rectangular area into two equal parts" and "the circular hole into two equal parts". ChatGPT's logical steps, especially regarding the different positions of the circle lack specificity and consistency: "near the edge" or "closer"; the fourth case indicates that part of the hole might be outside the rectangle [3]; the position of the line is not specified (and not substantiated) and the existence of other lines is not investigated.

In a lesson on symmetry, the teacher of a Year 11 class gave to the students the following problem:

A rectangular piece of paper has a whole circular hole somewhere on it. Draw one straight line and cut the piece of paper across this line so the two pieces you will get will have the same area. Give a solution to the problems for different positions of the circular hole. Think if there are more than one straight lines, which can solve the problem. Justify your response mathematically and show your working in your notebook.

Students are allowed to use their tablets for this task and Peter, one of the students, starts working on the problem individually. First, he does some quick scribbles in his notebook as you can see below:



He scratches his head and turns to his tablet. He switches on the ChatGPT application, and he types: "Solve the problem: A rectangular piece of paper has a whole circular hole somewhere on it. Draw one straight line and cut the piece of paper across this line so the two pieces you will get will have the same area." The exchanges of Peter and ChatGPT can be seen in the video [4].

Then he turns to his notebook and writes:

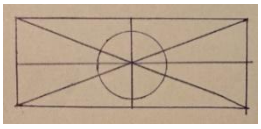
I must divide the entire area of the paper, including the circular hole, into two equal parts with a straight line. The area of the rectangle is A_{rect} and the area of the hole is A_{circle}. The remaining area is $A_{rect} - A_{circle}$. I have to split this area of the rectangle and the area of the circle into two equal parts: $\frac{1}{2}(A_{rect} - A_{circle})$.

The line divides the rectangular paper into two regions, each with an equal amount of rectangular area and an equal fraction of the circular hole's area. I must find the centre of mass (centroid) of the paper-hole system, considering the hole's negative area. The line should go through the centre of mass that divides the rectangular area into two equal halves and also divides the circular hole's area symmetrically (this happens naturally due to the line passing through the centroid).

Any line passing through the centre of the hole and dividing the rectangle into two equal halves (e.g., a horizontal, vertical, or diagonal line through the centre) will work.

The choice of the line depends on the relative location of the circle and the centroid of the combined system.

I found four options that can be seen in the diagram below:



You are the teaching assistant of the class, and you saw what Peter did …

Questions (in each of the items below justify your response, also with references to the Mathematics Education research literature, where needed):

1. Give your answer to the problem and explain your answer mathematically.
2. What may the teacher's aims be for giving the students this problem?
3. What issues do you see in Peter's approach to the problem?
4. What would you say to Peter?

Figure 1: Peter's work with ChatGPT Assistance (*mathtask)*

When Peter asks for a diagram again, ChatGPT, provides an image with the circle at the centre of the rectangle, this time in 3D. It claims that "the circular hole [is] in an off-center position" though. ChatGPT follows a probabilistic model and the (lack) of accuracy of the response is not surprising. There are elements in the response, beyond this (lack) of accuracy that may trigger participants' attention: (a) the vagueness of the response; (b) interdiscursive shifts between mathematical (symmetry, area) and

physics (centroid, centre of mass) of the discussion around the centroid together with the symmetry; c) misleading visual mediation in both provided diagrams; d) the (non) relevance of the response to the Year 11 context of the situation; and, e) the presentation of false or nonsensical information as mathematically correct (see notation, structure of the argument, word use). Peter's response replicates the above characteristics in an uncritical use of ChatGPT's response. Participants are invited to solve the problem, i.e. to engage with the mathematical activity themselves; to identify teacher's aims for using this problem, i.e. to see the problem in a teaching context; to identify issues in Peter's actions; and, to respond to Peter.

LEIGHTON'S AND NATALIIA'S RESPONSES

Give your answer to the problem and explain your answer mathematically

Leighton responds to the problem with reference to the ChatGPT video. His response draws on mathematical narratives around "symmetry", "rectangles", "lines", etc. that students learn in GCSE (General Certificate of Secondary Education) mathematics (14–16-year-olds). Although Leighton's description is not precise, he engages with mathematical routines appropriate for GCSE students and supports his argument with appropriate visual mediators (Figure 2):

> There are two cases which are mentioned in the ChatGPT video, namely: when the centre of the circle lies at the same point as the midpoint of the rectangle and when it doesn't. So, the area of both a rectangle and a circle are equal on either side of any line of symmetry, students learn this during GCSE. The area of the paper with the cut-out circle needs the area of the cut-out section equal on either side, along with the area of the rectangle equal on either side of the same line. Any straight line that goes through the centre of the circle will be symmetrical [sic] and for this line to make two equal areas by splitting the rectangle, it must pass through the midpoint of the rectangle.

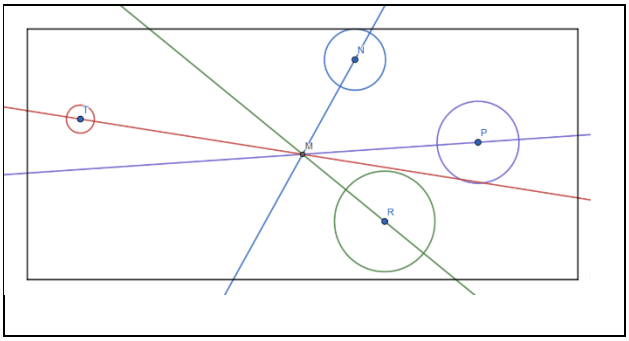


Figure 2: Leighton's drawing

Nataliia, also approached the problem for different positions of the circle:

> In conclusion, if the circle is symmetrically placed (e.g., at the center of the rectangle), horizontal, vertical, and diagonal lines through the centre will work. If the circle is somewhere on the rectangle, then only specific lines passing through the centre of mass which I have showed above [Figure 3].

However, her approach requires finding the "centre of mass":

For a straight line to divide the paper into two equal size, it has to cut through their center point. The centre of mass for a uniform rectangle is (L/2, W/2) which is evenly distributed along both axes. And for a circle, the centroid is its centre. Hence, any straight line passing through the centroid of the composite area, will divide the paper into two regions with equal total area.

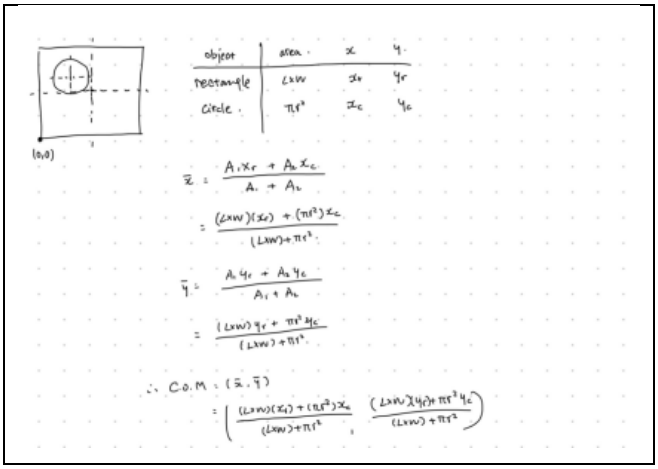


Figure 3: Nataliia's working

Nataliia's response draws on narratives, such as centre of mass, centroid, composite area – and routines – such as identification of centroid (Figure 3) that are endorsed by the Physics discourse, as ChatGPT did, but not seen in the Year 11 curriculum. The response to the next question gives more insight into Leighton's and Nataliia's meaning about the problem.

What may the teacher's aims be for giving the students this problem?

Leighton's response is consistent to his approach to the problem and to the context of the teaching/learning situation as the word use of "symmetry", "geometry" and "problem solving" indicates:

> The teacher's goal is to enhance students' understanding of symmetry and area in geometry. The task encourages students to explore how symmetry can divide shapes with holes, fostering problem-solving and critical reasoning. It requires the students to expand their understanding beyond simple shapes, introducing more complex geometric scenarios.

While in Nataliia's response to the same question aims are presented in a bullet list, some of them are quite generic and aspirational: "[t]he task encourages students to discuss their reasoning and justify their solutions, reinforcing mathematical communication". Some of the aims are not relevant to Year 11 context of the incident: "the problem mirrors real-world applications of centroids in physics and engineering, such as balancing systems or designing structures". It seems that the mathematical problem and its didactical potential are discussed by Nataliia through *ChatGPT's* input, and not in the context of the context of mathtask's learning/teaching situation.

We will close now with what Leighton and Nataliia see as issues in Peter's response and what they would say to him. I present the responses to those two questions together to avoid repetitions.

What issues do you see in Peter's approach to the problem? What would you say to Peter?

Leighton appreciates Peter's initial explorations with drawings. Then, he notices that Peter posts the entire problem on ChatGPT and claims that ChatGPT introduces "a complex concept prematurely [for Peter]", then he adds that:

> the introduction of centre of mass on top of the geometry, symmetry and problem solving will cause a year 11 to get cognitive overload. In the video Peter asks the AI whether the circle can occur elsewhere than the centre of the rectangular piece of paper, which the AI responds to with "yes"; however, in Peter's response he hasn't included this, therefore overgeneralising his response.

In this response to Peter, Leighton attempts bringing him back to the Year 11 curriculum, however, with no further reflection on the impact of ChatGPT in Peter's response and learning experiences:

> I would tell Peter that the concept of centroid is something that we don't cover at GCSE and that if he is interested in the concept then it will be covered in A-Level, then I'll ask about what is the line of symmetry for a circle, this is to see whether he has the understanding that any straight line that goes through the centre of a circle is a line of symmetry.

Nataliia in consistency to her earlier response, especial about "centroids", commends Peter's "[u]nderstanding of [k]ey [c]oncepts": "Peter correctly identified the need to calculate the centroid and understood the importance of dividing both the rectangle and the circular hole" and his "[l]ogical [r]easoning: [Peter] realised that the line must pass through the centroid to divide the areas equally". She also, identifies "areas for improvement", such as "[o]ver-reliance on AI technology: Peter switched to ChatGPT early in the process, potentially limiting his independent problem-solving"; "[i]ncomplete [e]xplanation: while Peter mentioned key concepts like the centroid and negative area, his explanation lacked clarity and detail"; "[l]imited [e]xploration: he identified only four possible solutions and didn't explore whether other lines might work for different placements of the hole"; [n]otation and [c]larity: his mathematical language was inconsistent, mixing casual and formal terms, which might confuse others". The issues are presented again in bullet list and are quite descriptive with no attempt for interpretation.

DISCUSSION

In this paper, I discuss undergraduate student agentive participation in mathematical discourse when GenAI (in the case here ChatGPT) is involved through the investigation of the question: *To what extent, and in what forms, is undergraduate students' agentive participation in mathematical discourse manifested in their responses when ChatGPT is involved?* Such participation is seen though responses to fictional teaching/learning situation (MathTASK methodology, Biza & Nardi, 2023a) in which ChatGPT appears as a conversational partner (Gíslason, 2025). Observations

in the responses of two undergraduate students, Leighton and Nataliia, indicate manifestations of agentive participation (or lack of those manifestations). In summary manifestations include respondents' *acceptance* (or not) of the *legitimacy* of the input of ChatGPT as an interlocutor in the problem-solving activity, and their *critical interrogation* (or not) of this input. Also, we observe respondents' *attention* (or not) to the *logical gaps* and the *lack of specificity* of ChatGPT's response, especially across prompts. Further to the previous observation, another manifestation relates to whether respondents' *questioning* (or not) the *reliability* of ChatGPT responses, especially against hallucinations. A final manifestation regards respondents' *contextualisation* (or not) of ChatGPT within the teaching/learning situation. ChatGPT's problem-solving is not necessarily context specific, it may draw on any discourse or across discourses. Such interdiscursive flexibility may generate opportunities, e.g. by proposing alternative approaches. However, it may also create conflicts when such alternative approaches cannot be accessed by the solver or if they are out of context. Of the two examples of undergraduate students presented here, Leighton's response demonstrates more manifestations of agentive participation in comparison to Nataliia's. I see Leighton having more agency of what (and why) he endorses from ChatGPT's input.

NOTES

1. Generative Artificial Intelligence is a type of Artificial Intelligence (AI) that creates new content, such as text, images, and programming code, by learning patterns from large datasets.

2. ChatGPT is a generative artificial intelligence application (chatbot) that uses Generative Pre-trained Transformer (GPT) technology to generate text, speech, and images in response to user prompts. It is developed by OpenAI and released to the public in November 2022 (https://chatgpt.com/).

3. Having the circle partly inside the rectangle is an interesting problem, which goes beyond the problem discussed here.

4. For the purpose of this task, a link to the video was provided of the screen videorecording of the exchanges with ChatGPT. In this paper those exchanges can be found in the Appendix.

AKNOWLEDGMENT

I would like to thank the Research in Mathematics Education group at the University of East Anglia and, especially, Elena Nardi for their ongoing support. Finally, I am grateful to the students who chose my course, their responses are always an inspiration.

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APPENDIX

Screenshot of the dialogue with ChatGPT (based on GPT-4 technology), free version, accessed in November 2024.

Solve the problem: A rectangular piece of paper has a whole circular hole somewhere on it. Draw one straight line and cut the piece of paper across this line so the two pieces you will get will have the same area.

To solve this problem, the key insight is to utilize the concept of dividing the entire area of the paper, including the circular hole, into two equal parts with a straight line. Here's how you can conceptualize the solution:

1. **Understand the Total Area**: Let the total area of the paper be A_{rect}, and the area of the circular hole be A_{circle}.
   - The effective area of the paper is $A_{\text{effective}} = A_{\text{rect}} - A_{\text{circle}}$.
2. **Objective**: Divide $A_{\text{effective}}$ into two equal parts using a straight line. Since $A_{\text{effective}}$ is symmetric about the cut line, the hole's contribution to the two halves must also balance.
3. **The Key Step**: Choose a straight line such that:
   - The line divides the rectangular paper into two regions, each with an **equal amount of rectangular area and an equal fraction of the circular hole's area.**

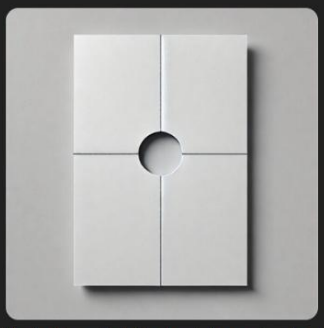
Practical Solution:

- Find the **center of mass** (centroid) of the paper-hole system, considering the hole's negative area.
- Draw a line through the center of mass that divides the rectangular area into two equal halves.
- Ensure this line also divides the circular hole's area symmetrically (this happens naturally due to the line passing through the centroid).

This ensures the two parts will have the same effective area.

If you want a diagram or specific calculations based on dimensions, let me know!

Can I have a diagram please



Here's a diagram illustrating the solution: a rectangular piece of paper with a circular hole and a straight line dividing both the rectangle and the circular hole into two equal parts. Let me know if you need further clarification!

Are there any other positions for the circular hole on the rectangular piece of paper?

Yes, the circular hole can be in various positions on the rectangular piece of paper, as long as the straight line can divide the paper and the hole into two equal-area portions. Here are some considerations for different positions of the hole:

1. Center of the Rectangle

- The hole is centered within the rectangular piece of paper.
- Any line passing through the center of the hole and dividing the rectangle into two equal halves (e.g., a horizontal, vertical, or diagonal line through the center) will work.

2. Off-Center Positions

- If the hole is not centered but lies entirely within the rectangle, there are still straight lines that can divide the areas equally. For these positions:
  - The line must pass through the **centroid** of the combined system (rectangle minus the circular hole).
  - The line must be oriented to cut the circular hole symmetrically (e.g., it might pass near or through the edge of the circle).

3. Touching an Edge

- The hole may touch one edge of the rectangle.
- The line should still pass through the centroid of the effective area (which shifts closer to the rectangle's center due to the loss of material).

4. Corner Positions

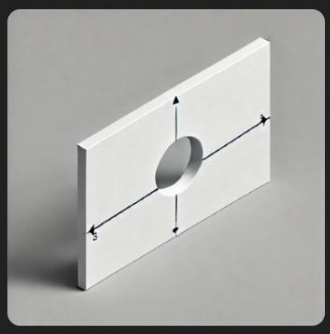
- If the hole is near or partially overlapping a corner, the dividing line must still ensure equal areas.
- This is geometrically possible but requires precise alignment of the line relative to the areas of the hole and the remaining rectangle.

General Rule

- For any position of the hole, the dividing line must:
  1. Split the **rectangular area** into two equal parts.
  2. Split the **circular hole** into two equal parts.
- The choice of the line depends on the relative location of the circle and the centroid of the combined system.

Let me know if you'd like to see a diagram of other configurations!

Can I have a diagram?



Here's a diagram showing the circular hole in an off-center position, with a straight line dividing the rectangular area and the circular hole into two equal parts. Let me know if you'd like further clarification or variations!

Digital Tasks and Automated Feedback - A Scoping Review

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Keywords: digital tasks, feedback, e-assessment.

MOTIVATION AND THEORETICAL BACKGROUND

The ongoing digitalization of university teaching presents digital task formats and tools providing opportunities and challenges for mathematics education. Digital tasks, such as interactive tasks, online homework, or e-assessments, are widely used to foster flexible, individualized learning opportunities. Automated and instant feedback enables a conceptual understanding and self-regulated learning.

In the last years, there has been much research on both digital tasks and feedback mechanisms in mathematics education. On the one hand, there is a lot of knowledge about how digital tasks can be designed, used, and developed for effective learning (Clarke et al., 2014; Kulik & Fletcher, 2016). On the other hand, there are theories regarding the effectiveness of feedback (Hattie & Timperly, 2007) used for many new approaches to design useful feedback.

However, integrative overviews of both digital tasks and feedback are lacking, leaving open questions about their combined educational functions: On which educational theories are the designs of digital tasks based, and which empirical findings support their effectiveness for student learning? By addressing those questions, we want to provide a comprehensive summary for educators about functions and evidence base of digital tasks.

RESEARCH QUESTIONS

Our scoping review seeks to demonstrate how the combination of digital mathematics tasks and automated feedback is conceptualized, implemented, and investigated in current mathematics education research. Our report is open to additional questions that may emerge during the review process. Core research questions include:

- What types of digital mathematics task formats and forms of automated feedback are being used and studied?
- What didactic functions do digital tasks have?
- How are digital tasks and automated feedback combined and integrated in learning environments?
- Which theoretical frameworks ground the analysis of digital tasks and feedback in higher mathematics education?
- Where are research gaps for effective integration of those elements?

METHODS

The review follows the PRISMA-ScR guideline for Scoping Reviews to ensure transparency and replicability. We systematically search multiple databases, including ERIC, Scopus, and PsycInfo, using Boolean search combinations such as (math* AND task* OR test OR feedback AND online OR digital).

The studies are eligible if they were in English and peer-reviewed articles published from 2015 onwards. They need to focus on mathematics education in higher or upper secondary education to ensure comparability in task formats and didactical functions. Relevant studies must either report empirical findings, conceptual analyses, or theoretical frameworks relevant to digital tasks. First, we exclude irrelevant papers while screening the abstracts and titles and in a second step the full texts. Findings will be synthesized using visual mapping techniques to highlight trends, relationships, and research gaps.

POSTER FOCUS

The poster will provide an overview of the methodological approach by briefly outlining the screening process as illustrated by a PRISMA flow chart. It will also present key findings from the mapping of current research, addressing the main questions mentioned above. Initial results indicate that didactic functions of digital tasks shift with a stringer focus on self-regulation and interactive engagement. Digital tasks are frequently used for communication between students and/or teachers about the mathematical content. While theoretical models highlight the importance of feedback that supports self-regulation, many implemented designs remain on a rather simple task level. Through the synthesis, we highlight the empirical research and provide structured typologies of digital tasks and feedback.

We will also point out research gaps and open questions that emerged during the review process. These gaps will be proposed as central points for (our) future research and serve as a basis for discussion among conference participants. By combining an overview of methods, theories and empirical experiences, the poster aims to contribute to the development of digital learning environments in upper secondary and tertiary mathematics education.

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Construction of the vector space concept through programming

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We report on a study situated within current discussions on computational thinking in mathematics education. The study contributes to research that uses computer programming to support students' engagement with proof and to deepen their understanding of vector spaces. Drawing on APOS Theory, we analyze how undergraduate students interpreted and modified Octave programs to test vector space axioms, with special attention to the mental constructions that emerged through their programming activity.

Keywords: Vector space, computational thinking, programming, APOS Theory.

Academic discussions in mathematics education over the years have emphasized computational thinking (CT) as a way of organizing mathematical activity through practices such as abstraction, decomposition, generalization, and algorithmic reasoning. Within this context, programming has been explored not only as a technical skill but as a medium for representing and manipulating mathematical structures. Programming environments allow students to operate explicitly on mathematical objects—such as sets and operations—while making their properties visible. This dual representational and operational nature creates opportunities for students to engage with mathematical ideas through action, reflection, and generalization (e.g., Dubinsky, 1984; Kallia et al., 2021; Papert, 1996; Tall, 1983; Weintrop et al., 2016).

From an APOS theory perspective (Arnon et al., 2014), programming activity can support the construction of mathematical knowledge by performing Actions, coordinating Processes, and encapsulating Objects. Executing or modifying code may be interpreted as Actions, while generalizing procedures reflects the development of Processes. When such Processes are treated as entities within computer programs, they may be encapsulated as Objects. In this sense, programming provides a setting in which key APOS mechanisms—such as interiorization and encapsulation—can be supported through the manipulation of mathematical objects.

CONSTRUCTING THE CONCEPT OF VECTOR SPACE

The concept of a vector space plays a central role in linear algebra, serving as a unifying and generalizing structure that encompasses a wide range of mathematical objects, including tuples, matrices, functions, polynomials, and sequences. However, this level of abstraction makes it difficult for students to understand this concept. Research in mathematics education has shown that students experience difficulties when working

with vector spaces, particularly due to the formal nature of the theory and the need to coordinate definitions, operations, and axiomatic properties (Dorier, 1995; 1998).

Research has emphasized the importance of considering prerequisite concepts—such as sets, equality, and binary operations—as part of the construction of the concept of a vector space. Additionally, the use of non-standard vector spaces and unconventional operations has been proposed to promote deeper reflection on the defining properties of these structures (Can Cabrera et al., 2022; Kú et al., 2008; Parraguez & Oktaç, 2012; Trigueros et al., 2024). Such approaches require students to move beyond memorized procedures and to engage in reasoning about the structural relationships that define a vector space.

In this context, programming provides an environment for constructing the concept of a vector space. By requiring students to define sets, implement operations, and test axiomatic properties, programming makes explicit the structural components that underlie the concept. Moreover, the possibility of generating and modifying examples—including non-traditional vector spaces—provides opportunities to explore the role of axioms and to observe how different structures satisfy or fail to satisfy the defining properties of vector spaces. This, in turn, opens the possibility of analyzing students' constructions through their programming activity, thereby providing access to how they coordinate sets, operations, and axioms.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

This study is grounded in APOS Theory (Arnon et al., 2014), a framework that explains how students construct mathematical concepts through four mental structures: Actions, Processes, Objects, and Schemas. An Action is a transformation on previously constructed objects that require external cues or memorized procedures. When students reflect on an Action and can carry it out mentally without external prompts, it becomes a Process. Processes may be coordinated or reversed, and when a Process is conceived as a unified whole, it is encapsulated into an Object, which can itself become the focus of new Actions. These mental structures interrelate to form a Schema, a coherent system that evolves as students build new connections or modify existing ones.

In this study, APOS Theory is used as an analytical lens to interpret students' activity when engaging with programming tasks. From its inception, APOS Theory recognized programming as a potential tool for fostering these constructions. Programming environments provide opportunities for students to operate on mathematical objects in explicit and structured ways, which aligns with the types of transformations described in APOS. For example, executing or modifying a program can be interpreted as performing Actions, while generalizing a program or anticipating its behavior reflects the development of Processes. Moreover, using programs as entities that can be manipulated, combined, or reused supports the encapsulation of Processes into Objects.

The connection between APOS Theory and computational thinking is relevant in this context. Computational thinking involves practices such as abstraction, decomposition, generalization, and algorithmic reasoning, which are closely related to the mental

mechanisms described in APOS. In particular, the transition from specific computations to generalized procedures in programming aligns with the interiorization mechanism, while treating procedures as manipulable entities resonates with encapsulation. Thus, programming activity can be seen as a setting in which computational thinking and APOS mechanisms converge, supporting the construction of mathematical knowledge through structured interaction with mathematical objects.

This perspective is pertinent in the study of vector spaces, where students must coordinate sets, operations, and axiomatic properties. Programming tasks that require the construction of sets, the definition of operations, and the verification of axioms provide a context in which these elements can be explicitly manipulated and related, allowing for the analysis of students' mental constructions through their observable activity on the computer.

This study is guided by the following research questions:

1. How do undergraduate students interpret and modify programming related to the construction and analysis of vector spaces?

2. What mental constructions, in terms of APOS Theory, can be identified in students' programming activity when working with vector space axioms?

METHODOLOGY

This research examines how university students construct the concept of a vector space through programming. It follows up on earlier work (Can Cabrera et al., 2022; Trigueros et al., 2024) that introduced technology in initial course sessions, though its implementation was cut short by the COVID-19 pandemic. For this new phase, the course was redesigned using APOS theory. We refined the original activities based on the previous findings and used them in this study, following the APOS theory's cyclic methodology.

For the present study, the APOS teaching methodology was adopted, incorporating activities based on a validated genetic decomposition for the concept of vector space (Can Cabrera et al., 2022; Trigueros et al., 2024). Their purpose was to support the mental constructions required for learning vector spaces. These activities were designed to be solved collaboratively in small groups. In addition, seven programming activities were introduced using *GNU Octave* (https://octave.org), through which students generated sets of vectors, defined operations, and verified vector space axioms using code. The selection of GNU Octave as the programming environment was motivated by its ability to handle vectors and matrices through algorithmic implementation, while emphasizing conceptual understanding rather than the technical details of a programming language. Furthermore, as open-source software, it removes economic barriers to access.

Activities were designed for students to work with different $\mathbb{Z}_p^n$ sets and operations so they could check the validity of the axioms for all elements in each set. APOS theory was used to analyze students' work on the activities. We found it helpful to incorporate

ideas from computational thinking to complement the analysis of students' teamwork and conclusions. For instance, the designed activities (see, for example, Figure 1) allow students to execute code without modification at the initial stages and subsequently to modify and adapt programs to new vector sets. They also encourage students to treat functions as mathematical objects, associated with the axioms defining a vector space (Weintrop et al., 2026).

Example of the programming activities: The code in Figure 1 generates a list containing all elements of the set $\mathbb{Z}_5^2$, arranged in rows. Modify this code: (a) to adapt it into a function capable of generating sets of the form $\mathbb{Z}_p^2$; and (b) generalize it into a function to generate the list of elements for sets of the form $\mathbb{Z}_p^3$ and $\mathbb{Z}_p^5$.

```
1  k=1;
2  for i=0:1:4
3      for j=0:1:4
4          V(k,1)=i;
5          V(k,2)=j;
6          k++;
7      end
8  end
9  V
```

Figure 1. Octave code to generate vectors.

Students were asked to analyze pre-written code used to test the vector space axioms and to modify it for different non-standard sets of vectors and for operations defined in alternative ways. Through code analysis and modification, students will generate lists of elements from various $\mathbb{Z}_p^n$. sets. These tasks were designed to foster a deep understanding of how these vectors are constructed, and the defining roles of the parameters n and p. The activity thereby helps students interiorize the Actions on a set through a structured, practical Process which can be later encapsulated into an Object.

The analysis of student teamwork was framed by APOS theory and complemented by computational thinking as mentioned before. We focus on their explanations and on how the strategies they adopted helped them analyze their results and connect them to their overall constructions of the Vector Space concept.

Students' work was video-recorded, allowing us to follow the discussions of the different teams, and their written work was collected. The instructor (a co-author of this paper) assisted in selecting three teams that used different approaches when reconstructing programs to verify selected axioms. Researchers independently analyzed the data, then discussed their findings and carefully revisited the data, reaching a shared interpretation of the students' discussions and the key details involved.

The analysis focuses on how different strategies helped students design their programs, to test the vector space axioms, and how these processes related to both the notion of proof and their overall constructions of the Vector Space concept. We selected students who played leading roles within their teams to illustrate the different strategies used and analyze them through the lenses of APOS theory and computational thinking.

Finally, we discuss the potential advantages of incorporating programming into instruction —specifically, how it may influence students' understanding of what constitutes a proof, as well as their grasp of the axioms and the vector space concept more broadly.

The teaching strategy implemented in all sessions was the ACE Cycle, which consists of cycles in which students work collaboratively in small groups on activities (A). Then a whole-class discussion with the teacher follows (C). These two steps were repeated several times during each session, and homework exercises (E) were handed out at the end of each class.

RESULTS

Constructing Actions trough Data Practices

Vector space was the last topic covered by the teacher. He presented the Octave program to students and asked them to read a code developed to define vectors in $\mathbb{Z}_5^2$(Figure 1) and discuss it in their teams. Students began discussing where and how they needed to change the code for each of the sets the teacher suggested. The teacher helped them when necessary. Students focused on the Actions of reading the code and looking for the lines that needed to be changed. Most of the teams discussed why those lines were important and how they related to the construction of vectors in $\mathbb{Z}_p^n$, all of which involved reflection on their Actions. The teacher asked them to modify the code to define matrices and vectors with elements in $\mathbb{Z}_p^n$. Students showed interest in these tasks, although they were new to them. Being able to modify code requires the interiorization of students' Actions into a Process, and most of them evidenced this construction in their explanations.

Constructing computational programs to construct Processes

The teacher asked students to work in their teams by adapting a given code to compute the sum and product of a vector by a scalar in $\mathbb{Z}_p$. Students discussed the changes needed for different sets. Although they faced some difficulties, their discussion enabled them to do so using specific $\mathbb{Z}_p$ set elements. Most teams were able to explain their reasoning. For example, Blanca explained "*The [function] vector… creates different vectors, but for* $\mathbb{Z}_p$ (Figure 2) *you put a value n…but it is really a value for p… and you get the vectors. For example, if n is 8, you get the set of vectors in set Z8...".* Blanca shows that Actions for repeating a block of code a specific number of times are needed, typically when the number of iterations is known beforehand, and that this is usually done by changing and executing the given code. It is clear that she understood the role of code in generating vectors in $\mathbb{Z}_p$. When asked about vectors in $\mathbb{Z}_p$, she responded: *"If it is* $\mathbb{Z}_p^2$*, it is fine; it generates two elements for each vector".* When asked what happens if it is five instead of two, she explained *"Here may be we create a cycle for…here where it is asking to form vectors n, besides n we need other variable, p, the number, the vector's elements…those have to be less that that limit and n is the number of elements in a row, or a vector…but how to do it for*

the exponent n, that is the number of possible elements in the vectors". Here, Blanca explains the roles of *n* and *p*, the entry parameters that define the set, and can modify the code for different *p*'s. Her team manually generated codes for specific cases (Figure 2), demonstrating that she has constructed a coding Process. She also demonstrates she has interiorized the role of a *cycle for* iterating a block of code a specific, known number of times.

```
function vect=vectors_n(n)
k=1;
p=n-1;
for i=0:p
    for j=0:p
        for f=0:p
            B(k,1)=i;
            B(k,2)=j;
            B(k,3)=f;
            k++;
        end
    end
end
vect = B:
```

(a)

```
function vect=vectors_n(n)
k=1;
p=n-1;
for i=0:p
    for j=0:p
        for f=0:p
          for g=0:p
            for r=0:p
              B(k,1)=i;
              B(k,2)=j;
              B(k,3)=f;
              B(k,4)=g;
              B(k,5)=r;
              k++;
            end
          end
        end
    end
end
vect = B;
```

(b)

Figure 2 Blanca's modifications to generate (a) $\mathbb{Z}_p^3$ and (b) $\mathbb{Z}_p^5$

By analyzing other coding examples, students demonstrated their understanding of the code's implicit algorithm. An example of the interiorization of Actions into a coding Process manifested by most students is: *"it takes... asks for two values in the product a and b are multiplied, and then it divides the result by 11, and it gets the residue" … "It is as if we were adding vectors or matrices"*.

Once students finished and discussed the activities on understanding and changing code to operate on elements of different sets, the teacher introduced vector spaces during a whole-class discussion. He then asked them how they would demonstrate the validity of the vector space axioms, including examples related to addition associativity and to neutral and inverse elements for different sets.

Encapsulating Vector Space into an Object

In other sessions, students worked collaboratively on activities in which they were asked to modify a given Octave code for subsets of $\mathbb{R}^n$ so it could be used to verify vector space axioms for different sets and various definitions of addition and scalar multiplication. Then, students worked on a new set of activities, including matrices with elements in $\mathbb{Z}_p$, which was new to them since they had used vectors in $\mathbb{Z}_p$ so far. Students were involved with this task, which was unexpected and challenging for them. The teacher then called for a whole-class discussion where he asked different students to explain how their teams had solved those tasks:

Teacher: Can you explain how you changed the code to test associativity, or explain how you worked with 2x2 matrices with elements in $\mathbb{Z}_p$?

Luis: Well, in activity 5, we created a function to generate all the 2x2 matrices with elements in $\mathbb{Z}_p$... we used four nested cycles, each going from 0 to p-1... for each matrix entry. Then... mmm... we did two functions: one for binary addition in the matrices addition module p and other for the product by a scalar, using also the p module. Then… we changed the verification codes we had done before… I mean codes for is closed and is associative, so we could work with matrices and not only with vectors…mmm… Also, we wrote functions to verify the neutral element and the additive inverses... these added functions counted the favorable results... Finally... we tested all with examples of sets such as... as $\mathbb{Z}_5$ and $\mathbb{Z}_3$ ... to see if they worked…

Teacher: Good, and how did you modify the code to test associativity?

Luis: Well... mmm... for the is_asociative code... we had to adapt it so it could work with matrices instead of vectors. Mmm, then…we changed the way to access to the set V elements... that now are matrices... and we used our function BinarySum with parameter p to perform the operations module p… inside the code. Then we calculated each side of the equality that is (writing on the blackboard) (A $\oplus$ B) $\oplus$ C and A $\oplus$ (B $\oplus$ C)... and then we just compared if all the results were the same, to be able to conclude if the operation was or not associative.

Teacher: Can you explain to us how you calculated each side of the equality you mentioned?

Luis: ...well... inside the nested cycles, for each matrix triplet A, B, and D … we first calculated W which is the sum of A and B, then... Y, which would be the binary sum of W with D… that would be the left side (writing) [(A $\oplus$ B) $\oplus$ D]... Then... we calculated X which is the BinarySum for B and D, I mean, the other matrices, and then we have (writing) Z = BinarySum (B, A, Y) ... which would be the right side (writing) [A $\oplus$ (B $\oplus$ D)] ... Finally... we compared if X equaled Z for all the combinations... and if they always coincided then we said that it was associative.

Teacher: Well, but what was the role of that function? How did you use it?

Luis: Mmm this function...mmm... we used it to verify one of the properties of vector space… if the sum was associative…We used... mmm... executing the associative law with the set of matrices we generated... for example, with gConjunto(5)... and our sum function 'BinarySum'... with p=5 value and the result told us whether the operation was associative or not on that set. In our case, we did it and got yes, it is associative.

Teacher: Ok, now could you explain what you did to verify the existence of a neutral element?

Luis: Of course... mmm for it we proposed that matrix N = [0,0,0,0] was the candidate to be a neutral...Then in the function is_exists_neutro... we used all

the set V and for each matrix A we calculated A plus N using BinarySum and verified if the result was equal to the original matrix A. Then we counted how many matrices complied with this, and if the number of their elements was equal to the size of V, we concluded that there was a neutral element and we could prove it.

Teacher: And why did you use that matrix N? Where did it come from?

Luis: Well, we selected it because we thought, for the sum, the neutral element should be a matrix that did not change the given matrix when added to it. So, as we are working in module p, the matrix should have mmm all its elements zero, five zeroes. I mean, it was the natural option, since adding zero to any element gives you the same element… and the module p is preserved.

The teacher chose another student to explain the work on his team:

Teacher: Now, Carla, can you explain what your team did to prove the existence of additive neutral elements? Carla started to explain, and Rodrigo, a teammate, interrupted her explanation:

Carla: When we started, we thought that…mmm the inverse could be defined with a negative number, something as… as -b(1,1) or –b (2,2)... but then Rodrigo said that in Zp there are no negative numbers, so we had better use the complement…p minus the value of the element.

Rodrigo: Yes, I remembered that we have done that before with vectors in Zp exponent n... there the vector's inverse is obtained by doing p minus each of its components, s…for matrices we applied the same idea...now for each of its four entries.

Carla: That is right... as a team we found the solution that the inverse of matrix B was the matrix (Writing) d = [p-b(1,1), p-b(1,2); p-b(2,1), p-b(2,2)]... and then we verified that adding B with d resulted in the neutral matrix zero, zero, zero. zero…and then we verified that adding B and d the result was the neutral matrix for all the set elements.

In this dialogue, students demonstrate their ability to debug code in order to solve the problem. They can also incorporate subroutines from existing code, as well as reuse and adapt previous programs to new situations. Luis's explanation suggests that he and his team had constructed the notion of proof at least as a Process. The security he shows when explaining his team's ideas suggests that they have developed an Object conception of coding, as other students may have. Carla's team showed the construction of a vector space Process for vector spaces and a Process for coding, since they were still struggling with them.

As Luis's team members, other students demonstrated that they had possibly encapsulated the concepts of vector space and proof. They were able to define a general rule that could be applied to all elements in the set whenever possible. They could also determine when this was not possible, and they showed they could generalize what

they had done in the former sessions on $\mathbb{Z}_p^n$ sets to $\mathbb{Z}_p$ matrices, which had not been discussed during lessons.

Other teams showed they still had difficulties understanding these concepts. Also, some specific functions in the codes, but all students were able to construct at least Process conception of vector space axioms and of vector space, The use of programming proved to have helped students to be aware of the need to test each axiom for all the elements of a set and to prove all the axioms in vector space definition to determine if a given set was a vector space.

CONCLUDING DISCUSSION

Students worked hard to interpret the teacher's suggested computer program. They were able to adapt the coding to align with the proposed activities, both during class sessions and as homework. Our results showed that students were interested in understanding code from the Octave program and that, although it was not direct, they were able to describe what the code did and how to modify it to solve the activities presented.

It was possible to see how most students developed different strategies for writing new code and adapting them to the task. They explained what the code was about and how it is used to prove vector space axioms. These results indicate that students are developing computational thinking. This development was evident in their language and in their ability to explain how they adapted the code to each task. The emergence and development of students' computational thinking, as we defined it, took longer for some students. However, by the end of the experience, all of them could modify code and describe what it did in specific situations.

Moreover, as they worked on coding and its application to solving problems in vector spaces, students advanced in their understanding of the axioms. They also developed their awareness of the role of proof in determining whether a given set, with standard or non-standard definitions, was or was not a vector space.

When analyzing students' productions in terms of APOS theory, we observed how students talked about the role of proof and about the meaning of each of the vector space axioms. The use of coding helped them to understand when a proposition was or not true and when to use the code they created to solve problems. One important learning for students was to distinguish whether they could use a "for cycle" code to prove some vector space axioms and why it could not be used for other properties.

Students' development in programming helped them transition from understanding concepts as Actions to interiorize them as Processes. It is almost impossible to write adequate code without reflecting on Actions until they are interiorized into a Process. Some students also showed the encapsulation of some Objects when they needed to do Actions on them. Nearly a third of students showed the construction of axioms and vector spaces as Objects, while most students interioraized vector spaces as Processes. One team struggled with coding and proof but they showed the construction of Actions.

The results of this study contribute to the literature by presenting examples of how computational and mathematical thinking need to be intertwined when students write programs to prove the vector space axioms for different sets. Combining these two types of thinking broadens students' understanding of both proof and vector spaces as unifying concepts.

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GenAI in Mathematics Teacher Education: In-Service Teachers' Experiences with AI-Generated Problem Variation

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In Sweden, despite policy mandates for integrating AI literacy within disciplinary contexts, empirical research on GenAI applications in university mathematics education remains limited. This study addresses this gap by examining how seven in-service teachers engaged with GenAI's content generation capabilities within a university-level geometry course. Participants completed a pedagogical design task requiring them to generate differentiated geometry problems across varying difficulty levels using GenAI. Through thematic analysis of post-task reflections, we examine how participants navigate GenAI capabilities alongside mathematical-pedagogical judgment when designing materials. Findings provide insights into GenAI integration in university mathematics teacher education.

INTRODUCTION

In mathematics education at tertiary level, Generative AI (GenAI) tools such as ChatGPT or Microsoft Copilot offer educators new possibilities for creating instructional materials (Bui et al., 2025). Unlike domain-specific technologies (e.g., dynamic geometry software) or general-purpose tools requiring programming conventions, GenAI combines natural language interaction with flexible content generation, from producing mathematical problems to adapting difficulty levels and contextualizing pedagogical content (Tankelevitch et al., 2024).These capabilities position GenAI as a potentially powerful resource for pedagogical design, the process through which teachers create and adapt learning materials to meet instructional goals and student needs. However, realizing this potential requires understanding how teachers navigate GenAI's generative capacity alongside their professional judgment.

However, realizing GenAI's potential requires more than technical accessibility. In particular, geometry presents particular challenges for current GenAI systems, which struggle with spatial reasoning, diagram interpretation, and multi-step geometric proof construction (Bui et al., 2025). Designing geometry word problems thus places teachers at a critical intersection as they must exercise pedagogical judgment over content that GenAI generates. More broadly, word problem design represents a complex pedagogical task integrating mathematical content knowledge, understanding of student thinking, and instructional design expertise (Brown, 2011; Pongsakdi et al., 2016).Geometry intensifies this complexity, as teachers must simultaneously balance spatial reasoning demands, real-world contextualization, solution pathway

accessibility, and mathematical rigor (Harris, 2023). The combination of GenAI's geometric limitations and the pedagogical demands of geometry problem design renders teachers' evaluative judgments particularly visible, making this domain a strategically revealing context for examining how practitioners negotiate between AI-generated content and professional standards.

In-service teachers pursuing advanced credentials in teacher education programs represent a distinctive population for examining GenAI integration. Unlike pre-service teachers, they bring established classroom practices and subject didactical knowledge developed through professional experience, and must develop new pedagogical approaches for AI-mediated instruction. Their expertise provides clear evaluative criteria against which GenAI outputs can be assessed, making visible how experienced practitioners negotiate between technological affordances and pedagogical standards.

Globally, recent studies have investigated the integration of GenAI within mathematics teacher education (Busuttil & Calleja, 2025; Kim et al., 2025). In Sweden, despite the call for developing AI literacy and critical evaluation of AI-generated content in subject-specific settings (Skolverket, 2024), research on GenAI in education remains limited and largely general, overlooking subject-specific needs (Bergdahl & Sjöberg, 2025). Major gaps remain regarding in-service teachers' experiences using GenAI as part of their professional development in mathematics teacher education

Addressing this gap, this paper reports findings from a broader research project on GenAI integration. Within a mathematics geometry course, seven in-service mathematics teachers completed an end-of-course assignment in which they used GenAI tools to design real-life geometry problems at varying levels of difficulty. We focus on the participants' reflections as they engaged in this problem-variation task, designed to draw on GenAI's strengths in content generation. By analysing their reflections on this problem-variation task, we examine how teachers navigate the intersection of GenAI capabilities and pedagogical judgment when designing instructional materials. Our focus is on teachers' reported experiences and evaluative reasoning, not on assessing the mathematical or pedagogical quality of the tasks produced. Accordingly, we ask: *How do in-service mathematics teachers enrolled in a university mathematics teacher education course reflect on using GenAI's content-generation capabilities to create real-life geometry word problems?*

THEORETICAL FRAMEWORKS

Understanding how teachers reflect on GenAI for pedagogical design requires theoretical perspectives that address both technology appropriation and domain-specific expertise. Technology Acceptance Model (TAM), particularly the extended

model TAM 3(Venkatesh & Bala, 2008), offers a robust framework for examining how users perceive new technologies in terms of usefulness, ease of use, and contextual factors that moderate adoption. However, TAMs are domain-agnostic and cannot capture the discipline-specific judgments required to evaluate the pedagogical quality of AI-outputs. Complementarily, the theoretical lens of Subject Didactical Knowledge (Bayrhuber & Frederking, 2024) foregrounds the specialized knowledge teachers possess for transforming subject matter into teachable forms, knowledge that integrates content expertise with understanding of student thinking and instructional design within a specific discipline. For mathematics, this includes considerations of cognitive demand, solution pathways, pedagogical components and authentic contextualization (Brown, 2011). We argue that teachers' reflections on GenAI use are shaped by this intersection: their established subject didactical knowledge provides evaluative criteria, while their emerging perceptions of GenAI's utility and limitations inform acceptance and future use. This dual-lens approach guides our analysis of teachers'reflective accounts.

METHODOLOGY

Research design

This study was conducted within a geometry course in a mathematics teacher education program at a state university in southwest-central Sweden, offered from April to June 2025. It was ethically reviewed according to applicable regulations (HNT 2025/181). The course enrolled two groups of students: pre-service teachers and in-service mathematics teachers pursuing additional credentials. The course's primary aim was to teach geometry content, with a small GenAI component incorporated alongside the regular curriculum without altering the mathematical content. At the beginning of the course, students participated in a one-hour session introducing GenAI technologies (e.g. GenAI characteristics, limitations and strength), including tools, mathematics- and geometry-focused prompting techniques, ethical considerations, and recent research in GenAI and mathematics education. Following this, students continued the course as usual; however, during each of the six weekly lectures, they were exposed to one geometrical task using GenAI tools, which they then discussed together in class.

Data collection

At the end of the course, students submitted their final assignments, which included tasks requiring the use of GenAI tools and reflections on their experience. This study draws on the reflective sections of the assignment in Table 1. Participants were free to use any GenAI tools to which they had access. Some participants used the free version of ChatGPT (GPT-4o), while others used Microsoft Copilot.

"You are standing in front of a tall tree and wondering how one can calculate the height of this using geometry". Use a GenAI tool to find two or more geometric solutions to solve the task above, as well as to construct two similar problems (real-life related) to use in school mathematics. Also use GenAI tools to adapt one of your new tasks to three different levels: easy, medium and difficult. Comment on which students your two problems are suitable for, for example suitable grade levels in school or necessary prerequisite knowledge. As a conclusion, reflect on whether and how the AI tool helped you find solutions that you might not have thought of yourself, and to construct new problems. Write the reflection from two different perspectives, both as a mathematics learner and as a mathematics teacher.

Table 1: Task description

Participants

Participants were seven in-service mathematics teachers (4 females, 3 males; >33 years old), with varied teaching experience. Most reported little prior experience with GenAI tools. Participants received research information via Canvas, provided consent, and only consenting individuals were included in the analysis.

Qualitative analysis

The reflections were extracted and imported to Excel for manual coding. The estimated total words were about 3200 and average about 460 words/ participant. We employed a hybrid thematic analysis (Naeem et al., 2023), allowing codes and themes to emerge inductively from the data, while also being informed by established frameworks. Specifically, we drew on TAMs (Venkatesh & Bala, 2008), including constructs such as perceived usefulness and output quality, as well as Subject Didactical Knowledge (Bayrhuber & Frederking, 2024), particularly pedagogical appropriateness and curricular reasoning, to guide the coding and synthesis process. The themes were not predetermined by these frameworks; rather, they emerged through iterative engagement between theoretical constructs and participants' accounts. To ensure coding reliability, all reflections were independently and thoroughly read by two researchers, followed by independent coding, with agreement established through comparison and discussion of emergent themes until consensus was reached.

RESULTS

The analysis suggested two central themes that illustrate how participants experienced and evaluated GenAI's role in supporting differentiated problem design. The first theme (GenAI's Content Generation Abilities) captures teachers' perceptions and experience of GenAI's generative capacities. The second theme (Subject Didactic

Evaluation of GenAI's Abilities) captures teachers' critical assessments of outputs against pedagogical and mathematical standards.

Theme 1: GenAI's Content Generation Abilities

This theme describes how the participants' perceptions and experiences of GenAI's content generation abilities unfold. The theme is built on four sub-themes: 1) Discovery, 2) Creative potential, 3) Content quality, and 4) Efficiency.

Discovery

The participants describe how unexpected strategies and variety of approaches solving the mathematical task came up when using GenAI. The output added value to their own suggestions to solutions.

> Teacher 4: When I used the AI tool to explore different ways to calculate the tree's height, I was reminded of several strategies I knew but also introduced to variations that I might not have thought of myself, for example using leaning sticks and combining trigonometry with similarity.
>
> Teacher 5: The way I directly came up with myself are the shadow method […] but here I got suggestions with trigonometry and to use a mirror and through that similarity to solve the problem.

Creative potential

Using GenAI also opened for taking novel approaches and innovative methods when engaging with the mathematical content.

> Teacher 7: Being able to switch between more classical methods (like shadows and similarity) and digital measurements (angle apps) gave me new ideas for laboratory and outdoor-based instruction.

The participants situate the creative potential in educational settings and warrant it by the user's familiarity to handle the tool.

> Teacher 5: [T]his requires that one actively works with AI and that one supports the students in how one can use AI to develop.

Content quality

Perceived usefulness of GenAI's tools was not only connected to mathematical accuracy, but also the appropriateness in relation to a teaching situation; to both apply problems in real-life situations and make concepts clearer by showing theory behind geometric methods.

> Teacher 4: It helped me to concretize abstract concepts and create tasks that both train basic skills and challenge the students further […] new variants on classical problems and suggested clear solution paths.

Efficiency

When describing their experiences using GenAI, the time-saving aspect regarding planning practice comes forward. Given the need to assess AI-output, participants point to productivity gains as well as increased volume of content (potential drawbacks).

> Teacher 3: In this task, AI helped me to quickly compose tasks and alternative solutions that would have taken me longer time to work out […] However, it is required that one as a teacher controls tasks etc. and possibly that one makes adjustments for the individual student groups.

Still, there is a catch; the teacher needs to have enough GenAI-related competence to interact with the tool to gain this efficiency:

> Teacher 6: I chose to end the attempts to get correct instructions via ChatGPT, since this took more time than writing them myself from scratch.

Theme 2: Subject didactic evaluation of GenAI's abilities

This theme uncovers participants' critically assessment of GenAI's capabilities and limitations within the specific pedagogical context of mathematics education. Participants evaluate pedagogical adaptation and variation and puts in relation to different levels of engagement shown in three sub-themes: 1) Surface-level assessment, 2) Nuanced pedagogical reasoning, and 3) Professional knowledge integration.

Surface-level assessment

When describing general utility, efficiency, and convenience of GenAI as a tool, both teacher and learner perspectives were evaluated. The participants value the dialogic nature of GenAI when solving mathematical tasks.

> Teacher 3: As a student, AI gave guidance and how the problems can be solved without presenting a complete solution which gives me as a learner the chance to try different techniques to solve the tasks and thus develop my own learning.

From a teacher's perspective, the usefulness generating new tasks was highlighted:

> Teacher 4: [T]he AI tool was very valuable for quickly generating realistic, varied and level-adapted problems that connect mathematics to the students' everyday life.

The participants also assessed the usefulness in relation to development of their own mathematical knowledge. Here different experiences occur:

> Teacher 2: For my own learning as a learner in mathematics, I didn't get much out of this conversation.
>
> Teacher 1: I now understand both the problem and the proof and can explain it to a student.

Nuanced pedagogical reasoning

The participants made sophisticated evaluations of mathematical appropriateness and curricular connections of the output in relation to student progression

> Teacher 6: [S]uitable for investigative working methods or for applying already learned knowledge and can thereby contribute to making otherwise theoretically heavy instruction more varied and motivating for students.

Since the reflections also covered insights of how interactions with GenAI might unfold in practice, they considered the generated output in relation to its potential impact on students' learning processes.

> Teacher 1: The tool simulated a conversation with a knowledgeable supervisor. It showed how to build a proof step by step and how to navigate complex concepts through questions.

The participants emphasise teacher's responsibility regarding mathematical content.

> Teacher 2: [I]t's important to thoroughly consider what one wants to get out of the tasks and carefully ensure following the curriculum for the subject.

Professional knowledge integration

Engaging in subject-didactical tasks appears to open opportunities for reflections on further developing professional knowledge. The participants' experiences suggest that – when GenAI capabilities are sufficiently advanced – teachers' specific pedagogical expertise, professional judgement, and subject-specific knowledge play a central role:

> Teacher 7: AI inspired me to develop my own tasks at different levels […]which improved my understanding of progression […] It felt like my own problem-solving competence grew when I received suggestions that I could then analyse, modify and further develop. […] AI also suggested concepts like "trigonometric modelling", which led to me starting to investigate new mathematical concepts that could enrich both my own teaching and the students' understanding.

> Teacher 2: I felt that I got quite a lot out of it in my capacity as a teacher. I think it was clear that one can use GenAI to come up with ideas for tasks and suggestions for solution methods.

DISCUSSION AND CONCLUSIONS

This study explored how mathematics teachers enrolled in geometry course within a teacher education program in Sweden reflected on using GenAI tools to complete a problem-solving assignment, in which they identified multiple solution strategies, created similar real-world problems, and adapted tasks to various difficulty levels.

Leveraging GenAI for content generation and professional learning

The task was explicitly designed to leverage one of GenAI's established strengths: content generation. Participants reported that AI-outputs facilitated discovery of novel solutions, with reflections describing how AI-generated variations prompted them to consider problem structures they would not have designed independently. This creative scaffolding, however, was valued precisely because participants could evaluate outputs against their classroom experience. These align with prior research that demonstrating GenAI's capacity to expand teachers' pedagogical repertoires and stimulate reflective practice (Bui et al., 2025). GenAI also supported efficiency, allowing rapid generation of multiple task variants. However, our data suggested that effective use depended on participants' familiarity with the tool(s), echoing findings that GenAI is most effective when mediated by professional judgment. Furthermore, these observed benefits also came with potential trade-offs. Participants consistently emphasized the need to evaluate and adapt GenAI outputs to their pedagogical goals, several noted that refining AI outputs required more iterative work and time than completing the tasks themselves.

The task was designed to examine whether GenAI tools can support the adaptation of problems to real-world contexts and multiple difficulty levels, highlighting their potential to foster adaptive expertise - the ability to apply knowledge and skills flexibly in novel situations (Verschaffel et al., 2009). Designing open-ended mathematical problems that support adaptive expertise traditionally requires substantial time and resources (Bui, 2023; Verschaffel et al., 2009), yet our results suggested that GenAI facilitated this process by quickly generating diverse, non-routine tasks. Participants valued GenAI's capacity to generate various problem variations, particularly for connecting abstract geometry concepts to real-world contexts. Their reflections suggested consistent attention to student accessibility-evaluating whether AI-generated problems aligned with their students' everyday experiences and curriculum relevance. This evaluative stance, drawing on classroom expertise, distinguished their engagement from uncritical adoption. These suggest that GenAI holds potential to inspire innovative teaching practices, foster flexible mathematical thinking, and create a wider range of opportunities. By generating diverse problems and novel solutions, GenAI can support teachers in task designing that challenge students to explore mathematics from multiple approaches. However, realizing these benefits raises important concerns (Busuttil & Calleja, 2025), as using GenAI involves potential drawbacks highlighting the need for further research.

The role of subject-didactic expertise

Our findings suggest that effectively leveraging GenAI for creating teaching materials may critically depend on the combination of strong mathematical knowledge and, centrally, subject-specific didactical expertise (Bayrhuber & Frederking, 2024), along with familiarity with tools. While gaps in any of these areas appeared to limit effective use, the data particularly highlight that subject-didactical knowledge is essential for evaluating, adapting, and integrating AI-generated outputs into meaningful tasks. This observation remains tentative and warrants further empirical research, but it highlights that GenAI's potential is realized only when guided by pedagogical judgment, reinforcing its augmentative rather than substitutive role in teaching practice.

University teacher education as a productive context

The structured university coursework shaped participants' reflections in distinctive ways. Unlike informal professional development, the course - and its GenAI component - prompted explicit articulation of pedagogical reasoning when creating and evaluating AI-generated materials. This suggests that structured tasks combining GenAI use with guided reflection can foster both subject-didactic expertise and critical AI use in mathematics teacher education. Future research should explore how such programs can develop pedagogical, mathematical, and technological competencies over time, identify which pedagogical knowledge domains are most critical for effective GenAI integration, and examine how these tasks can be adapted for students with varying levels of GenAI competence and pedagogical expertise.

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Epistemic Orientations and the Teaching–Research Gap in Mathematicians' Use of Programming

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Keywords: Mathematicians, Programming, Mathematical Practices, Activity Theory, Communities of Practice

INTRODUCTION

Programming plays an increasingly important role in mathematical practice, yet research indicates that mathematicians use programming far less in undergraduate teaching than in research (Broley et al., 2018). Despite growing interest in programming and computation in mathematics education, little is known about how mathematicians' views on the legitimacy of programming in research shape their decisions about integrating computation into teaching. In this poster, we examine how mathematicians articulate the role of programming across practices and how these views help explain systematic differences in engagement. Specifically, we ask:

1. What epistemic orientations toward mathematical objects and tools do mathematicians express in describing their research practices?
2. How do these orientations shape their views on the legitimate use of programming in research and in teaching?
3. How do these orientations help explain the gap between programming engagement in research and in teaching?

THEORETICAL FRAMEWORK AND METHODS

We conducted semi-structured interviews with 15 mathematicians representing a broad range of specialisations and seniority at a university with two decades of institution-wide programming integration. Interviews (90–120 minutes) covered research and teaching practices, and institutional context. Transcripts were analysed iteratively through discussion within the research group and wider scientific community. Our analysis combines Cultural-Historical Activity Theory (CHAT) and Communities of Practice (CoP) (Engeström, 2015; Wenger, 1998). CHAT frames mathematicians' research and teaching as two interacting activity systems directed toward distinct but related objects and helps identify epistemic tensions and contradictions across settings. CoP provides concepts such as legitimate participation, reification, and boundary crossing, allowing us to operationalise CHAT as an analytic lens linking discourse on the lived experience of "being a mathematician" with a broader systemic focus. A key analytic move was to organise participants by research orientation, using an epistemic matrix with axes for object orientation (theoretical vs. real-world purpose) and tool orientation (theoretical vs. computational) (cf. Suh et al.,

2022). This matrix guided a second phase of deductive coding, where we compared expressions of legitimacy for programming in teaching and research within and across orientations. In the poster, we will provide an explicit table illustrating how relevant constructs of CHAT were operationalised using CoP as an analytic checklist to analyse the interviews. The epistemic matrix will be defined and used to present the findings.

FINDINGS AND DISCUSSION

The epistemic matrix yielded four orientations, each reflecting a distinct configuration of epistemic rules for research practices: *classical pure, classical applied, computational pure, and computational applied*. For example, some classical pure mathematicians used programming for light-touch 'pre-epistemic' support of analytic work that they viewed as neither programming nor mathematical work. In contrast, computational applied mathematicians integrated analytic and computational approaches as mutually supporting and legitimate methods for solving real-world problems. Despite these differences, engagement with programming in teaching was lower across all orientations. Some expressed concerns that computation could be detrimental to analytic understanding; others saw no reason to involve programming where pen and paper sufficed. The gap was most pronounced in talk about assessment, where computation was near invisible. Our findings suggest that the gap between research and teaching is due not to a break in epistemic norms, but to epistemic continuity: mathematicians carry research-based norms into teaching. We argue that the gap is related to an object–outcome shift in teaching, suggesting a reframing of Burton's (2004) account of discrepancies between mathematicians' teaching and research practices, and discuss implications for education.

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Exploring how first-year students evaluate the AI task-solving process

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This study investigates the role of ChatGPT as a tool for university freshmen enrolled in Linear Algebra courses. We analyse the accuracy and coherence of ChatGPT's responses to a specific task, and investigate how learners evaluate these answers. Drawing on an a priori analysis of ChatGPT's solution, we create an analytical grid about the main critical aspects identified and analyse students' written evaluations relying on it. The results present the main issues identified in ChatGPT's responses and report how students addressed these aspects in their evaluations.

Keywords: Linear Algebra, qualitative study, ChatGPT, logical inference, formative assessment.

INTRODUCTION

The growing presence of Artificial Intelligence (AI) systems in educational practice raises the need to examine how students and teachers are appropriating such tools and to what extent their use supports or transforms mathematical learning processes. Despite the rapid adoption of generative AI in higher education, recent reviews point out that its integration is often weakly grounded in established pedagogical frameworks (Pepin et al., 2025). Although the range of AI-based technologies is vast and rapidly and continuously evolving, in this paper we refer to those conversational agents based on large language models (LLMs), such as ChatGPT, as they are the ones that have currently forcefully entered the everyday use of students and teachers (Pepin et al., 2025). This invites reflection on what should be considered as evidence of understanding, appropriate help, or authorship in students' work. In this scenario, universities face the challenge of leveraging the opportunities offered by such conversational agents. Recent studies have also highlighted critical limitations of generative AI systems in mathematical contexts. Bagno et al. (2023) showed that ChatGPT can produce solutions to Linear Algebra tasks that appear correct to students but contain significant inaccuracies and logical gaps, particularly in the handling of inference. Similarly, Plevris et al. (2023), in a comparative study of large language model-based agents, found that these can generate coherent and convincing responses, but their answers to complex mathematical problems or advanced logical tasks may not be reliable. In particular, they point out issues of inconsistency, and the tendency of such agents to provide different answers to the same problem, or not coherent answers with respect to the question to be faced. These findings suggest that the apparent coherence of AI-generated solutions may mask deeper issues in mathematical reasoning, making students' critical evaluation skills particularly important.

The present study focuses on the use of ChatGPT in the context of first-year Linear Algebra courses. We carried out a previous investigation (Albano et al., in press) on the accuracy and coherence of ChatGPT's responses to various formats of a specific task (open-ended, multiple-choice, and single-item argumented questions). The results highlighted the limitations of ChatGPT in evaluating items proposed as answers, both in terms of the approach used (which essentially remains the same independently by the format) and in handling items with a specific structure (i.e., logical implication). In this paper, we move to explore how students evaluate ChatGPT's responses, according to co-constructed and shared criteria of success, within a formative assessment framework (Black & Wiliam, 2009). In particular, we consider tasks with the single-item argumented question format, in which an open-ended question and a possible given answer with its justification are submitted to ChatGPT. This allowed us to shed light on the types of reasoning activated by students when interpreting and critiquing ChatGPT's answer. Thus, our research question is: (RQ) How do students evaluate the solving process generated by ChatGPT to a Linear Algebra task?

CONCEPTUAL BACKGROUND

Teaching-learning of Linear Algebra is characterized by the need for a strong integration between a structural and conceptual knowledge of mathematical objects and the ability to carry out procedures to transform them (Donevska-Todorova, 2016; Dorier & Sierpinska, 2001; Sfard, 1991; Skemp, 1976; Star, 2005). Linear Algebra is cognitively demanding because students must operate simultaneously on some objects, such as vectors and transformations, and perceive them as elements of broader structures. This entails reasoning at a trans-object level (Dorier & Sierpinska, 2001), integrating procedural fluency with structural insight. Teaching and learning Linear Algebra thus involve, among the others, three main challenges, i.e.: 1) *coordinating multiple modes of thinking* - commonly described as abstract, algebraic, and geometric (Donevska-Todorova, 2016; Dorier & Sierpinska, 2001); 2) *managing different semiotic representation systems* (Duval, 2006; Stewart et al., 2019); 3) *developing cognitive flexibility,* moving between local manipulations and a broader unifying view. The choice of the task, focused on logical inference, is due to two main reasons: the first one is the widespread students' difficulty in mastering implication (Durand-Guerrier, 2023), and the second is that has been recognized that, although ChatGPT delivers accurate and well-motivated answers in some mathematical areas, it often fails in logical inferences (Bagno et al., 2023).

METHODOLOGY

Context and participants

The study was conducted in two different Italian universities, with engineering freshmen attending a course in Linear Algebra during the 2025-26 academic year. The first sample included 38 freshmen of Management Engineering, and the second one 22 freshmen of Building Engineering-Architecture. All students participated voluntarily. The main topics treated in the two courses are the same. Students were used to tackling

open and closed-ended questions, and also questions with items such as "*A because B*". The only difference is that in the first context, vector spaces were introduced before matrices and linear systems, whereas in the second they were approached in the opposite order. We expected that, regardless of geographical context or instructional background (i.e., the order in which topics were taught), students would react similarly to ChatGPT's answers (version based on the model GPT-4o).

The educational path

The educational path is set within the frame of formative assessment, and in particular, relies on the first strategy by Black & Wiliam (2009), concerning sharing and understanding learning intentions and success criteria. The path involved various phases asking students to: 0) list the features of an "understandable" solving process; 1) evaluate ChatGPT's solving process for a single-item argumented question; 2) answer multiple-choice questions about specific aspects of ChatGPT's answer (theoretical references, procedures, logical chain); 3) submit to ChatGPT the task and interact with it until to obtain a satisfactory solution. In this study, we focus on phase 1), in which students received the task shown in Figure 1 together with the solution produced by ChatGPT. The task is formed by three components: a *core question*, i.e., the sentence "Given the following question, clearly and fully justify whether the item proposed as a possible answer is correct"; an *open-ended question*, i.e., "State whether the vector v is a linear combination of vectors u and w"; an *item provided as a possible answer*, i.e., the sentence "No, because the three vectors are linearly independent".

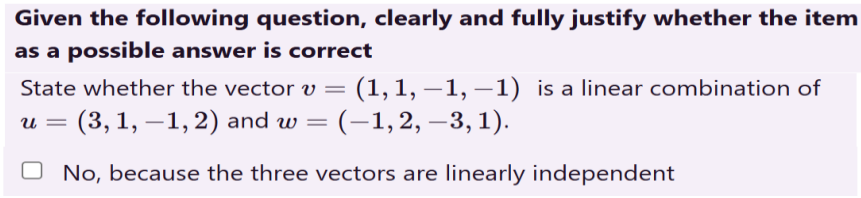


Fig. 1. Task submitted to ChatGPT

Students were asked to evaluate ChatGPT's response based on the following questions, aimed at investigating their perception of the solving process generated by ChatGPT: Q1) Does the answer to the question satisfy you? Why? Q2) Would you solve the question differently? How? Why? Q3) Do you think the solution addresses the requirements of the question? Why? Q4) Do you find the solution comprehensible (clear, correct, and complete)? Why?

In tune with research on narratives, Q1 is broad to allow respondents to focus on what they perceive as most significant. Q2 and Q3 aim to stimulate students' critical thinking to avoid the widespread phenomenon of overreliance on AI-based agents (Pepin et al., 2025). In particular, Q3 makes students focus on consistency of ChatGPT's answer with respect to the question, which is an issue in research on AI (Plevris et al., 2023). Finally, Q4 directly addresses the first strategy of formative assessment, clarifying the criteria for success shared with the students in solving a task, that are clearness, correctness and completeness.

A-priori analysis of the solution generated by ChatGPT

In order to analyze ChatGPT's task-solving process, we focus on the approach adopted in the solution, and the coherence of the argumentation. In particular, we examine both the procedural aspects of the solution and the justifications provided, with specific attention to the logical inferences used to derive conclusions. A first general remark concerns the approach used. In fact, the task is evaluative rather than solution-oriented: the aim is to assess whether the given item is correct as a possible answer. Instead, the approach taken by ChatGPT begins by solving the open-ended question, and only after completing the solution does it address the actual task request, which pertains to assessing the correctness of the item proposed as an answer to the core question.

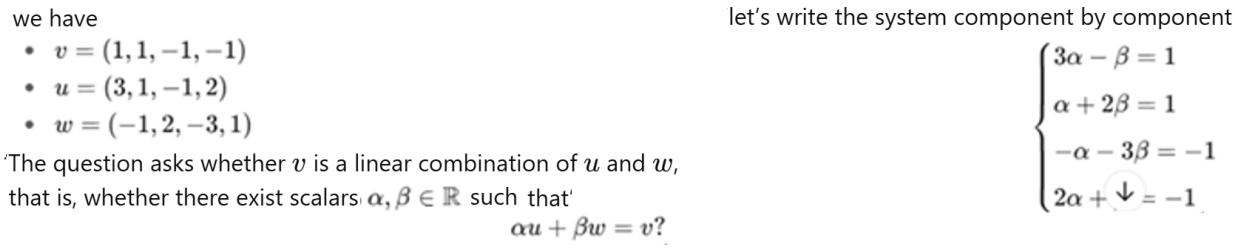


Fig. 2. ChatGPT's answer - part 1

Going into details on the resolution of the open-ended question, this is reformulated by implicitly invoking the definition of a linear combination (the vector equation is then converted into scalar equations, with an explicit reference to the components of the vectors involved, Figure 2). Then the resulting linear system is solved by substitution; this ultimately leads to the conclusion that the required scalars do not exist (Figure 3).

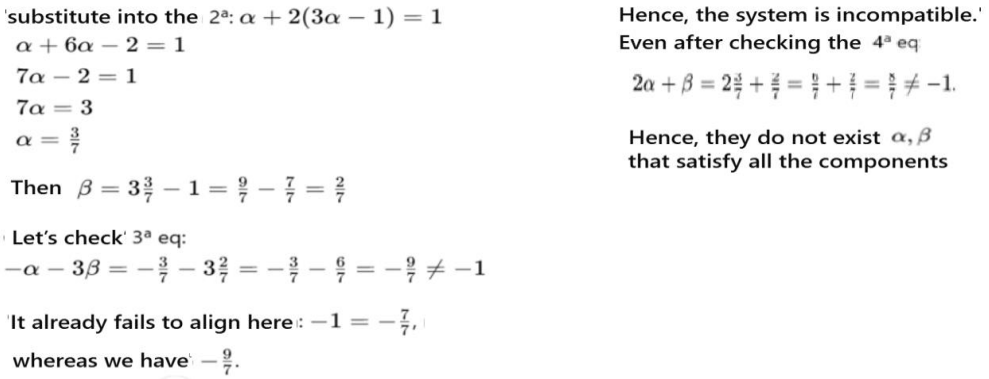
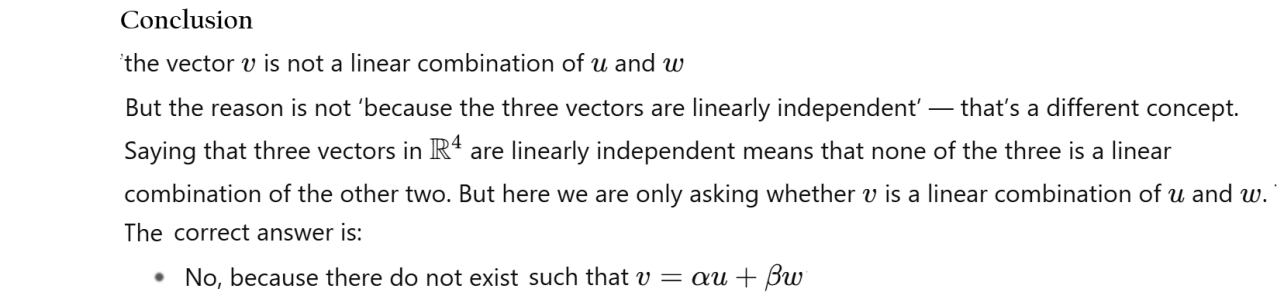


Fig. 3. ChatGPT's answer - part 2

Concerning the linear system, it is worth noting that it is not necessary to solve the system, but only to determine whether it is consistent; moreover, the substitution method used to solve it is rather rough and not appropriate for the university level. Let us move on to analysing how the task is handled. The item proposed as an answer supports the conclusion that the vector *v* is not a linear combination of *u* and *w* (a conclusion that ChatGPT also reached through its own solution process), giving as its justification the linear independence of the three vectors involved. It is precisely in the evaluation of this justification that a mistake emerges from the logical point of view in

the mathematical sense (Figure 4). Indeed, while on the one hand it correctly states that saying three vectors are linearly independent is equivalent to saying that none of them is a linear combination of the other two, on the other hand, it fails to grasp that this equivalent statement implies that *v* is not a linear combination of *u* and *w*. There, therefore, appears to be an issue in handling logical implication and quantifiers—specifically, in moving from "there does not exist an *x* such that P holds" to "for every *x*, not P holds", and finally to "for a particular *x*, not P holds". It is worthing to note that LLMs-based agents do not perform mathematical logical reasoning, but rather generate responses based on probabilistic pattern matching (Sambrotta, 2025). It therefore becomes more and more crucial to make students in a mathematics course aware of the difference between mathematical logical reasoning and the kind of reasoning that underlies the responses of LLM-based agents.


Conclusion

'the vector v is not a linear combination of u and w

But the reason is not 'because the three vectors are linearly independent' — that's a different concept.

Saying that three vectors in $\mathbb{R}^4$ are linearly independent means that none of the three is a linear combination of the other two. But here we are only asking whether v is a linear combination of u and w.'

The correct answer is:

- No, because there do not exist such that $v = \alpha u + \beta w$

Fig. 4. ChatGPT's answer - part 3

Let us note that the verification of the linear independence of the three vectors is not taken into consideration. Finally, rather than assessing the truth of the proposed item, it offers its own answer item, which is constructed based on the solution process previously carried out. According to previous studies (Bagno, 2023), it emerges that ChatGPT provides a plausible response, but including logical inaccuracies; moreover, it is a not fully coherent solution with respect to the question posed.

Method of analysis of the students' protocols

Based on the critical issues that emerged from the a-priori analysis of the ChatGPT response discussed in the previous section, we created a grid as a tool for analyzing student assessment. In Table 1, for each row, we have indicated the critical element in ChatGPT's solution and what we look for in the students' evaluations.

The analytical grid was used to code students' responses, according to a qualitative coding approach (Cohen et al., 2017). Each category, identified through a label, corresponds to a specific aspect of the evaluation process. A code was assigned to an answer when the student explicitly referred to the corresponding aspect in their evaluation. In particular, a response was coded as identifying a logical bug (LB) when the student explicitly pointed out an incorrect inference, a missing justification, or an incorrect use of a mathematical concept. The researchers independently coded all the students' answers and then discussed to reach an agreement. The coding framework has been iteratively refined as the analysis proceeded. This inductive approach makes some not predetermined aspects to emerge, as we will discuss in the final section.

Label	Critical element in ChatGPT's solution	What we look for in the students' evaluation
MTR	Missing theoretical references	sentences stating that a reference to some theoretical element is missing or that such a reference is only implicit
RR	Elementary (rough) solution of the linear system	sentences that point out the possibility of alternative solution methods (besides the substitution method) that are more appropriate for the university level (for example, row-reduction or Rouché–Capelli + Cramer)
RvsC	Solution vs consistency	sentences that highlight that it is unnecessary to solve the linear system, but it suffices to verify its consistency, or sentences that refer to the use of the Rouché–Capelli theorem
LB	Logical bug	sentences in which it is recognized that the truth of the statement 'none of the vectors is a linear combination of the others' implies, as a special case, the truth of the statement 'v is not a linear combination of u and w'
NR	Irrelevance of the response	sentences that note that the response provided by the chatbot refers to the core question itself, rather than to the validity of the item proposed as an answer to the core question

Table 1: Grid for the analysis of the students' evaluations

RESULTS OF THE ANALYSIS

To answer the RQ, we analysed students' responses to questions (Q1-Q4).

An example of the coding process is the following:

> S37: ChatGPT used the standard method (setting up and solving the linear system). I could solve it in a slightly more effective and rigorous way by using matrices and the Rouché–Capelli theorem. This would be preferable because it avoids explicitly finding the solutions.

This answer has been coded with RvC, because the last sentence suggests that it is not necessary to find the solutions of the linear system, and as RR, because it refers to a different way to deal with the linear system. The codes shed light on the types of reasoning activated by students and their understanding of ChatGPT's task solution. Students indicated with labels S1-S38 belong to the first sample, students with labels S39-S60 belong to the second sample. The analysis of the students' answers allowed us to highlight that most students (15 out of 38 in the first sample, 21 out of 22 in the second) felt the need for more theoretical references (MTR). Many students (15 in the first sample and 10 in the second) recognized that establishing the system's consistency is enough to answer the question, without solving it (RvsC). In some cases (4 in the

first sample, 7 in the second), they proposed alternative methods to tackle the system (RR), e.g., by using the Rouchè-Capelli theorem or the Gaussian elimination method for linear systems. Only a small number of students (8 in the first sample, 3 in the second) declared that the final answer proposed by ChatGPT is not relevant with respect to the request (NR). Only 2 students in the first sample recognized the logical bug about the inference and quantifiers (LB). Let us examine some examples of students' answers in detail.

Student S27 perceived the need for a more explicit statement about the definition of linear combination; in other excerpts, the need for the explicit transformation from a vector equation to scalar equations arose.

> S27: [Q1, MTR] The solution should have begun with an introduction stating that the vector v is a linear combination of u and w if there exist two scalars a and b such that v=a u+bw.

S1 and S21 clearly stated that ChatGPT performed unnecessary calculations to solve the linear system, since evaluating its consistency is sufficient to answer the request.

> S1: [Q1, RvsC] If the linear system turns out to be inconsistent, the vector v cannot be a linear combination of the two vectors [u and w].

> S21: [Q4, RvsC] It would have been sufficient to show that the system has no solutions.

Some students reported a preference for a more conceptual approach, relying more heavily on theoretical references or alternative interpretations. In some cases, they invoked the Rouché-Capelli theorem or the Gaussian elimination method to either solve the linear system or simply assess its consistency. Student S22 referred to the geometric interpretation of the linear dependence of vector v on vectors u and w, viewing it as lying in the plane spanned by u and w.

> S22: [Q2, RR] I would choose a more conceptual rather than numerical approach. Instead of immediately setting up the system of equations, I would try to understand whether v belongs to the plane generated by u and w by analyzing the relationships among their components.

S4 recognized both the connection between linear independence and linear combination and the logical bug in ChatGPT's answer, concerning the critical handling of the logical implication and quantifiers.

> S4: [Q1, LB] It is incorrect to say that the linear independence of the three vectors is a different concept, since if one vector is not a linear combination of the other two, then those two are certainly linearly independent from each other; therefore, if the three vectors are linearly independent from one another, none of them will be a linear combination of the other two.

S11's answer displayed a high degree of competence and awareness about the connections between pieces of knowledge. Indeed, in planning the solving strategy, she suggested relying on linear independence to conclude that vector v is not a linear

combination of vectors *u* and *w*. In particular, she mentioned the theorem concerning the connection between linear independence and linear combination, correctly stating the quantifiers. Regarding procedural aspects, she used the rank to establish the linear independence of the vectors, instead of solving the linear system, as ChatGPT did.

S11: [Q2, LB&RR] One could solve the question differently by appealing to linear independence: if three vectors are linearly independent, then none of them is a linear combination of the other two; hence, in this specific case, v would not be a linear combination of u and w. [...] you should form the matrix A=[v u w] and compute its rank — if it equals the number of vectors, namely 3, then they are linearly independent.

Student S46 stated that ChatGPT did not answer the request, but did not fully justify. Similarly, other students argued that the answer is inconsistent, since the justification provided is the linear independence of the three vectors.

S46: [Q1, NR] ChatGPT does not answer the request of the task, although the final conclusion is correct

DISCUSSION AND CONCLUSION

Based on the analysis previously presented, we will try to provide some answers to the research question that guided this study. The results show that students are generally able to assess procedural and structural aspects of the solution, such as the correctness of steps or the choice of methods. In particular, they frequently noticed the lack of explicit theoretical grounding and the unnecessary resolution of the linear system, also suggesting more appropriate techniques such as Rouché–Capelli theorem or Gaussian elimination. However, the logical-deductive dimension remains a delicate issue. In fact, only a minority explicitly recognised the logical inconsistency in ChatGPT's resolution, that is, the fact that ChatGPT does not recognise the equivalence between "None of them is" and "Each of them is not". This suggests that while students may confidently judge procedural or structural aspects of a solution, they may be less sensitive to logical nuances involving quantifiers or implications, according to previous study (Durand-Guerrier, 2023). It appears to be particularly relevant in the context of Linear Algebra, where understanding relies heavily on the correct definitions, conditions, and logical inference. Furthermore, the analysis of the students' answers highlighted some logical aspects not considered in our a-priori analysis. Two further data-driven themes (Cohen et al., 2017), sub-items of the already identified 'logical bug', emerged from qualitative analysis, concerning the relationship between the two theoretical notions coming into play. The first sub-item (labelled C) concerns students (9 in the first sample and 10 in the second) who explicitly recognize the theoretical connection between linear independence and linear combination, contrary to what ChatGPT states. They often correct ChatGPT's claim that the two notions were unrelated. The transcripts by S4 and S11 in the previous section already show comments on this issue, as well as the following:

S54: [Q1, C] ChatGPT claims that linear independence is a different concept, whereas in fact vector *v* is not a linear combination of the other two precisely because they are linearly independent.

The second sub-item (labelled U) concerns students (7 in the first sample and 4 in the second) who focused solely on the core question, that is, whether *v* can be written as a linear combination of *u* and *w*, arguing that the verification of the independence of the three vectors is irrelevant for the task at hand. This is shown in the transcript below:

S18: [Q1, U] The question only asks whether *v* can be obtained from the first two [vectors], not to check overall independence.

The latter sub-item suggests that the students' focus is only on the open-ended question and not on the item to be evaluated. In doing so, students miss the opportunity to assess the correctness of the item by relying solely on conceptual relationships, which could foster more concise and meaningful solutions and support the development of relational understanding. Unfortunately, this is in tune with ChatGPT's approach. However, the two new sub-themes highlight the richness of students' interpretative strategies and reveals nuances not initially considered in the a-priori analysis.

In the future, we plan to analyse the other phases of the educational path, concerning students' interaction with ChatGPT. We aim to determine whether this interaction leads to an improvement in the students' mastery of the logical chain of a solving process. In addition, findings of this first phase of the experiment suggests us further research lines. Indeed, students' use of inadequate approaches, such as examining the item proposed as the answer only after solving the open-ended question, as well as difficulties in managing implications, raises new questions for research: Could the origin of students' use of inadequate approaches be related to the now widespread use of AI-based chatbots as tutors? Could the use of AI-based chatbots as tutors amplify, instead of helping to overcome, learning difficulties that are already intrinsically present in managing logical deductions?

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A Preliminary Survey Focusing on Calculus Students' Practices and Beliefs Regarding Their Use of Generative Artificial Intelligence

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While calculus courses remain a gatekeeper for undergraduate STEM students in the U.S., there is a growing phenomenon of students seeking support from generative artificial intelligence (GenAI) tools. However, little is known about how students are utilizing these tools to assist in their calculus studies. Therefore, we developed and piloted a survey to assess students' organic study practices and beliefs regarding the use of GenAI. We detail an initial validation of the scales and include a preliminary analysis which revealed that usage was strongly predicted by perceived learning benefits but showed no significant relationship with beliefs about GenAI's reliability. This suggests that students use GenAI regardless of trust and thus highlights the need for incorporation of AI literacy in mathematics education.

Keywords: Calculus, University Mathematics, Artificial Intelligence, Practices, Beliefs.

INTRODUCTION

Technological advances are rapidly increasing with the advent of generative artificial intelligence (GenAI) becoming freely available and ubiquitous with large language models (LLMs) like ChatGPT and Google Gemini being integrated into our daily lives. Naturally, the conversation surrounding the use of artificial intelligence has also entered mathematics education circles. While many of these discussions have centered around instructors' use of AI in various aspects of their practice such as planning, assessment, and professional development (e.g., Buchholtz & Huget, 2024; Byrne et al., 2025; Pepin et al., 2025), there is little research at present focusing on how students are organically employing GenAI tools in their mathematics studies. With this regard, our informal observations in undergraduate calculus classrooms suggest that students are actively engaging with GenAI technology to assist with their calculus studies. That is, students have communicated to us that they consult applications like ChatGPT for various related activities such as assisting with homework and studying for tests. However, our students are candid that they sometimes do not understand the GenAI output and ask us in our role as instructors to help interpret. Thus, there is an existing phenomenon of students employing GenAI tools to assist with their calculus studies. It is not altogether surprising that students in calculus courses are seeking help from new technologies as calculus has served as a gatekeeper for many STEM students in the United States over the decades [1]. Therefore, students may perceive GenAI as a valid resource that provides them with the necessary tools they need to be successful in such a high-stakes course.

Motivated by our observations, we were curious to explore this phenomenon further. In particular, we wanted to understand how students use GenAI tools like ChatGPT in their own calculus studies and the beliefs students have regarding the use of such tools in this context. To this end, we developed and piloted a survey instrument designed to measure aspects of students' use of GenAI in their calculus studies. In this paper, we present the details of the instrument development as well as the preliminary findings from the survey with the aim of better informing researchers, instructors, and curriculum developers in this new world of AI.

LITERATURE REVIEW

In recent years, researchers have investigated the feasibility of utilizing various forms of GenAI in the teaching and learning of mathematics. These studies have theorized and provided some evidence to the affordances and disadvantages regarding the incorporation of GenAI with mathematics teaching and learning. Namely, GenAI may greatly supplement instructional practices because of its capability to provide personalized and real-time feedback for learners (e.g., Pepin et al., 2025; Schorcht et al., 2024). This is incredibly attractive to mathematics instructors, particularly to those teaching gateway mathematics courses like calculus. While mathematics departments are experimenting with new ways to adapt to an increasingly diverse calculus population (Biza et al., 2022) and mitigate issues of large class sizes (Johnson et al., 2022), it is important to identify new avenues of individualized support to students. For example, Pepin et al. (2025) conducted a scoping review and noted several themes among the literature. GenAI may support administrative aspects of the classroom such as generating and grading assessments. As well, the technology can help facilitate active learning sessions by organizing small groups, assigning roles to group members, and by providing real-time feedback to team members. Additionally, GenAI has the potential to "work" directly with students as a didactical "conversational partner," helping students solve problems and provide reflective questions to help them understand the rationale for their work.

On the other side of the same coin, GenAI tools have some limitations. That is, GenAI technology is not subconsciously bound by the same curriculum constraints of a mathematics instructor. For instance, Yoon et al. (2024) observed ChatGPT confusing a student by providing a proof by contradiction to a mathematical statement when the student was not yet familiar with that proof technique. Arguably, an instructor with knowledge of the students' academic background (likely based on the course in which the student was enrolled) would not employ unfamiliar techniques unless it was didactically sound. Another concern common among students and instructors is the reliability and transparency of outputs produced by GenAI. GenAI models such as ChatGPT 4.0 have been observed to experience varying degrees of false reporting, "hallucinations," and engaging in incorrect mathematical reasoning (Pepin et al., 2025; Schorcht et al., 2024). In fact, Siebold et al. (2024) surveyed 54 preservice teachers (PST) and found that "over 80% of the PSTs were concerned about the accuracy of LLM outputs" (p. 376). This was also a concern among Yoon and colleagues' (2024)

participants who perceived ChatGPT as a "knowledgeable but fallible tool" (p. 1547). Such views about GenAI tools in mathematics suggest that students who use GenAI tools in their mathematics studies should have a critical eye toward the outputs. Additionally, the literature consistently cites a variety of ethical concerns among all users of GenAI (e.g., Pepin et al., 2025; Schorcht et al., 2024; Siebold et al., 2024; Yoon et al., 2024). Many are reluctant to utilize the tool for fear of privacy violations and environmental impact. Moreover, there is some concern among students that using GenAI to assist with homework problems and other assignments may be a violation of academic integrity (Yoon et al., 2024). Thus, students who have such concerns might encounter a dilemma. While GenAI may provide beneficial assistance for calculus-related studies, users risk being influenced by inaccurate responses and possibly engaging in unethical practices.

Beyond these considerations of ethics and mathematical inaccuracy, there is also concern among educators and researchers that students could trust GenAI with little to no skepticism or critical evaluation of mathematical outputs (Kasneci et al., 2023). In fact, several researchers cautioned against an over-reliance on GenAI tools. Rather, students should engage in responsible use of the technology like one of Yoon et al.'s (2024) participants, Jiho, who used ChatGPT "as a source to obtain ideas that serve as a starting point" (p. 1540). Similarly, Pepin et al. (2025) advised that the technology should be balanced with "activities that promote active problem-solving and self-reflection" (p. 28) to mitigate such concerns. Thus, with the inevitable inclusion of GenAI tools in the teaching and learning of mathematics, particularly in the context of gatekeeper courses in the U.S. like calculus, it is imperative to develop a deeper understanding of how students are already navigating their calculus learning experience with GenAI tools so that instructors and researchers may better prepare them for the future.

METHODS

Participants and Context

The focus of this study was to develop a survey that examines undergraduate calculus students' current practices and beliefs about GenAI and how they use GenAI tools in their calculus coursework. We invited students who were enrolled in an undergraduate calculus course (Calc. 1, Calc. 2, Calc. 3, or Business Calculus) at a large public university in the Southeastern U.S. during the Spring or Summer 2025 semesters. University calculus courses in the U.S. typically cover a range of topics over a sequence of three courses. Calc. 1 focuses on single variable differentiation. Calc 2 focuses on methods of integration, sequences, and series. Calc. 3 focuses on a wide range of topics in multivariable calculus. Business Calculus may be taken by non-STEM majors and only covers applications of single variable calculus for business and economics contexts. We received 189 responses; analytic samples ranged from 91 to 114 depending on model-specific missingness. Several factors may have influenced the low response rate, including not offering an incentive as well as sending emails to students after the semester had completed while students were on semester break. Respondents

represented a range of majors with over 90% comprising STEM and Business majors. Additionally, all reported responses met the inclusion criteria that respondents indicate that they had used GenAI in their calculus studies at least once.

Procedure

Participation in the survey was voluntary, and the study protocol was approved by the university's institutional review board. The survey was administered online via Qualtrics. After providing informed consent, students were prompted if they had ever used GenAI in their calculus studies. In order to complete the remainder of the survey, respondents must have positively responded to this item. Students then completed three main sections: demographic and background information, practices and beliefs, and an open-ended applied calculus problem to be completed utilizing GenAI, specifically ChatGPT. For the latter section, students provided their ChatGPT input and responded to several open-ended questions about their experience and understanding of the applied problem. The analysis of the applied problem section is beyond the scope of the current paper and is not discussed further. We only report survey development and preliminary analysis between *practice* and *beliefs* items.

The initial item pool included 28 *practice* and 23 *beliefs* items which were inspired partly by Siebold et al.'s (2024) survey documenting pre-service teachers' attitudes regarding the use of AI in a mathematics education context as well as the authors' own observations and experiences with calculus students who reported using AI to aid in their coursework. The *practice* items were designed to assess how undergraduate students engage with GenAI tools in their calculus-related studies and how often they do so. *Practices* were rated on a five-point frequency scale (1: always, 2: frequently, 3: sometimes, 4: rarely, 5: never) and originally covered five dimensions: *AI usage and context*, *solving calculus problems*, *practice and specific use*, *academic support*, and *communication with professors*. Following item screening and parallel and exploratory factor analysis (EFA), we focused on 14 items which explained three dimensions: *solution-focused*, *conceptual study*, and *assessment-focused* practices. Similarly, the *beliefs* items were designed to assess students' perception of the helpfulness of using GenAI in their calculus-related studies. *Beliefs* were rated on a five-point agreement scale (1: strongly agree, 2: agree, 3: neutral, 4: disagree, 5: strongly disagree) and originally covered three dimensions: *effectiveness and learning impact*, *ethical considerations and academic integrity*, and *challenges and limitations*. Following item screening and EFA, we focused on 11 *beliefs* items which explained two dimensions: *perceived learning benefits* and *reliability and clarity* beliefs.

Analysis

All analyses were conducted in R using the psych (Revelle, 2025), lavaan (Rosseel, 2012), and ggeffects (Lüdecke, 2024) packages. We followed a multistep survey refinement and validation process. First, item distributions were inspected for skew, near-zero variance, and response patterns indicating potential misfit. All five-point Likert items were treated as ordinal for factor-analytic models and continuous for

regression models, consistent with common practice in educational measurement. We conducted EFA separately for the *beliefs* and *practice* items. Minimum residual extraction with oblimin rotation was used, and parallel analysis guided factor retention. For the practice items, EFA supported a three-factor model representing *conceptual study*, *solution-focused*, and *assessment-focused* practices. For the *beliefs* items, parallel analysis suggested a two-factor model capturing *perceived learning benefits* and *reliability and clarity* beliefs. Four negatively phrased reliability items were reverse coded so that higher scores reflected greater perceived reliability.

Next, we conducted confirmatory factor analyses (CFA) using the WLSMV estimator for ordinal indicators. The CFA specified a three-factor and two-factor structure for *practices* and *beliefs*, respectively, consistent with the EFA. All items included in the CFA demonstrated strong standardized loadings, and both models showed acceptable global fit given the modest sample size. Latent correlations indicated that the three *practice* domains were positively related but distinct and that *perceived learning benefits* and *reliability and clarity* beliefs were largely independent.

Composite scores were then computed by averaging the items retained in the final CFA models, and all composites were standardized (z-scores) for interpretability in subsequent analyses. To examine relations between *practices* and *beliefs*, we estimated three separate linear regression models. Each model predicted one standardized practice composite: *conceptual study*, *solution-focused*, and *assessment-focused* practices. Predictors included standardized *perceived learning benefits* beliefs, *reliability and clarity* beliefs, and *prior calculus experience*. The latter was coded as a three-level factor (1: none, 2: high school, 3: college/university). All models used listwise deletion on predictors and outcomes. We report standardized regression coefficients (β), confidence intervals, and model R^2 values. Effects were interpreted using standardized betas and Cohen's f^2 to characterize the magnitude of explained variance associated with each model.

RESULTS

Item-level missingness was common in both the *beliefs* and *practice* scales. For the *beliefs* items, complete-case CFA estimation required the exclusion of participants with no responses on any *belief* item, resulting in N = 91 for the final CFA. For the *practice* items, complete cases yielded N = 114 for the CFA. Listwise deletion in the regression models yielded N = 91 to N = 96, depending on the pattern of missingness across predictors and outcomes. Figures 1 and 2 display students' reported practices and beliefs, respectively, for our sample. Percentages on the left-hand side represent combined negative responses (i.e., never & rarely, strongly disagree & disagree). Percentages on the right-hand side represent combined positive responses (i.e., always & frequently, strongly agree & agree).

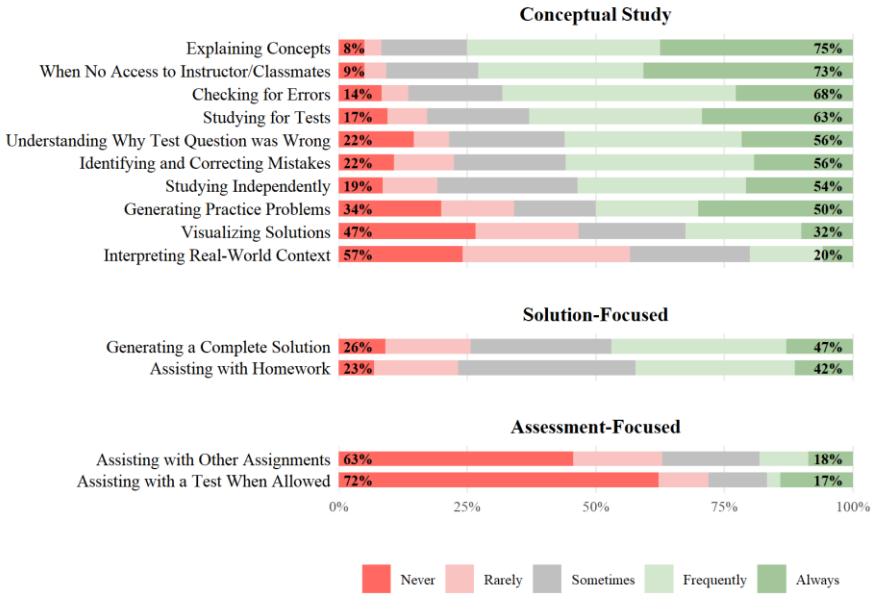

Figure 1: Students' reported frequency of GenAI practices

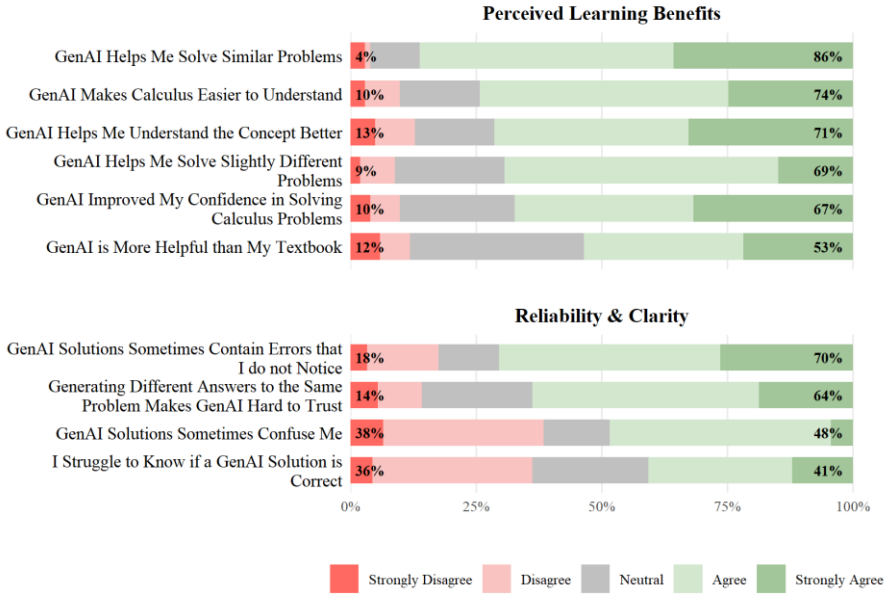

Figure 2: Students' reported agreement of GenAI beliefs

Measurement Models

A three-factor confirmatory factor model was estimated for the *practice* items. WLSMV estimation was used to accommodate the ordinal nature of the items. The model demonstrated mixed fit (CFI = .899, TLI = .876, SRMR = .095) with an elevated RMSEA (.115) that likely reflects the modest sample size. Standardized factor loadings were strong across all three factors and ranged from .53 to .94. The three latent factors were moderately and positively correlated (r = .53 to r = .62), indicating that although the practices are related, they represent meaningfully distinct ways in which students

use GenAI. Similarly, a two-factor confirmatory factor model was also estimated for the *beliefs* items. The model demonstrated reasonable fit (CFI = .953, TLI = .937, SRMR = .10) with an inflated RMSEA (.115) that is consistent with expectations for small samples and models containing many ordinal indicators. Standardized factor loadings for the learning-benefits items ranged from .74 to .89, and loadings for the reliability items ranged from .54 to .75, indicating that all retained items functioned well as indicators of their respective constructs. The correlation between the two latent belief factors was small ($r = .20$, $p < .026$), suggesting that students' perception that GenAI enhances their learning is largely independent of their beliefs about its clarity and reliability.

Relationship Between Practice and Beliefs

We conducted three separate standardized linear regression models to examine how *beliefs* and *prior calculus experience* predicted each *practice* domain. All predictors and outcomes were z-scored, enabling direct interpretation of coefficients as standardized effects. The first model predicting *conceptual study practice* explained 43% of the variance. *Perceived learning benefits* emerged as a strong positive predictor for this model ($\beta = .62$, 95% CI [.43, .74], $p < .001$). This suggests that students who believed that GenAI improved their understanding of calculus concepts and similar problems were more likely to use the tool for studying, conceptual understanding and error-checking tasks. In contrast, neither reliability beliefs nor prior calculus experience were significant predictors of *conceptual study practice* ($p > .13$). The second model predicting *solution-focused practice* explained 16% of the variance. Again, *perceived learning benefits* significantly predicted this practice ($\beta = .39$, 95% CI [.18, .55], $p < .001$). Thus, students who perceived learning benefits from using GenAI reported somewhat more frequent use of GenAI to generate solutions or assist with homework. *Reliability and clarity beliefs* and *prior calculus experience* were not associated with solution-focused use. Finally, the third model predicting *assessment-focused practice* explained 21% of the variance. Two predictors were significant. Namely, *perceived learning benefits beliefs* positively predicted *assessment-focused practice* ($\beta = .30$, 95% CI [.10, .48], $p < .003$), indicating that students who viewed GenAI as helpful were also more likely to use it in graded or test-related contexts when allowed. Additionally, *prior calculus experience* was also a significant predictor. That is, students with no prior calculus experience reported significantly lower *assessment-focused practice* compared to students with high-school or college calculus backgrounds ($\beta = –.31$, 95% CI [–1.79, –.35], $p < .004$). This may reflect differences in familiarity or confidence with using technological tools in general to assist in calculus studies, though this interpretation remains hypothetical. *Reliability and clarity beliefs* were unrelated to the *assessment-focused* practice.

DISCUSSION

This study had two primary aims. Our first aim was to develop, refine, and validate measures capturing how undergraduate calculus students use GenAI tools (*practices*) and what they believe about their engagement with these tools (*beliefs*). After

experiencing a rather modest response rate, we conducted refinement and validation analyses in which a three-factor model for *practices* (consisting of *solution-focused*, *conceptual study*, and *assessment focused* practices) and a two-factor model for *beliefs* (consisting of *perceived learning benefits* and *reliability and clarity* beliefs) emerged. Using these refined models, we focused on our second aim, which was to provide a preliminary analysis of the survey results. We examined the relationship between students' *practices* and *beliefs* and included one covariate, prior calculus experience, in our linear regression models. The regressions provided a nuanced perspective of how these aspects are related. Unsurprisingly, *perceived learning benefits* significantly predicted all three GenAI practices. Additionally, the regression models also suggest that students who have not had a calculus course engage with GenAI less often than those who have prior experience. The latter phenomenon could be due to multiple factors including unfamiliarity with terminology and tools, suggesting that students should receive training to incorporate GenAI as an academic support of their mathematics learning like any other new technology.

Possibly more notable was the lack of a significant relationship between students' practices and their beliefs regarding reliability and clarity of GenAI. The literature has extensively documented the mathematical limitations of GenAI and LLMs, specifically their tendency to "hallucinate" or provide plausible but incorrect reasoning (Pepin et al., 2025; Schorcht et al., 2024). Figure 2 suggests that students are generally aware that GenAI tools can provide inaccurate responses as well. In some cases, students admit that it is difficult for them to determine if the response is valid. The literature also supports that students are aware of the limitations of mathematical accuracy (e.g., Siebold et al., 2024; Yoon et al., 2024). Regardless, we did not find a significant association between *reliability and clarity beliefs* and any of the *practice* domains in this sample, though the models may have been underpowered to detect modest effects. Based on our regression models, if students believe that GenAI helps them learn calculus, then they will use it anyway. Thus, their skepticism about accuracy does not appear to be a barrier to usage. This could suggest that another phenomenon is present in which students critically evaluate outputs rather than blindly accept them as truth.

Students have expressed trepidation of employing the use of GenAI in their studies because of the possible inaccuracies. For instance, Yoon et al. (2024) interviewed a participant who was reluctant to use GenAI to assist with writing a proof because it used the unfamiliar proof by contradiction technique. Interestingly, the researchers reported that the student's final answer was based on proof by contradiction, suggesting that it was largely informed by the GenAI output. Thus, other students may be engaging in similar practices. To some degree, students are critically evaluating responses and making judgements about whether or not the responses make sense. This may point to a need to reconsider how GenAI use is framed in educational contexts, though this interpretation extends beyond the present data. The widely accepted term "artificial intelligence" denotes an independent "intelligence" at work. Yet, this is far from the way it is being employed in many contexts, especially within calculus studies. Rather,

students are "augmenting" the intelligence by providing direction and critically evaluating responses with their own intelligence. This suggests that the term "artificial intelligence" may be a misnomer in the educational context; students are not outsourcing their intelligence, rather they are augmenting it to overcome barriers of a gatekeeper calculus course. Therefore, we suggest that mathematics educators pivot toward AI literacy as students are already using the technology to assist with their studies. If students are prioritizing utility over reliability, instruction must explicitly address how to prompt GenAI as well as critique, interpret, and verify its beneficial but error-prone outputs.

Limitations and Future Directions

Several limitations should be considered in interpreting these findings, given the small sample size and item-level missingness. The sample size for factor analyses and regression models was modest, limiting the stability of our estimates. Additionally, the small sample may not represent the full calculus student population, especially those with less interest in or familiarity with GenAI. As this was just a pilot study and preliminary analysis, we plan to distribute the survey to larger samples as well as to modify the scales to better capture several dimensions such as students' ethical concerns with GenAI. Finally, we plan to analyze students' open-ended survey responses in which they interacted with ChatGPT to solve an applied calculus problem, and we will eventually conduct clinical interviews in which we can observe students "augmenting" with the technology. Investigating further in this manner should provide some insight into how students make decisions about how they determine if an output is reliable and correct. Despite the limitations listed, the present study contributes an early empirical framework for understanding how students think about and use GenAI in their calculus studies. The refined scales and observed associations provide a foundation for future research and practical guidance for instructors navigating the evolving role of GenAI in mathematics education.

NOTES

1. We recognize that calculus being a gatekeeper for STEM students may not a phenomenon in other countries. Therefore, our reference to calculus as a gatekeeper throughout this paper refers specifically to the U.S. context.

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Can AI Support the Development of Resilience, Perseverance and Metacognition in Learning Mathematics?

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This study presents the design of a learning framework that explicitly teaches "how to learn mathematics", with a focus on developing autonomy, resilience, and evaluative judgement. The framework is grounded in the concept of productive struggle and in theories of metacognition and self-regulated learning, addressing a gap where these skills are often assumed but not explicitly taught. The design incorporates AI-generated responses as a reflective tool, enabling students to engage in structured critique and evaluate reasoning. The framework is being implemented with undergraduate mathematics students. Ongoing evaluation draws on student reflections, surveys, and focus groups to explore how learners engage with the framework. Early insights suggest that this approach can strengthen confidence, build resilience, and promote deeper engagement with mathematical thinking.

Keywords: Learning how to learn, maths confidence, resilience in Higher Education, growth mindset, metacognitive skills, productive struggle.

INTRODUCTION AND CONTEXT

Metacognition and self-regulated learning are foundational to effective mathematical learning, yet they are rarely taught explicitly in undergraduate curricula. This gap can hinder problem-solving, contribute to mathematics anxiety, with potential implications for progression and attainment. In digital learning contexts, emerging approaches suggest that tools such as AI-generated solutions can be used to support reflective and evaluative learning processes. **RQ:** How can a pedagogical framework that integrates AI-generated responses support the development of resilience, perseverance and metacognition in undergraduate mathematics learners?

DESIGN OF THE PEDAGOGICAL FRAMEWORK

Engaging with challenge requires learners to explore their own strategies, revise their thinking, and persist through difficulty. Building on this perspective, this study proposes a pedagogical framework that supports the development of key learning capacities, including resilience, perseverance, confidence, and metacognition.

The Framework: CURATE Learning Cycle

The CURATE Learning cycle is a structured way to think about learning through productive struggle.

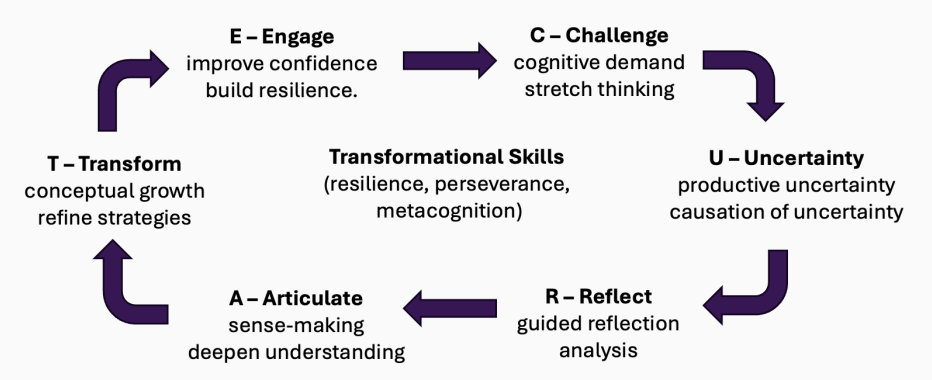

Figure 1: CURATE Learning Cycle

Role of AI.

AI is used to explore how students engage with mathematical struggle. Students are presented with mathematical proofs that are incomplete or contain gaps in justification, and they are required to interact with AI by asking targeted questions in order to refine, and make sense of the proof. Students' engagement with the task is guided through the CURATE Learning Cycle.

METHODOLOGY AND INSIGHTS

A design-based research approach is used to iteratively develop and refine the framework in practice. The approach consists of:

- Focus groups: Explore students experiences and perceptions about struggle and how to address it.
- Surveys: Capture changes in confidence and learning approaches before and after introducing the framework and associated theories.
- Observed activities: Analyse engagement in metacognitive processes.

Emerging Insights.

- Students showed increased awareness of their thinking patterns and processes.
- Engagement in evaluating mathematical reasoning and identifying gaps in learning.
- Positive responses to collaborative and hands-on activities.
- Indications of increased confidence and willingness to engage with challenge.

Those findings are preliminary and part of ongoing evaluation.

Contribution.

This study

- proposes a framework to explicitly teach how mathematics is learnt and explored.
- introduces a structured approach to identifying and addressing uncertainties in mathematics.
- integrates metacognitive development within mathematical practice.
- demonstrates how digital tools (e.g., AI) can support reflective learning.
- Encourages the development of confidence and resilience in response to challenge.

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Large Language Models as Co-designers in the Creation of 'Sci-problems': An Analysis of Interactions

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Keywords: calculus for non-mathematicians, reality problems, ATD, case studies, artificial intelligence.

SCI-PROBLEMS CO-DESIGNED BY AN LLM

A *sci-problem* $S \rightarrow M$ is a problem set in a scientific domain S that requires tools from a mathematical subdomain M (Rizzo, 2025): it may be views as the tertiary-level analogue of school word-problems, where the context is scientific rather than everyday, under the assumption that solving such a problem may render M relevant to a student primarily interested in S. A sci-problem needs to *look* like a real problem in S, but it does not need to be one, its goal being to help students make sense of M.

Designing a proper (hard) sci-problem requires not only solid content knowledge (CK) and Pedagogical Content Knowledge (PCK) of Mathematics, but a level of scientific content knowledge $\mathrm{CK}(S)$ which typically exceeds what could be expected from most calculus instructors. Hence, an interdisciplinary team would normally be required.

Large Language Models (LLMs) are particularly effective at semiotic transformation; they can explain specialistic content to non-specialists and, in the proper setting and under realistic institutional constraints, may constitute a viable substitute for an interdisciplinary team in the design of sci-problems (Rizzo & Vergallo, 2025).

Here we extend our previous case studies to the interactions between calculus instructors (and teaching assistants) and LLMs in the design of sci-problems for a specific Calculus course, focusing on the evolution of praxeologies and on how the level of $\mathrm{CK}(S)$ shapes such interactions.

DIDACTIC SETTING

The didactic actors

First year students in a Natural Sciences programme, quite weak in Quantitative Literacy (PIQL (White Brahmia et al., 2021)), high Maths anxiety level (BMAS (Núñez-Peña & Guilera, 2023)), historically high failure rate in Calculus. Teaching staff: A: Experienced calculus instructor, PhD in Arithmetic Geometry. B: PhD candidate in Maths Education, M.Sc. in Natural Sciences. C: Recently tenured instructor, PhD in Algebraic Geometry. D: same, but PhD in Maths Education.

Didactic codeterminacy

The Calculus course operates under very weak institutional constraints: number of credits, the timetable, and a generic expectation of mathematical learning; content

and structure are almost entirely determined by its instructors. At the same time, the course is laterally codetermined by parallel courses in botany, chemistry, zoology, biology and physical geography.

SAMPLE INTERACTIONS

Hallucinations are an intrinsic feature of LLMs, and controlling them is a major issue in their educational utilisation. Colleague C asked ChatGPT to prepare a problem involving integration. A function was proposed to represent the concentration $C(t)$ of a chemical intermediate, measured in mol/ℓ, with t denoting the time in hours. The LLM then proposed computing the *total concentration* as $\int_0^{+\infty} C(t)\mathrm{d}t$.

> C: The concentration is in mol/L, while the integral you are asking to compute is with respect to time — are you sure this makes sense?

The praxeology $[T/\tau/\theta/\Theta]$ emerging in this human–LLM interaction is an epistemic control praxeology, in which the task T consists in identifying and correcting LLM hallucinations (epistemic failures), and the technique τ relies on dimensional analysis. This technique presupposes a non-trivial level of $\mathrm{CK}(S)$.

A different epistemic stance emerges in colleague B's interaction:

> B: The formulas you proposed in the problems — did you invent them, approximate them, derive them from real data, or something else?

This passage enacts a *praxeology of epistemic traceability*, in which hallucinations are addressed by questioning the provenance and epistemic status of the formulae rather than their formal correctness.

These two episodes reveal two distinct control strategies: scientific validation and epistemic traceability. Together, these praxeologies exemplify a broader landscape for the integration of LLMs into tertiary-level service mathematics.

ACKNOWLEDGEMENTS

This work has been partially supported by GNSAGA–INdAM. The authors thank S. Lippiello and M. Carli for an early access to the Italian translation of PIQL.

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TWG5: Teaching university mathematics: practices, innovation, and professional development

Influence of Initial Didactics Training on the Teaching of Quadratic Functions and Equations: A Case Study in French-Speaking Belgium

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Keywords: quadratic functions, quadratic (in)equations, initial teacher education, semiotic representation registers

CONTEXT

We present research on teachers' practices related to the teaching of quadratic equations, inequalities, and functions in French-speaking Belgium. A specific feature of these contents is that they are taught within the same school year, which may differ across countries (Chaachoua et al., 2024). A first study of reported practices (Dumonceau, 2023) showed that the teaching of quadratic topics does not simply consist of a juxtaposition of the frameworks (Douady, 1987) of Analysis and Algebra, and that teachers seem to choose teaching scenarios that allow these two frameworks to be articulated. It was subsequently determined that the sample comprised only teachers who had undergone didactics training. We therefore investigate to what extent initial teacher education in mathematics influences their teaching practices.

PROBLEMATIC

An epistemological, curricular, and cognitive analysis of the content, referred to as a "relief study" (Robert in Vandebrouck, 2013), enabled us to describe the possible conceptualisation of the notions to be taught (Dumonceau, 2023). It highlights the diversity of representation registers (Duval, 1993) and the need to move between them to address the tasks prescribed for students. Although challenging for students, such work supports the conceptualisation of mathematical notions (Gueudet & Robert, 2022), and it is also a result recognised as robust in mathematics education research (EMS, 2011). This raises questions about teachers' didactical knowledge, particularly regarding semiotic representation registers, and more broadly about the knowledge required for teaching mathematics, as described in the Mathematical Knowledge for Teaching model (Ball, Thames & Phelps, 2008). Previous studies show that prospective teachers often complete their studies without a deep understanding of the content they will teach (e.g., Burroughs et al., 2023). At the University of Mons, the initial training programme enables students to undertake courses and participate in projects within the domain of didactics of mathematics aimed at developing didactical knowledge. However, some teachers do not benefit from this training (career changers, absence of training at the time). In this context, our research question is: *How can specific teaching on the concept of representation registers influence teaching practices for quadratic (in)equations and functions?*

To address this question, we extended the previous study (Dumonceau, 2023) by conducting interviews with five teachers without didactics training. The interview

guide focuses on chapter organisation, “Lessons” section, proposed exercises, and students’ difficulties. These data are complemented by an analysis of teaching materials and compared across teacher profiles.

CONTENTS OF THE POSTER AND PERSPECTIVES

The poster will present the first results of this study. It will highlight teachers’ choices regarding chapter organisation, comparing teachers with and without didactics training. It will also analyse the “Lessons” section and the exercises proposed, emphasising differences in the diversity and complexity of examples, as well as in the use of representation registers and their conversions. These results will be illustrated by excerpts from interviews and teaching materials. Finally, it will discuss the impact of didactics training on teaching choices and outline perspectives for analysing classroom practices.

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Building the Definition of Limit
A Discursive Analysis of Classroom Practice

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Lecturing is the main teaching mode at university, but significant communication gaps persist between university mathematics instructors and their students. This study examines how the use of visual mediators shapes an instructor's introduction of the definition of the limit of a sequence in a first-year analysis class. We analyse video recordings of the lecture through a commognitive lens. We show that the instructor gradually transitions to mathematically literate discourse through staggered changes in terminology and visual mediators. Moreover, we demonstrate how coordinating multiple visual mediators enables the instructor to highlight different aspects of discourse, emphasizing either what changes or what remains constant across representations.

Keywords: real analysis, lecturing, commognition, classroom observation.

INTRODUCTION

While lectures have long been the dominant instructional approach in university mathematics, research on lecture teaching is still nascent (Speer et al., 2010). Work in the past decade has shown that lecturing in mathematics is remarkably consistent: instructors typically write formal mathematical results on the board while providing meta-commentary–like arguments, explanations, or "informal mathematics"–aloud (Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017). Moreover, while instructors' preference for lectures partly stems from practical concerns (like the time required to implement other approaches), many instructors favour this teaching approach for more conscientious reasons (Melhuish et al., 2022). For example, they argue that the lecture format ensures precision and is essential for modelling mathematical behaviour for students. Indeed, evidence suggests that the lecture acts as a model of mathematical activity (Viirman, 2021), but what is modelled can vary between instructors (Pinto, 2017).

What remains unclear is whether students grasp the modelling enacted by their instructors. Studying a lecture on a proof, Lew et al. (2016) showed that what students and their instructor identified as mathematically relevant in the proof presentation overlapped very little. Moreover, Fukawa-Connelly et al. (2017) found that students primarily take notes of what is written on the board, and rarely capture the meta-commentary the instructors provide. Therefore, even when instructors try to convey specific meanings, these may not reach their students, suggesting potential breakdowns in communication.

This may be particularly troublesome at the transition to university, where shifts in mathematical notation (which becomes increasingly symbolic), in mathematical activity (from procedural work to more abstract and formal reasoning) and in the relationship between students and instructors can contribute to misunderstandings (De Guzmán et al., 1998; Gueudet, 2008). Although these discontinuities have been documented, little is known about which aspects of the lecture contribute to or help bridge them. The

overarching goal of this research project is thus to examine features of lectures and their impact upon what can be understood by students in these lectures.

In this paper, we introduce a first study in this context, analysing the use of multiple representations during a lecture. Other theoretical frameworks have examined the importance of multiple representations for mathematical understanding (Duval, 2017), but our focus is on how lecture features facilitate communication between instructors and students. Consequently, we are not concerned with the role of representations in mathematical activity, but rather with their communicative function within lectures. Adopting a discursive (commognitive) perspective allows us to examine how these representations operate simultaneously at mathematical and communicative levels.

THEORETICAL BACKGROUND

In the commognitive framework (Sfard, 2008), mathematics is viewed as a discourse – a form of communication between participants about mathematical objects. This discourse is characterised by its *word use* (vocabulary), its *visual mediators* (the visible objects used in communication), and its *narratives* (the statements about mathematical objects and their relations), said to be endorsed when accepted by the mathematical community. Finally, the discourse is marked by the recurrent patterns of behaviour in which its participants engage, called *routines;* these may involve manipulating objects (as when doing long division) or creating new patterns of discourse (as when carrying out a proof). Notably, the commognitive framework has been used to analyse university mathematics instruction. Previous lecture analyses have demonstrated recurring patterns (routines) in how instructors teach, both in terms of their pedagogical approaches and mathematical behaviour they model (Karavi & Mali, 2026; Viirman, 2015, 2021).

Within commognition, learning may occur in two ways: by expanding the endorsed narratives about an object (*object-level*) or by expanding the types of narratives and routines permitted within the discourse (*metalevel*). In a real analysis course, for example, object-level learning involves understanding concepts like sequences or continuity, while meta-level learning involves knowing when and how to use definitions and proofs. Metalevel learning is especially challenging because it involves a shift, not only in the object of talk, but in how one talks about it.

In situations that involve metalevel learning, students' mathematical discourses may be at odds with those of their instructors. Because metalevel learning entails changes to the discourse itself, the rules guiding students' discourse may no longer suffice for interpreting that of the instructor. In these cases, where incommensurable discourses are at play, we talk about *commognitive conflict*. Such conflicts surface, for example, when the same mediators are used according to different rules or when discussants use the same visual mediators to endorse conflicting narratives.

In the first year of university, for instance, students have been found to use the rules of secondary school mathematics to operate on the symbols (visual mediators) learned at university (Thoma & Nardi, 2018). Similarly, instructors and students both use the same visual mediators to communicate about limits, but students endorse a limit-as-

process narrative while instructors endorse a limit-as-number narrative (Güçler, 2013). Finally, the discrepancy in what instructors and students consider mathematically relevant in a proof may reflect different framings of the same discursive activity, shaped by each participant's perception of the discourse at play. These discrepancies in student and instructor discourse suggest that commognitive conflict may be at play during this transition.

While this suggests that students may struggle to understand their instructors, commognitive conflict is not one-sided. Because advanced mathematical discourses subsume more elementary ones, experts may struggle to return to earlier points of understanding to the same extent that novices struggle to adopt more sophisticated forms of discourse. The question, then, is whether there are aspects in instructor discourse that particularly facilitate or impede communication.

One aspect that warrants consideration instructors' use of visual mediators. From a commognitive viewpoint, using visual mediators can help resolve commognitive conflicts (Sfard, 2008, p. 153). When interlocutors' discourses seem incompatible, visual mediators (graphs, gestures, or symbols) can provide a shared reference point, while explicit description of scanning procedures can make participants' interpretations more transparent to one another. Accordingly, in our analysis, we focus on the use of multiple representations, which we hereafter refer to as visual mediators.

The role of visual mediation

It is well established that the use of multiple visual mediators is crucial in mathematics teaching and learning (Arcavi, 2003). Different representations highlight different aspects of mathematical objects or situations, potentially supporting student understanding. Some researchers even argue that fluency in multiple semiotic systems as well as between them is a requirement for mathematical understanding (Duval, 2017). Still, managing multiple representations can be challenging. Interpreting individual representations takes time and practice, and understanding the links between representations is itself demanding. Consequently, the potential benefits of using multiple representtations during lectures may be mitigated by the interpreting challenges they present.

To examine how visual mediators shape lecture-based teaching, we focus on the case of the formal definition of the limit, where differences between instructor and student discourse have been previously documented. Its introduction marks an important shift in mathematical activity, from the procedural task of finding a limit to the more formal task of justifying that a given value is indeed the limit. Hence, this paper explores how visual mediators are used in, and support, this definition. We address the following research questions:

RQ1: How are word use, visual mediators, endorsed narratives, and routines coordinated in the discourse surrounding the presentation of the definition of limit?

RQ2: How do visual mediators support or constrain the endorsement of narratives and the enactment of routines within the discourse?

METHODOLOGY

This study is part of a larger project drawing on classroom observations and interviews with instructors and students. We focus here on the analysis of a single lecture episode. Offering a detailed description of a classroom episode, this analysis is not intended for generalisation. Instead, it aims to offer a rich description that informs a framework for comparing the discourses of instructors and students.

The study took place in Real Analysis courses for engineering students at a large technical university in Europe. This course is one of two required mathematics courses in the first semester and is widely regarded as particularly challenging. At the start of the Fall 2025 semester, all eight instructors teaching the course were invited to participate; five agreed, four men and one woman, all of whom held teaching (not academic) positions at the university. Each had at least five years of teaching experience, and all but one had taught Real Analysis at least twice before.

For each instructor, four two-hour lessons were observed during the semester, two of which were video recorded. We analyse video data from a single instructor's segment in the second lecture, where he introduces the definition of the limit of a sequence:

> Let $(a_n)_{n\geq 1}$ be a sequence and $l \in \mathbb{R}$. We say that $(a_n)_{n\geq 1}$ tends to l if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}^*$ such that for all $n \geq N$, $|a_n - l| \leq \varepsilon$.

This instructor's presentation was selected because it relied heavily on visual mediation and therefore lent itself to the type of analysis we were interested in. The analysed episode begins when the instructor first introduces the concept of limit and ends when the entire definition has been fully explained.

To conduct the structured commognitive analysis of the episode, we first divided it into shorter segments. For each of them, we noted the vocabulary and visual mediators used by the instructor, including whether the vocabulary was written on the board. We then identified the narratives he endorsed. Since the time frame is too short to discern meaningful repetition patterns, we do not focus on routines here.

RESULTS

Coordination of discourse elements

Before working on the definition, the instructor defines sequences and presents four examples—each comprising an algebraic expression, a list of numerical values, and a corresponding sketch, engaging multiple representational modes from the outset. He then transitions to introducing the definition of the limit. **Table 1** details the flow of his presentation, including the words and visual mediators used, as well as the corresponding endorsed narratives. Overall, the presentation progresses from an initial intuitive introduction of the concept to a graphical example supported by a metaphor, then to a verbal articulation of the idea, and finally to the formal definition.

During his presentation, the instructor's discourse gradually aligns with advanced mathematical discourse, as seen in his word choice, visual mediators, and the narratives

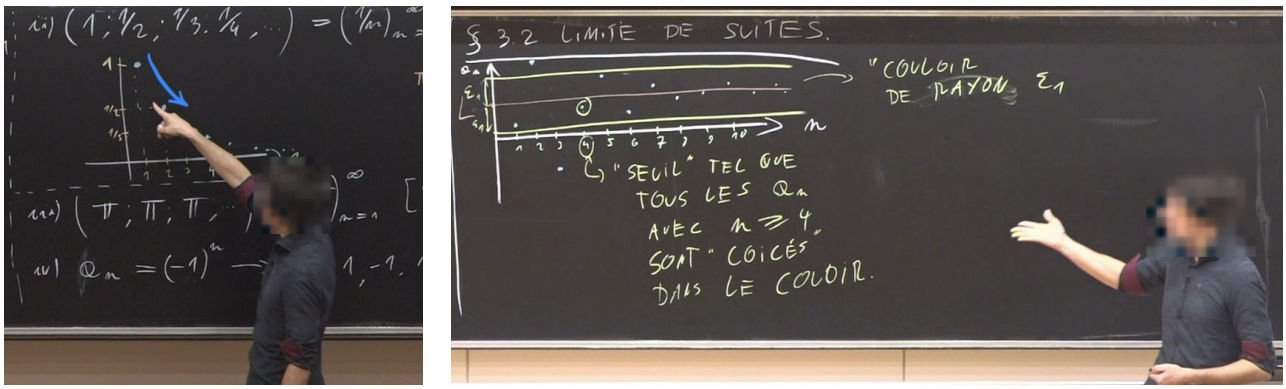
Description: The instructor…	**Words (in bold if written on the board)**	**Visual mediators**	**Endorsed narrative**
…introduces an "intuitive" definition of the limit as "getting closer to a value"	"behaviour of the points", "points get flat/get closer to the (x-) axis", "limit is the number toward which the sequence gets closer"	Graph, numbers, pointing	The limit is a number that the points get closer to
…states the objective of the class as **constructing a definition** of limit.	-	–	
…draws a 2-D plane, fixes a value L on the y-axis and sketches a non-monotonic sequence that approaches L.	Metaphor ("horizontal line", "points", "around L", "get closer", "higher/lower", "closer", "here") Mathematical ("limit", "sequence", "an", "number", "value", "limit", "**L**")	Graph, gesture	
… introduces the metaphor of a "hall of width epsilon" on the graph and finds the corresponding threshold.	Metaphorical ("approaches"; "**hall (of width/radius epsilon)**", "a certain moment", "**threshold**", "**as of which**", "points", "**stay/be stuck inside the hall**") Mathematical ("elements of sequence", "index", "n", "**an**", "**n** ≥ ")	Graph, colour coding, pointing, gesture, writing	For this epsilon, we can find a threshold after which all points will be stuck in the hall
…repeats the metaphor and the procedure on the graph with a different width epsilon.	Metaphorical ("hall (of size ε)", "stay/be stuck inside the hall", "**threshold**") Mathematical ("index", "elements of sequence", "**an**", "**n** ≥")	Graph, colour coding, pointing, gesture, writing	
… states orally that the same can be done with yet a smaller width.	Metaphorical ("hall", "threshold", "stay within the hall") Mathematical ("index", "elements", "sequence")	Gesture, pointing	
… states limit definition verbally	Metaphorical ("width of the hall", "threshold", "as of which", "stay inside the hall")	Gesture, graph, pointing	For <u>any</u> epsilon, we can find a threshold after which all points will be stuck in the hall
… writes a verbal version of the formal definition using the metaphor terms ("hall", "threshold")	Metaphorical ("**width/radius of hall**", "**threshold**", "**as of which**", "**stuck in the hall**") Mathematical ("**tend to L**", "**for all**", "$\boldsymbol{\varepsilon}$", "**there exists**", "$\boldsymbol{N}$", "**elements of the sequence**")	Writing, symbolic	
… states that the mathematical ("analytic") definition will be a translation of the verbal definition in mathematical terms	Metaphorical ("radius", "hall", "inside the hall") Mathematical ("tend to", "converging", "limit", "L", "for all", "ε", "sequence", "a_n", "there exists")	Pointing	
… writes the formal definition, using quantifiers	Metaphorical ("for every radius", "there is a threshold", "as of which", "be/be stuck in the hall")	Writing, symbolic, graphical,	

and interval notation for the "hall".	Mathematical ("**sequence**", "(a_n)", "**converge/tend to**", "**limit**", "**L**", "real number", "$\mathbb{R}$", "for all **(∀)**", "$\varepsilon > 0$", "positif", "**there exists (∃)**", "N", "integer", "$\mathbb{N}$", "elements of the sequence", "in the interval", "$a_n \in]l-\varepsilon, l+\varepsilon[$", "$\forall n \geq N$")	gesture, pointing	
… writes a second version of the formal definition, replacing the interval by an absolute value.	Metaphorical ("") Mathematical ("absolute value", "$\forall \varepsilon > 0$", "$\exists N \in \mathbb{N}^*$", "$a_n$", "interval of radius ε", "$\lvert a_n - l \rvert < \varepsilon$", "$\forall n \geq N$")	Pointing, colour coding, symbolic	
… writes a third version of the formal definition, using a logical implication sign.	Metaphorical ("threshold") Mathematical ("implication", "for all", "there exists", "integer", "absolute value", "smaller than", "an", "L", "epsilon", "N") Mathematical ("logical implication", "$\forall \varepsilon > 0$", "$\exists N \in \mathbb{N}^*$", "logical proposition", "$\Longrightarrow$", "$\forall n \geq N$", "$\lvert a_n - l \rvert < \varepsilon$",)	Pointing, colour coding, symbolic	

Table 1. Sequence of Events in the Lecture Section on Sequences and Limits

he endorses. Since he uses a metaphor to build the definition, his vocabulary initially relies on everyday terms ("hall", "threshold") to refer to the elements in the definition, both orally and in writing. Mathematical terms ("ε", "N") are first written down alongside the metaphorical terms as the instructor produces the verbal definition of the limit. Once introduced, they remain in the subsequent speech and written work, while metaphorical terms are gradually phased out, first remaining only in the oral discourse, then disappearing altogether.

The visual mediators–graphs, gestures, and written notation–also evolve. Initially, the discussion focuses on the drawn graph, with gestures and speech indicating scanning procedures. As the definition is developed, attention shifts to verbal explanations supported by annotations to the graph (Fig. 1b) and then to a written definition using the metaphorical terms. The formal definition is then presented primarily in symbolic notation, with only the defining sentence still expressed in natural language.



(a) Gesture in reading graph **(b) Verbal annotations on the graph**

Figure 1: Dr B's mediating procedures.

Early in his presentation, the instructor briefly endorses a narrative aligned with the limit-as-process intuition ("the function approaches L") but quickly shifts to a limit-as-number perspective. Unlike other instructors, however, he does not reject the process view; instead, he frames his activity as an analysis of the situation where a limit exists, explicitly showing how one moves from an intuitive account to a formal definition.

The changes seen in word use, visual mediators and narratives are staggered. For instance, when the visual mediators shift from primarily graphical to primarily verbal, the metaphorical terms remain in use, even in writing. Similarly, the metaphorical terms begin to disappear only once the mathematical terms have been introduced in writing alongside them. By staggering shifts in vocabulary and visual mediators, the instructor transitions to an advanced mathematical formulation of the limit while maintaining continuity with earlier accounts of the limit. Through an initial agreement on a shared narrative and gradual changes in symbols, representations, and terminology, the new knowledge remains anchored in what is implied that the students already know.

Moreover, shifts between mediators are drawn out explicitly. Through gesture (Fig. 1a), the instructor demonstrates the scanning procedures involved in interpreting the graph and highlights when two visual mediators refer to the same mathematical object. This is also made explicit through writing (Fig. 1b), where color-coded annotations on the board make graph-decoding routines explicit: the same terms are always written in the same colour (red for the limit L, yellow for ε and N).

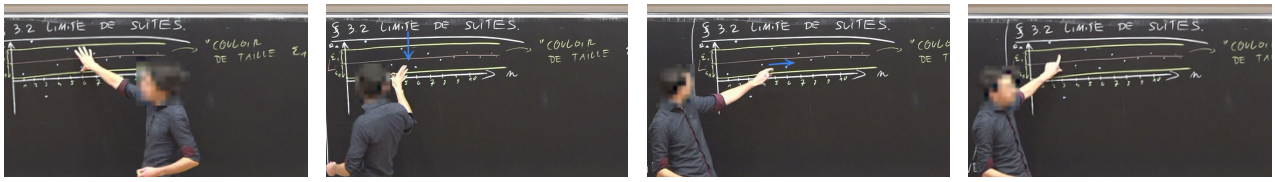
How visual mediator use constrains and supports discourse

Having explored how these representations are coordinated, we now look at how this coordinated use shapes the instructor's endorsement of narratives and enactment of routines during his definition presentation.

First, the instructor's use of a single graph anchors the presentation. Rather than focusing on the different ways functions may approach a limit, the discussion is grounded in the analysis of this single graph. This is noticeable, since the apparent insistence on this simple concept draws attention to why the instructor continues to discuss it. And, because only one graph is used, this attention shifts to how what is said about the graph changes. This highlights a shift in focus: the instructor is not developing narratives about which functions tend to a limit, but about how one can articulate what it means for a function to tend to a limit. The focus thus moves from constructing and endorsing new narratives about mathematical objects to constructing defining narratives.

Second, the explicit statement of and translation between different visual mediators requires the instructor to repeatedly invoke the same logical structure. Early on, the instructor introduces the narrative underlying the formal definition, which frames the remainder of the lesson. The same sentence structure is used across the presentation:

> For [this/an/any] [hall/hall width/epsilon], we can find a [time/integer] as of which all [points of the graph/elements of the sequence] will be [stuck in the hall/inside an interval]."



For this width ε_1	I have, as of a certain moment,	all the points that will	be stuck inside this hall
$\forall \varepsilon_1$	$\exists N_1$	$\forall\, n \geq N_1$	$a_n \in (L-\varepsilon_1, L+\varepsilon_1)$

Figure 2: Dr B's enacted routine for the definition of a sequence through gesture (top), metaphorical terminology (middle), and mathematical terminology (bottom).

The narrative is repeated multiple times and across several visual mediators (Fig. 2), each time closely adhering to the same structure. It is first developed orally through reflection on the graph, where the action "fix an ε, find an N" is demonstrated for a specific ε with the support of a metaphor. The instructor then repeats the process for a second ε and then indicates that the same process can be repeated for a third, setting the stage for generalisation. The verbal definition builds on this generalisation, and the final formal definition is presented as a direct translation of that verbal description. Finally, three versions of the formal definition are written down, each using slightly different notational choices. The homogeneity in structure therefore highlights how different aspects of the definition are realised in each of the different visual mediators.

This repetition spans too short a scale to be considered a routine, yet it is significant because it serves a double scaffolding function. First, it mirrors the logical structure of the formal definition itself. Second, it models the type of activity that the students must later engage in when working with the definition (for example, when proving convergence). Thus, the repetition in this presentation draws a clear link between the narrative the instructor endorses and the routines in which students will need to engage.

DISCUSSION

To date, the commognitive framework has proved useful in analysing instructors' discourse in university teaching. However, relatively few studies have examined the role of visual mediation in such discourse, despite recognition of its potential to bridge between incommensurable discourses. We analysed an instructor's introduction of the definition of limit, determining how visual mediators were coordinated and how they shaped the meaning he conveyed. We found that the progressive transition to literate mathematical discourse relied on the staggering of changes in discursive elements, coordinated through gesture and colour coding. Moreover, we found that the use of visual mediators enabled him to repeatedly endorse the definition's narrative while drawing students' attention to the changed nature of the mathematical task at hand.

This analysis reveals two key ways in which visual mediators shape discourse. When representations change, attention can be directed to what remains invariant across them; when they remain fixed, attention shifts to what changes in the discourse. Using several visual mediators in parallel requires the instructor to explicitly articulate the

connections between them while restating his endorsed narrative. This repetition makes the underlying structure of the discourse visible and can enhance students' understanding by reducing the cognitive effort needed to translate between mediators (Duval, 2017). Conversely, restricting the number of visual mediators allows the instructor to reuse the same representation while varying the discourse, underlining situations where the nature of mathematical activity itself is changing, supporting metalevel learning.

Notably, the presentation of this definition aligns with the "examples-first" teaching routine identified by Viirman (2021), but diverges in that the definition is constructed by reflecting on a single example rather than being inspired by one or several of them. This constitutes an important difference, since it models a key feature of advanced mathematical practice: that new layers of discourse can emerge through meta-level reflection rather than through pattern spotting alone.

While the discursive moves we identify seem characteristic of this instructor's practice (as confirmed by other classroom observations), the analysed segment is not typical, either at our university or more broadly. Moreover, since we have not analysed student data yet, we cannot yet make claims about the role of visual mediators in bridging commognitive conflicts. However, this analysis provides a detailed account of how visual mediators are coordinated during a lecture and how this can shape the meaning the instructor is able to convey, thus providing a framework for comparing instructors' discourse with each other and with students. Future work will analyse other instructors' lectures using similar methods to examine how their use of visual mediators shapes discourse differently, and student interviews to assess whether this influences understanding. Overall, this work contributes to our understanding of how elements of the lecture shape discourse, and identifies features that may be leveraged to make lectures more intelligible to students.

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Towards beliefs on proof in mathematics: a survey of future elementary school teachers in Switzerland

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This paper analyzes the beliefs of proof among first-year bachelor's students, future primary school teachers, in the context of current thinking in proof research. Using a questionnaire tailored to this audience, we study the functions of proof that they recognize, as well as their criteria for evaluating the rigor and validity of different types of proof. The results show a clear gap between their ability to identify intellectual, deductive proof as rigorous and the empirical or perceptual approaches relevant to pragmatic proof that they spontaneously mobilize. Explanatory and verification functions dominate, while more social dimensions appear only marginally. These results highlight the need to reinforce, in initial training, the active and diversified appropriation of the activity of proof.

Keywords: belief, proof, primary school teacher training, mathematical practice

INTRODUCTION

Proof lies at the heart of mathematical activity, yet persistent difficulties in learning and teaching it remain across grade levels and countries. Mathematics education research has highlighted a significant discontinuity between secondary and tertiary education internationally (e.g. Gueudet, 2008; Selden et al., 2010) and teacher education (both pre-service and in-service) often appears insufficient or inadequate. At the international level, few studies have examined teachers' practices or their conceptions (or their beliefs) of proof and its teaching (e.g. Mariotti et al. 2018; Stylianides, 2016). Our work builds on discussions within the international mathematics education community, particularly following Ouvrier-Buffet's (2023) collaborative proposal at CERME 13 concerning undergraduate students' conceptions of proof elicited through questionnaires. This issue has already been investigated with freshmen in Belgium and pre-service teachers in France (De Vleeschouwer & Ouvrier-Buffet, 2024). In this paper, through a questionnaire adapted from previous similar studies (e.g. Healy & Hoyles, 1998; Stylianou, 2015; Ouvrier-Buffet, 2023), we focus on the belief of proof, – in accordance with Beswick (2005) who "refer to anything that an individual regards as true" (ibid., p. 39) –, in a hardly investigated population: first-year Bachelor students entering primary teacher education in Switzerland (who have not taught yet). The instrument includes analyses of multi-proof items, questions about student's school experiences with proof in mathematics classes, and questions about the functions of proof. Our research questions are: *What beliefs of proof do prospective primary teachers display when examining different mathematical proofs? Which functions of proof do they identify in mathematics, and what do their stated beliefs*

relate to the mathematical practice of proof they experienced during their previous schooling?

THEORETICAL BACKGROUND AND DIMENSIONS OF PROOF ADDRESSED BY THE QUESTIONNAIRE

Ouvrier-Buffet (2023) developed a questionnaire based on a bibliographical study to determine how researchers in mathematics education have already investigated the undergraduate students' conceptions of proof (based on their statements, assessment and writing about proofs by comparing what they say with what they do). The study highlights the interest of Balacheff's (1988) theoretical framework and notes that few items in the literature explore students' judgements of given proofs and their global perceptions and conceptions of proof. Ouvrier-Buffet's (2023) proposal selects mathematical contents where mastering the involved mathematical concepts is minimised and chooses mathematical problems outside of formalism (so that the formalism is not an obstacle) and outside of curricula (to avoid obstacles or ready-made results and processes). Based on results in the literature about students' difficulties (mainly the significant students' difficulties even with deductive short proofs), our questionnaire explores beliefs (Beswick, 2005) focusing on "basic" components of deductive proofs, reusing some items and enriching previous surveys. In particular, we focus on the use of the instances of a mathematical statement, the evaluation of written proofs with multiple choice questions (as Healy & Hoyles 1998 and Stylianou et al. 2015, but with new features and mathematical domains), and the exploration of the students' global beliefs of proof (the way they think about proof) with open questions leading to kinds of proofs, functions of proofs (e.g. Hanna, 2000), and declarative students' tools to write/read/evaluate proofs. As a theoretical background, in order to justify the proofs we submit to the students and to analyse their answers, we use Balacheff (1988)'s characterisation of arguments (mainly naïve empiricism, generic argument and intellectual proof) and we play on the fact that a proof can be written (completely or partially) in a natural language, in a formal language, or can be visual.

METHODOLOGY

In response to Ouvrier-Buffet's (2023) call to investigate university students' conceptions of proof, we adapted the initial questionnaire (ibid.) for students enrolled in primary teacher education at HEP-VS. Drawing on Healy and Hoyles (1998), Stylianou et al. (2015) and Ouvrier-Buffet (2023), we designed a questionnaire containing problems aligned with mathematical content familiar to these students. The instrument comprises five parts: the first two administered in a paper-and-pencil format (40 min.) and the remaining three digitally (40 min.), enabling control over backtracking effects and precise measurement of response times.

- Part I: General beliefs of mathematics and mathematical activity (10 min)
- Part II: Instantiation of known theorems or formulas (30 min)
- Part III: Multiple-proof statement (30 min)

- Part IV: School experiences with proof and perceived functions of proof (10 min)
- Part V: Demographic info: gender, age, prior experience, nationality (2 min).

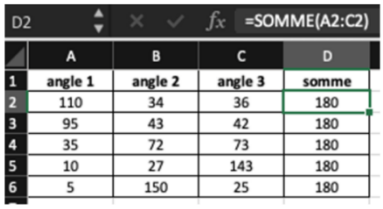
This questionnaire was administered at the beginning of the first year of the primary education program (Bachelor's degree) at HEP-VS. It involved 30 volunteer students with a wide variety of backgrounds. Completing the questionnaire took approximately one hour, including both the paper-and-pencil and digital sections.

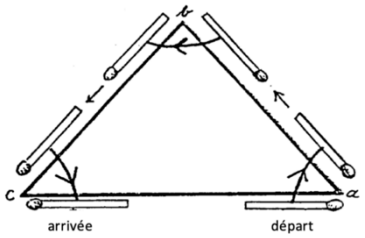
In this paper, we will focus primarily on Parts III and IV. Part III proposes to study proofs for two multi-proof statements, one in the field of arithmetic and the other in the field of geometry.

- Statement 1 – The product of three consecutive numbers is a multiple of 6.
- Statement 2 – The sum of the angles of a triangle is a straight angle.

For each statement, four possible answers are provided, and students are asked to check one of them : *Which answer is nearest to what you would have done?* or *Which answer would you choose to explain the "solution" to your classmates?* or *Which answer is the most "thorough" for you?* or *What is/are the correct proof(s) for you?* or *What is/are the incomplete proof(s) in your opinion?*

Below are examples of proof proposed for each of the statements (Table 1).

	Statement 1	**Statement 2**
Answer A	A number that has 2 and 3 for factors is a multiple of 6. If I have three consecutive numbers, one will be congruent to 0 modulo 3, so it will be a multiple of 3. Also, at least one number will be even and all even numbers are multiples of 2. So, if I multiply three consecutive numbers together, the answer must have at least one factor of 3 and one factor of 2. I can conclude that the product of three consecutive numbers is a multiple of 6.	By measuring the angles of various kinds of triangles and writing them in a spreadsheet, I have:  D2 =SOMME(A2:C2) A B C D 1 angle 1 angle 2 angle 3 somme 2 110 34 36 180 3 95 43 42 180 4 35 72 73 180 5 10 27 143 180 6 5 150 25 180 Then, I deduce that the sum of the angles of a triangle is equal to 180° (straight angle).
Answer B	$1 \times 2 \times 3 = 6$ $2 \times 3 \times 4 = 24 = 6 \times 4$ $4 \times 5 \times 6 = 120 = 6 \times 20$ $25 \times 26 \times 27 = 17550 = 6 \times 2925$ I can conclude that the product of three consecutive numbers is a multiple of 6.	Let *ABC* be any triangle. The angles are marked on the figure. A line parallel to *(BC)* is drawn through *A: p+c+r* =180° (straight angle) The alternate interior angles are equal: we deduce the following equalities

		$p=a$ et $r=b$. Thus: $a+b+c=p+c+r$ The sum of the angles of a triangle is equal to 180° (straight angle).
Answer C	Let n be a natural integer. $n \times (n + 1) \times (n + 2) = (n^2 + n) \times (n + 2)$ $= n^3+3n^2 + 2n$ The factor coefficients of the powers of n are: 1+3+2=6 I can conclude that the product of three consecutive numbers is a multiple of 6.	Observe how the matchstick moves around the triangle:  At the end, the matchstick is in the opposite position to the start. During this displacement, and by turning through the three angles of the triangle, the matchstick has made a half-turn. The sum of the angles of a triangle is therefore equal to 180° (straight angle).
Answer D	For three consecutive numbers, let P be the first of them. P is either – even: there exists an integer a such that $P = 2a$ by definition of an even number, – or odd: there exists an integer b such that $P = 2b-1$ by definition of an odd number. If P is even, then the product of three consecutive numbers is: $2a \times (2a+1) \times (2a+2)$: it is a multiple of 2. –If a is a multiple of 3, the proof is done. – If a is not a multiple of 3, then 2a is not a multiple of 3, but, in this case, $(2a+1)$ or $(2a+2)$ is a multiple of 3 and the proof is done. If P is odd, then the product of three consecutive numbers is: $(2b-1) \times 2b \times (2b+1)$: it is a multiple of 2.	We cut out the angles and place them side by side: we get a straight line. The sum of the angles of a triangle is therefore equal to 180° (flat angle).

	– If *b* is a multiple of 3, the proof is done. – If *b* is not a multiple of 3, then *2b* is not a multiple of 3, but, in this case, *(2b-1)* or (*2b+1)* is a multiple of 3 and the proof is done. I can conclude that the product of three consecutive numbers is a multiple of 6.	

Table 1: Different proofs proposed for statements 1 and 2.

Part IV examines students' beliefs about the role of proof in mathematics and how they encountered proof in their own schooling. Students were asked to indicate the frequency with which they experienced specific classroom situations using two 5-point Likert scales.

- Situation 1. In math classes, proofs were produced by the teacher alone.
- Situation 2. In math classes, I constructed proofs with my classmates.

For these first two situations, the options were: *very often*, *often*, *sometimes*, *rarely*, and *never*. For the next three situations, students selected from: *strongly agree*, *often agree*, *agree*, *neither agree nor disagree*, *disagree*, and *strongly disagree*.

- Situation 3. Proofs were a central focus of math lessons.
- Situation 4. Constructing proofs is fundamental in mathematics.
- Situation 5. One can study mathematics without being skilled at constructing proofs.

The final question in this section addressed the functions of proof. Students indicated what they believe mathematical proofs are used for, choosing among several options: *explaining*, *discovering*, *convincing*, *communicating*, *verifying*, *formalizing*, and *other* (with specification). To analyse responses from Parts III and IV and address our research question, we conducted a brief *a priori* analysis of the proofs proposed in Part III. This enabled us to formulate hypotheses that were subsequently confirmed or refuted using data from Part IV. The following section presents this analysis and our main results.

ANALYSIS AND RESULTS

Table 2 summarises the *a priori* analysis of the proposed proofs for Statements 1 and 2. Our analysis draws primarily on Balacheff's (1988) levels of proof, the level of language used (formal vs. natural), and the validity of the proof (correct vs. incorrect).

	Statement 1	**Statement 2**
Answer A	Natural language, arithmetic register, correct proof.	Naive empiricism / crucial experiment limit; graphical and numerical register, incorrect proof:

		empirical measurement not generalizable.
Answer B	Naive empiricism and crucial experience, excessive generalization based on numerical data from calculated examples; incomplete proof: only a few values have been processed.	Intellectual proof, formal language, alternating between natural and algebraic-symbolic registers. Correct proof.
Answer C	Intellectual proof, formalized language belonging to the algebraic register; incorrect use of the existence theorem: the sum of the coefficients has no relation to the divisibility of the product. Invalid reasoning.	Informal but formalizable and generic visual proof, manipulation, correct. Mathematical theory not explained, not formalized, outside the conventional and institutional framework (French and Swiss). (This example comes from a Hungarian textbook/course).
Answer D	Intellectual proof, natural language, completeness of cases and disjunction of cases, correct proof.	Visual proof, a form of naive empiricism, incomplete. Manipulation of cut-outs (common in textbooks/resources)

Table 2: Brief preliminary analysis of the various proofs presented in the statements

The average time spent on Part III was 6 minutes and 59 seconds, far below the 30 minutes allocated. This strongly suggests that students did not engage in an actual proving process for Statements 1 and 2, but instead responded intuitively.

Table 3 summarises students' responses in Part III. Overall, the results reveal predominantly empirical beliefs of proof. For both statements, the responses selected as closest to what students would do spontaneously are largely empirical or inductive. For Statement 1, example-based proofs (B: 10 responses) and superficial algebraic treatments (C: 10) prevail. For Statement 2, measurement-based (A: 14) or manipulation-based approaches (D: 11) dominate. This reliance on empirical approaches is confirmed when students choose a proof to explain to classmates: example-based reasoning is widely selected in arithmetic (B: 15), and manipulation-based reasoning in geometry (D: 19). These choices reveal a strong dependence on perceptual or numerical registers, at the expense of deductive reasoning.

However, when students judge rigor or validity, their choices shift markedly toward deductive proofs—those corresponding to Balacheff's (1988) category of intellectual proofs. For Statement 1, the formal proof (D: 17 responses) is considered the most rigorous; for Statement 2, the Euclidean proof (B: 19) is preferred. Proofs judged correct also predominantly rely on structured deductive reasoning, whether arithmetic (D: 20) or geometric (B: 18; D: 23). Conversely, proofs reflecting naïve empiricism are generally identified as incomplete (B: 22 for Statement 1; C: 21 for Statement 2), indicating students' awareness of the limits of perception or isolated examples for

generalisation. This suggests that students do not attribute a genuinely explanatory function to proof: examples may help explain, but are recognised as insufficient.

Overall, students display a tension between their spontaneous practice – rooted in empiricism – and their ability to recognise the institutional expectations of proof centered on deduction and explicit use of properties and theorems. This discrepancy may indicate an intermediate stage in the development of their beliefs of proof: they appear to understand what constitutes a proof, yet do not spontaneously mobilise it.

Table 3 presents the detailed results. Each pair (a; b) indicates the number of responses for Statement 1 (a) and Statement 2 (b).

	A	**B**	**C**	**D**
Which answer is nearest to what you would have done?	(7 ; 14)	(10 ; 4)	(10 ; 0)	(2 ; 11)
Which answer would you choose to explain the "solution" to your classmates?	(7 ; 6)	(15 ; 1)	(2 ; 3)	(5 ;19)
Which answer is the most "thorough" for you?	(0 ; 5)	(2 ; 19)	(10 ; 0)	(17 ; 5)
What is/are the correct proof(s) for you?	(16 ; 14)	(11 ; 18)	(20 ; 8)	(20 ; 23)
What is/are the incomplete proof(s) in your opinion?	(14 ; 12)	(22 ; 6)	(5 ; 21)	(2 ; 7)

Table 3: Results of Part III regarding responses to questions on statements 1 and 2.

The results of the first part of Section IV, summarised in Table 4, indicate that most students experienced proof as a predominantly lecture-based—or even monumentalised—activity (Situation 1). Fifteen students reported that proofs were produced “very often” by the teacher and eight “often,” whereas only one student selected “rarely” or “never.” Conversely, collaborative proof construction (Situation 2) appears uncommon: 17 students reported experiencing it “rarely” or “never,” and none selected “very often.” Proof is also not perceived as central to mathematics instruction (Situation 3): only seven students “strongly agree” or “agree,” while 23 express neutrality or disagreement.

Yet, despite limited exposure, students overwhelmingly acknowledge the importance of proof (Situation 4): 24 “strongly agree” or “agree” that constructing proofs is fundamental in mathematics. Regarding whether one can study mathematics without being proficient in proof (Situation 5), opinions vary, but a majority (14 “disagree” or “strongly disagree”) consider proof competence necessary. Overall, these results highlight a tension between mathematical practices of classroom centered on teacher-led demonstration and students’ recognition of the pivotal role of proof in mathematical activity.

	Very often	Often	Sometimes	Rarely	Never
Situation 1	15	8	5	1	1
Situation 2	0	4	9	12	5
	Totally agree	**Agree**	**Neither disagreeing nor agreeing**	**Disagree**	**Strongly disagree**
Situation 3	2	5	10	9	4
Situation 4	11	13	2	3	1
Situation 5	1	10	5	13	1

Table 4: First result of Part IV on students' perceptions of proof practices.

The second part of Section IV focuses on students' views of the functions of proof. As shown in Table 5, most students attribute explanatory (93%) and validating (90%) functions to proof, confirming a belief centered on clarification and verification. This finding refutes the hypothesis, suggested after Part III, that students do not see proof as explanatory. The function of *convincing* is also widely recognised (73%). In contrast, more discursive or structural functions such as *communicating* (30%) and *formalising* (43%) are less represented. Only 16% associate proof with *discovering*, suggesting a limited belief of proof as a means of exploration.

Among the four students who selected *other*, two specified additional roles (*confronting* and *reproducibility*). These responses suggest that proof is seen both as a tool for testing and comparing assertions or lines of reasoning, and as a process whose reproducibility ensures that results can be independently verified by any member of the mathematical community.

Function of proof	Number of responses	Percentage of students
Explain	28	93%
Discover	5	16%
Convince	22	73%
Communicate	9	30%
Verify	27	90%
Formalize	13	43%
Other	4	13%

Table 5: Second result of Part IV on the role of proof (function) by students.

DISCUSSION

Taken together, the results reveal a marked discrepancy between, on the one hand, the institutional belief of proof that students are able to recognise and, on the other, the mathematical practices they actually use or have encountered during their schooling.

In Part III, students clearly identify deductive proofs as the most rigorous and valid even when they are mathematically incorrect (e.g., proof D for Statement 1 with 17 responses, and proof B for Statement 2 with 19). However, when asked which approach they would adopt themselves or select to explain a solution, they overwhelmingly favour empirical, manipulative, or example-based strategies (proofs B and C for Statement 1; A and D for Statement 2). This tension suggests that while students possess a conceptual understanding of the criteria for proof validity, these criteria are not spontaneously mobilised in their own proving or explanatory practices.

The results from Part IV help illuminate this discrepancy. A majority of students reported that proofs were *very often* or *often* produced solely by the teacher (23 responses), whereas collaborative construction of proofs was described as rare or absent (17 responses). Likewise, proof is not perceived as the central activity in mathematics lessons, despite strong agreement that constructing proofs is fundamental (24 students *agree* or *strongly agree*). This indicates a learning environment in which proof is valued in principle but seldom practised, which may account for students' reliance on empirical approaches despite their recognition of the epistemic superiority of deductive reasoning.

Moreover, students' identification of the functions of proof (primarily *explanation*, 93%, and *verification*, 90%) reflects a belief of proof oriented toward clarification and validation, while communicative functions appear less prominent. Notably, the function of *verification* is understood by many students as checking examples rather than engaging in the semantic or syntactic evaluation of a proof. This gap between theoretical understanding, actual practices of proof, and institutional expectations constitutes a significant educational challenge, underscoring the need for instructional approaches that foster the effective production of proofs and a broader repertoire of justification methods.

CONCLUSION AND RESEARCH PERSPECTIVES

The results reveal a clear discrepancy between students' stated beliefs (according to which they correctly identify deductive proof, particularly syntactic and formal forms, as the most rigorous), and their actual practices of proof, which remain largely grounded in empirical or perceptual approaches. This misalignment can be partly attributed to school experiences in which proofs are predominantly presented by the teacher and rarely socially constructed. Although students acknowledge the fundamental role of proof, they prioritise only certain functions (*explaining*, *verifying*, *convincing*), while its social dimensions appear insufficiently recognised. For instance, functions such as *discovering* or *communicating* are perceived in a more superficial manner, and more advanced functions (such as confronting ideas or ensuring reproducibility) emerge only marginally.

These findings call for further research on teaching approaches that foster active student engagement in the production and discussion of proofs. They also highlight the need to develop a shared knowledge base concerning proof and proof processes, as

well as to reflect on the institutionalisation of this knowledge: What exactly should be institutionalised? Through which pedagogical means? And under what conditions? As a continuation of this exploratory work, we have slightly revised the questionnaire — modifying certain modalities — in order to administer it to a larger sample of students across different French-speaking contexts (France, Belgium, Switzerland).

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Gaining Syntactic Understanding through Semantic Reasoning in Symbolic Logic

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Keywords: transition-to-proof, symbolic logic, syntactic/semantic, conceptual understanding, qualitative.

INTRODUCTION

The importance of foundational symbolic logic in the teaching of proof at the tertiary level is generally recognized in the mathematics education community (Durand-Guerrier et al., 2009). The pedagogy can be approached from a "semantic" or "syntactic" point of view, or ideally, a mix of both. Here I will share research-informed tasks presented to a university-level transition-to-proof class taught via an inquiry-based learning approach that are designed to bridge the gap between the formality of symbolic logic (syntactic) and the natural logic with which students enter the course (semantic). Linguistic implications will also be addressed.

THEORETICAL FRAMEWORK

Weber and Alcock (2004, p. 210) propose the categorization of mathematical proving into two approaches: syntactic, in which "the prover works from a literal reading of the involved definitions" and semantic, in which "the prover also makes use of his or her intuitive understanding of the concepts." It is an underlying potential source of struggle and/or confusion for students to translate and transition between formal and informal logical language, and thus it is generally agreed that a balance should be struck between the semantic and syntactic approaches (Durand-Guerrier et al., 2009).

Additionally, it has been well established that students' taking an active role in their own learning helps solidify concepts (Freeman et al., 2014), which is particularly relevant in addressing variations of language when working within a semantic understanding. Epp (2003) notes that time spent in individual discussions between student and instructor would aid students in resolving their logical misunderstandings; given the impracticality of that on a large scale, small-group and whole-class discussions provide a reasonable alternative. The particular active learning perspective taken here is the theoretically- and empirically-grounded inquiry-based mathematics education approach (Laursen & Rasmussen, 2019), implemented using the author's course text containing the tasks under consideration (Schwell 2025).

RESEARCH QUESTIONS

The aim behind the construction of the tasks was to address an observed lack of semantic understanding in students' manipulation of logical symbols, which led to reduced benefit of the symbolic logic background in the subsequent learning of proof.

The following questions seek to determine if this deficiency was improved via implementation of the tasks.

1) To what degree are students able to use only their lived experiences and informal reasoning abilities to deduce truth values and logical equivalences?
2) How well can students naturally "discover" equivalent linguistic formations?

METHODOLOGY AND PRELIMINARY RESULTS

The author is a mathematician and has taught the course described over 25 times; the qualitative data are taken from written student solutions obtained over the past three semesters. The solutions were analyzed qualitatively using the following a priori codes:

SY: correct purely syntactic reasoning was demonstrated.

TR: correct transitional reasoning between semantic and syntactic understanding was demonstrated.

X: incorrect or no reasoning was demonstrated.

Solutions from over the course of the symbolic logic unit were analyzed. Preliminary findings show that early responses demonstrate a mix of all three categories, while later solutions show essentially only the latter two, and primarily TR. Sample student tasks can be seen here, and examples of coded student responses will be included in the poster.

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The role of analysing reasoning tasks in promoting critical thinking

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The importance of critical thinking - encompassing the core skills of analysis, evaluation and inference - for active engagement in the modern world should not be underestimated. We assert that 'analysing reasoning' tasks, which provide opportunities for students to make decisions about the correctness of a given piece of mathematics, are ideally suited to the development of these skills in mathematics. However, the prevalence of analysing reasoning tasks in textbooks seems to be very low as evidenced by the findings from a classification of a selection of tasks in four undergraduate textbooks presented here. To address this gap, we propose a set of principles for the design of analysing reasoning tasks which aim to effectively bridge the gap between theory and practice to facilitate the creation of such tasks.

Keywords: critical thinking, analysing reasoning, task classification, task design, calculus

INTRODUCTION

It is widely accepted that teaching and learning at secondary and higher levels of education - in all subjects, but in mathematics in particular - should go beyond the transmission of knowledge to involve the development of higher order skills such as critical thinking (e.g., Saroyan, 2022). This is increasingly important today to enable citizens in the modern world to make informed decisions in the face of growing levels of volatility, uncertainty, complexity and ambiguity (or VUCA) (Bennis & Nanus, 1985) and their exposure to a disproportionate amount of misinformation and disinformation through the internet and social media. In fact, critical thinking is one of the key competencies identified by UNESCO for the achievement of the seventeen Sustainable Development Goals (SDG) adopted by the United Nations (Rieckmann, 2017). Equipping students with critical thinking skills becomes even more significant given the rapid progress of generative artificial intelligence and the resultant concern that students might accept generated responses without questioning, together with the ongoing prospect of such responses containing biased information (Pepin et al., 2025).

Despite the fact that critical thinking is often highlighted in national educational policies and qualification standards, there is little indication that this translates into teaching, learning and assessment practices (Saroyan, 2022). For example, it has been found that, in Ireland, both secondary school students and undergraduates in introductory mathematics courses are mostly required to carry out procedural calculations and only seldom need to use any higher-order thinking skills (O'Sullivan, 2017; Mac an Bhaird et al., 2017). This reflects what is happening internationally. For instance, Tallman et al. (2016) examined 150 terminal Calculus examinations from

colleges and universities across the United States and found that these generally required only low levels of cognitive demand and did not necessitate the demonstration or application of a comprehension of the course's central ideas. Moreover, their analysis revealed inconsistencies between the findings of the study and the instructors' own perceptions of the conceptual focus of the examination questions they had set. In another study, Alayont et al. (2022) used Bloom's taxonomy to classify the levels of cognitive demand of 467 end-of-section problems from a widely-used Calculus textbook. Almost 70% of these problems, frequently used in students' homework assignments, were found to be at the lower levels of knowledge, comprehension and application while only three tasks were at the 'evaluation' level which involves making judgments about the value of a method or validity of an argument.

On the other hand, specialist mathematics students sometimes engage in higher order proof evaluation or proof validation classroom activities as a precursor to proof writing in 'introduction to proof' courses at university (e.g., Pi et al., 2024). Such activities, often completed in groups, aim to provide students with the tools necessary to read, understand and critically discuss abstract mathematics and may include a focus on the qualities possessed by a "good proof". Other studies have used similar tasks or activities to *research* what kind of skills and thinking are necessary for students to determine whether or not a proposed proof with which they have been presented is valid (e.g., Alcock & Weber, 2005). Here we will focus on mathematics tasks which promote critical thinking, not only for specialist mathematics undergraduates, but also for non-specialists. In particular, we are interested in mathematics tasks which can be used as homework on first calculus and introductory analysis courses. Students in calculus courses are often not expected to engage with proof, and thus our study will take a broader view of reasoning.

We take core 'critical thinking skills' to comprise analysis, evaluation and inference, (Facione, 1990) and assert that the development of such skills requires students to use their ability to reason and to be active learners rather than passive recipients of information. Saroyan (2022) found that, in order to be effective, higher-order skills such as these must be explicitly targeted as learning outcomes in curricula and require careful pedagogical planning at both the design and implementation stages. In a previous study (Breen & O'Shea, 2019), we proposed 'analysing reasoning' tasks as one of seven types of homework tasks which would encourage undergraduate students to move away from thinking about mathematics in a solely instrumental manner and to engage in productive higher-order thinking. Such 'analyzing reasoning' tasks require students to make decisions about the correctness of a piece of mathematics and can be used to draw learners' attention to mistakes, allowing difficulties and common misconceptions to be confronted rather than avoided. We propose that this type of task fosters reflective thinking and is ideally suited to the development of critical thinking skills in mathematics. Our goal here is to present a clear set of principles for the design of analysing reasoning tasks, which we have developed following our reading of the relevant literature. In addition, we provide some sample tasks to illustrate these design

principles. We use our definition of analysing reasoning tasks to study our research question:

How prevalent are analysing reasoning tasks in undergraduate calculus and introductory analysis textbooks?

THEORETICAL FRAMEWORK

There are many different definitions and characterisations of critical thinking in the research literature. One of the most influential is the definition which was developed by a group of 46 experts from a wide range of disciplines as part of a project initiated by the American Philosophical Association (Facione, 1990). This group stated that critical thinking involves the cognitive skills of interpretation, analysis, evaluation, inference, explanation and self-regulation. The group posited that of these the core skills were analysis, evaluation and inference. Table 1 outlines these skills in more detail.

Skills	**Subskills**
Interpretation	Categorisation; Decoding significance; Clarifying meaning
Analysis	Examining ideas; Identifying arguments; Analysing arguments
Evaluation	Assessing claims; assessing arguments
Inference	Querying evidence; Conjecturing alternatives; Drawing conclusions
Explanation	Stating Results; Justifying Procedures; Presenting Arguments
Self-regulation	Self-examination; Self-correction

Table 1: List of Critical Thinking Cognitive Skills and Sub-skills (From Facione, 1990, p. 12).

The core skills of critical thinking identified by Facione (1990) bear a strong resemblance to the higher-order thinking processes of Analysis, Synthesis and Evaluation in Bloom's taxonomy (1956). There, the thinking process of Synthesis is described as 'building a new whole by combining elements or creating new meaning', while the terms Analysis and Evaluation have slightly different meanings than those described by Facione (1990). In Bloom's Taxonomy, the term Analysis refers to 'breaking down information into parts to understand relationships, motives, or causes'. The thinking process of Evaluation is described as 'the ability to make judgments in terms of the value of internal evidence and/or external criteria'. This ability is clearly crucial in order to be able to think critically and we will concentrate on this thinking process when designing our mathematical tasks.

Our aim is to provide a set of design principles for the development of tasks that encourage critical thinking in the context of mathematics. Lin et al. (2012) suggest that

task design principles should be rooted in theory so as to take note of students' learning but should also be practical and help both in the design and evaluation of tasks. Their view is that such a set of principles forms an intermediate level theoretical framework (Ruthven et al., 2009), in that they bridge the gap between theory and the practice of design. The Facione framework is the theoretical underpinning of our principles of mathematical task design that we outline below.

Analysing Reasoning Tasks

We adopt Lithner's definition of reasoning here; that is *the line of thought adopted to produce assertions and reach conclusions in task solving.* (Lithner, 2008, p. 257). Lithner is clear that the reasoning need not be a formal proof and could even be incorrect. It is important to us that there is a distinction between reasoning and proof since the context we work in will include courses for non-specialist mathematics students which often do not focus on proof.

A piece of reasoning, whether it be a calculation, an informal argument, or a rigorous proof can be thought of in terms of Toulmin's model of argumentation (1958). A simplified version asserts that an argument (or for us a piece of reasoning) consists of three core elements: the conclusion, the data and the warrants. The conclusion is the claim or result that is to be shown. The data is the mathematical foundation to be used as evidence for this and might include the assumptions being made or some core piece of knowledge. The warrant is the explanation why the data leads to the conclusion, for example it might appeal to a rule, a definition, or another result (often in the form of a claim). If we wish to ask students to decide if a piece of reasoning is valid or not, we need to keep these three core elements in mind, and design tasks that highlight their roles.

We define an *analysing reasoning task* to be a task that asks students to critically examine a piece of reasoning. It may include tasks where students are asked to do one or more of the following: decide if a piece of reasoning presented is valid or not; check assumptions and/or detect bias; query claims; identify errors; or compare solution methods/arguments.

We see assumptions as being an important part of the data in an argument. Komatsu et al. (2024, p. 327) consider assumptions to be *statements that people use or accept (often implicitly) and on which their assertions are based*. They classify assumptions into two main groups: global and local. For them global assumptions include those related to axioms and definitions, while local assumptions concern conditions of a task or the meaning of lay terms mentioned in the task. They developed a set of design principles to highlight the role of assumptions, and in particular to make explicit that a conclusion is dependent on the assumptions made in the reasoning that led to it.

Checking the warrants in a piece of reasoning would include querying claims as well as identifying any errors. This task type includes finding mistakes in supposed proofs or calculations as well as critiquing a line of reasoning. Tasks such as these would be included in Sangwin's (2003) task type *Criticize a fallacy*. He asserted that this type

could develop undergraduate mathematics students' high-level reasoning skills. Watson and Mason (1998) also emphasise the importance of tasks that start with a prompt like 'Tell me what is wrong with …'.

We see asking students to make a decision as to whether a piece of reasoning is valid or not as an overarching task type. It may involve looking closely at assumptions, or claims and searching for errors on a line-by-line basis, or it may necessitate students considering the piece of reasoning as a whole and asking if it does what it set out to do. Our final task type of *compare solution methods/arguments* also asks students to consider the piece of reasoning as a whole. Swan (2008) used a similar label to denote activities in which students compare different methods for solving a problem, thereby enabling them to recognize that there are alternative routes through a problem. We also include here situations where students are asked to compare given pieces of correct and incorrect reasoning.

We note that the subskills identified by Facione (1990) of the core critical thinking skills of analysis, evaluation and inference include elements of our definition of analysing reasoning tasks (see Table 1). For example, the subskills of assessing arguments and querying evidence contribute to deciding if a piece of reasoning is valid or not by checking assumptions, querying claims and identifying errors.

DEVELOPMENT OF DESIGN PRINCIPLES

Our project aims to provide design principles which aid the development and evaluation of analysing reasoning tasks in the context of an undergraduate mathematics course. Bridging the gap between a theoretical definition of critical thinking and how that thinking might be developed in the classroom was the subject of an OECD study (Vincent-Lancerin et al., 2019). One of the outcomes was a conceptual rubric of critical thinking in the context of teaching and learning. This rubric aims to identify practices which should be encouraged such as 'Identify and question assumptions, check accuracy of facts and interpretations, analyse gaps in knowledge' and 'Identify strengths and weaknesses of evidence, arguments, claims and beliefs' (p. 28). These practices are not subject-specific and so there is a need to consider these skills in the context of mathematics. Given our definition of an analysing reasoning task above and the studies on which it is based, our design principles are opportunities to make decisions on the validity of reasoning, in particular to:

Identify and check assumptions. Here the term assumption refers to definitions, axioms, conditions under which a result is true, meanings of terms, and (often implicit) underlying statements.

Query individual claims made in reasoning. In our context, claims are mathematical statements made in the piece of reasoning.

Identify errors in reasoning and check logic in reasoning. In this design principle we draw attention to errors in computation or logic in particular.

Compare alternative solutions or arguments. Here tasks can consist of two or more pieces of reasoning (which may be correct or not), and students are asked to decide on their validity.

Note that we have not included the creation of a correct solution in our design principles, even though Facione (1990) lists 'conjecturing alternatives' as a subskill of the Inference section of the Critical Thinking Framework. This is obviously an important skill, but we do not think this fits well with our definition of an analysing reasoning task. Conjecturing an alternative solution or argument could be used as an extension to an analysing reasoning task.

CLASSIFICATION OF TASKS

We investigated whether analysing reasoning tasks were common in calculus and introductory analysis textbooks. We selected the two books recommended by lecturers in our universities in first Calculus modules (Larson & Edwards, 2024; Stewart, 2009) and the two recommended on Analysis modules (Abbott, 2001; Brannan, 2006). The textbooks were followed closely in the modules in question and are all very well established texts used in many universities internationally. The Calculus modules are considered to be service courses as they are aimed at students who are not studying for a degree in mathematics. In contrast, the Analysis modules are taken by students who are in specialist mathematics programmes. In each book we identified the chapter on Derivatives or Differentiation and all the tasks included in the exercise or problem sections in these chapters for classification purposes: there were 1688 tasks in total, some of which had multiple parts. Each question was examined by at least one of the authors for opportunities to critique a reasoning sequence presented in the task by (a) checking assumptions and/or detecting bias, (b) querying claims, (c) identifying errors, or (d) comparing solution methods/arguments. Tasks were classified as 'analysing reasoning' tasks only if they afforded opportunities for any of (a)-(d). To establish inter-rater agreement, all 144 tasks from the Analysis books and a subset of 464 tasks from the Calculus books were also classified independently by the other author. There was total agreement in the classifications assigned.

Results of the classification

Of the 1688 tasks examined, not one presented a piece of reasoning for critical examination by the student: thus, none were classified as analysing reasoning tasks. The tasks in Figures 1 and 2 below are included as examples.

Q22. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point (0,1).

Figure 1: Example of task from Calculus textbook (Stewart, 2009, p. 121)

Q5. Use Rolle's Theorem to prove that, if p is a polynomial and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct zeroes of p, then p' has at least $(n-1)$ zeroes.

Figure 2: Example of task from Analysis textbook (Brannan, 2006, p. 253)

In four of the tasks considered, the solver was asked to 'compare solution methods', but in each case the solution methods were to be completed by the reader themselves rather than being presented as part of the task. Thus, these tasks were not classified as 'analysing reasoning' tasks. One of these tasks is shown in Figure 3.

Thus, the results of our classification exercise show that analysing reasoning tasks are not prevalent in textbooks.

Q22. Find the derivative of the function $f(x) = \frac{x-3x\sqrt{x}}{\sqrt{x}}$ in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

Figure 3: Comparison of methods task from textbook (Stewart, 2009, p. 145)

SAMPLE TASKS

In this section, we present sample tasks which were created using our design principles outlined above. The Derivative Task shown in Figure 4 overleaf asks students to make decisions on the validity of the reasoning used as they compare three proposed 'solutions' to an exercise involving the obtaining of a derivative. In doing this, students are expected to identify the errors in two of the proposed 'solutions'. Anna claims that the product rule should be used to determine the derivative, a claim students are expected to query. This affords a confrontation with the common misconception that $sin(\sqrt{x})$ is a product rather than a composition of functions. In this task, one of the proposed solutions (Colm's) is correct as it is equally important for students to be able to recognise a correct reasoning sequence as an incorrect one. Also, we believe analysing reasoning tasks are more beneficial if students are not conditioned to always expect errors in the reasoning presented.

In Figure 5, the Differentiability Task again affords students the opportunity to check the validity of the reasoning presented. To respond correctly to this task, students must identify and question the assumption about differentiability made.

Textbook tasks can be used as a basis for the design of analysing reasoning tasks. In fact, the Derivative task in Figure 4 was a modification or redesign of a routine textbook exercise in which students were given $f(x) = sin(\sqrt{x})$ and asked to find $f'(x)$ (Stewart, 2009, p. 161).

A class was given $f(x) = \sin(\sqrt{x})$ and asked to find $f'(x)$. Different approaches were taken by different students. Compare the solutions below and decide if any is correct, giving reasons for your answer.

Anna's solution: We know that $\frac{d}{dx}(\sin(x)) = \cos(x)$ and $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

Therefore, using the Product Rule, $f'(x) = \sqrt{x}\cos(x) + \frac{\sin(x)}{2\sqrt{x}}$.

Ben's solution: We know that $\frac{d}{dx}(\sin(x)) = \cos(x)$ and $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

Therefore $f'(x) = \cos\left(\frac{1}{2\sqrt{x}}\right)$.

Colm's solution: We know that $\frac{d}{dx}(\sin(x)) = \cos(x)$ and $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

Therefore $f'(x) = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$.

Figure 4: Derivative Task

Decide if the solution presented below is correct or not. Give a reason for your answer.

Question: Consider the function $f(x) = \begin{cases} 2x & \text{if } x \leq 0, \\ 3x & \text{if } x > 0. \end{cases}$ Is f differentiable at $x = 0$?

Solution: The function $g(x) = 2x$ is differentiable for all $x \in \mathbb{R}$ (and thus for all $(x \leq 0)$ and the function $h(x) = 3x$ is differentiable for all $x \in \mathbb{R}$ (and thus for all $x > 0$), since both are linear functions.
We conclude that f is differentiable for all x and thus it is differentiable at $x = 0$.

Figure 5: Differentiability task

DISCUSSION

Critical thinking skills are becoming increasingly important to support citizens in making informed decisions in relation to the many challenges with which they will be faced in the modern world. However, there is little indication internationally that critical thinking skills are being developed through school and/or higher education (Saroyan, 2022). In particular, the textbooks examined here and other literature in the area indicate a missed opportunity for undergraduate mathematics students to develop the core skills of analysis, evaluation and inference by engaging with the reasoning used in a given piece of mathematics. Guided by the results of our textbook analysis and in alignment with the literature on critical thinking, we have presented design principles for what we term 'analysing reasoning' tasks which aim to address this discrepancy. These principles can be used both to design new tasks and to evaluate an existing set of tasks (Lin et al., 2012). We suggest that the use of such explicit design principles can help avoid disparities between intention and actuality in task design.

We plan to trial and evaluate tasks that we have designed using these principles through think-aloud task-based interviews. By including analysing reasoning tasks not only in our calculus and introductory analysis modules, but also across other topics we teach, we hope to foster a culture of questioning and inquiry among our students, promoting active learning, and empowering them to deal with uncertainties in the information with which they are presented.

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Sustainability and impact of study and research paths on university teaching practice

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In this study we examine how university teachers with no prior didactics expertise progressively appropriate and sustain an inquiry-oriented approach in terms of study and research paths (SRPs) after a single year of collaborative work with researchers. Using a design research approach, we analyse two SRPs in engineering and nursing degrees and trace the evolution of the participating teachers. Results show that, after one guided implementation, teachers can autonomously redesign the generating question to fit their disciplinary context while preserving the core SRP infrastructures that sustain chronogenesis, mesogenesis and topogenesis. Interviews indicate a shift from content transmission towards the orchestration of an inquiry-based collective study. The findings highlight the transformative potential of SRPs and the conditions for their sustainable introduction in university teaching practice.

Keywords: statistics, teacher professional development, Anthropological Theory of the Didactic, design research, study and research paths.

INTRODUCTION

The challenge of promoting inquiry-oriented approaches in the teaching of mathematics and statistics at university level has been addressed from multiple theoretical perspectives. For example, Artigue and Blomhøj (2013) have analysed how different well-established frameworks in mathematics education contribute specific and complementary tools for conceptualising inquiry-based mathematics education (IBME). They highlight the Anthropological Theory of the Didactic (ATD) as a particularly coherent approach, providing an epistemological model for organising inquiry around generating questions and the tools to address how these processes are rooted in their specific social and institutional environments. However, research has also documented the ongoing challenges university lecturers face in adopting IBME: implementing inquiry-oriented approaches requires a significant renegotiation of the didactic contract, a process that should not be underestimated (Artigue & Blomhøj, 2013). Furthermore, ensuring the long-term sustainability of instructional changes remains a critical challenge, as most studies on professional development only report on the initial adoption of new practices, with little evidence that the support provided was sufficient for lecturers to consolidate and maintain these changes over time (Laursen & Rasmussen, 2019).

Research in university mathematics education has already identified many challenges lecturers must face when implementing inquiry-oriented approaches, and some actions for professional development have been proposed (Gómez-Chacón et al., 2024). In this line, the LABINQUIRY [1] project focuses on the transfer and dissemination of inquiry

proposals developed within the framework of the ATD (Chevallard, 2015). Its main objective is to facilitate the implementation of study and research paths (SRPs) in university and secondary education by providing teachers with the didactic and pedagogical infrastructures needed to promote study processes based on open questions and collective inquiries.

Since 2020, the project has developed and refined a pilot SRP in statistics centred on the delimitation of Low Emission Zones (LEZ), implemented in three Catalan universities and across different engineering degrees. This SRP has shown strong transformative potential in the organisation of content and in the type of statistical activity promoted (Barquero et al., 2025). Through successive implementations, a didactic infrastructure has been co-created, composed of semi-structured phases, monitoring instruments (logbooks, interim reports, maps of questions and answers), and collaborative strategies among teachers and researchers (Lombard et al., 2024).

The study by Verbisck et al. (2024) constitutes a central contribution to the analysis of the epistemological and role-related change SRPs can produce in university teachers. Through the longitudinal follow-up of a statistics lecturer over four years of implementing the SRP, the study shows how the teacher moves from an approach centred on the transmission of content to a practice of co-inquiry with students. In this process, statistical knowledge ceases to be presented as a closed object and becomes a functional tool for study, linked to a real and socially relevant question. The teacher assumes the role of mediator and organiser of the study milieu, managing the dialectic of questions and answers and promoting the collective validation of students' productions. Moreover, the study highlights the emergence of new professional competencies, such as the creation of didactic devices (question maps, logbooks, reports) that are later integrated into the LABINQUIRY infrastructure, showing how epistemological change also translates into a reconfiguration of university teaching.

Despite the progress made by the project in creating didactic infrastructures for the implementation of SRPs at the university level, the studies conducted so far leave open a key question: *How do teachers who are not experts in didactics evolve after participating in these collaborative processes with researchers?* The works of Lombard et al. (2024) and Verbisck et al. (2024) emphasise the transformative potential of SRPs in university teaching, but also the need to understand the extent to which this epistemological change is consolidated beyond the initial support. Similarly, Barquero et al. (2025) highlight the construction of collaborative infrastructures between teachers and researchers as a necessary condition for sustaining changes in teaching practices, but do not yet address their effects.

THEORETICAL FRAMEWORK AND RESEARCH METHODS

The ATD proposes a view in which teaching does not consist of transmitting pre-established knowledge, but rather of organising a study milieu that enables students to construct answers to meaningful questions (Chevallard & Bosch, 2020). This principle gives rise to the *paradigm of questioning the world*, in which knowledge is mobilised

as a tool for inquiry. This is in contrast with the traditional *paradigm of visiting works*, in which knowledge is presented as a finished product to be studied (Chevallard, 2015). The instructional device that realises this paradigm is the study and research path (SRP). An SRP is a long-term, collective inquiry process structured around a generating question Q_0: an open, contextualised question of genuine interest to the study community (Bosch, 2019). Rather than working through a pre-sequenced exposition of content, students engage in a dialectic of questions and answers. Q_0 generates derived questions, which in turn require the mobilisation of theoretical tools, the construction of partial answers, and the formulation of new questions that reopen the inquiry (Winsløw et al., 2013). Throughout this process, students must also search for and critically evaluate pre-existing answers available in different media, integrating them into an evolving study milieu (Chevallard, 2015). The teacher's role shifts from that of a transmitter of knowledge to a supervisor of the inquiry: managing *chronogenesis* (the rate at which new questions are raised and new knowledge is introduced when the inquiry demands it), *mesogenesis* (the progressive construction and reorganisation of the study milieu: data, resources or theoretical tools, in response to emerging needs), and *topogenesis* (the redistribution of epistemic responsibilities, in which students play a central role in formulating questions, justifying decisions, and validating answers) (Barquero & Bosch, 2015). Implementing SRPs usually necessitates specific didactic infrastructures, such as semi-structured phases, logbooks, interim reports, and maps of questions and answers, to support the collective management of the inquiry process (Lombard et al., 2024).

In our study, we adopt a methodology framed within Didactic Engineering (Artigue, 2015), understood as an approach simultaneously oriented towards transforming educational practice and generating theoretical knowledge about the mechanisms that make such transformation possible. From this perspective, a didactic intervention is not conceived as a closed design, but as an iterative process of co-design (teachers and researchers), implementation, analysis, and redesign. We conducted our analysis of two cases of SRP developed at the University of Girona (Spain), both based on the principle of studying an open and contextualised generating question (Q_0). The implementations took place during the academic year 2024-2025, in statistics courses lasting approximately 7 to 10 weeks. In total, two university lecturers and around 200 students took part, organised into groups of 25-60 students. While the two lecturers had no prior expertise in educational research, they had different roles within the LABINQUIRY project. L1 had participated in the project for two academic years. In the first year, they implemented an SRP that had been fully designed by the research team, with some local adaptations and with the project team’s full support. In the second year, they designed and implemented the SRP analysed here (SRP1) independently, in a compulsory statistics course for the Computer Engineering degree. By contrast, L2 implemented SRP2 in an elective biostatistics course in the nursing degree programme. SRP2 was designed within the collaborative framework of LABINQUIRY, drawing on previous project experiences, and advice from the research team and L1. In both cases, the LABINQUIRY team was mainly responsible for designing the proposal, while the

lecturers implemented the SRPs independently and adapted the available tools to their specific teaching context.

A variety of data sources were used for the analysis, allowing the triangulation of information from different participants and stages of the implementation. Teaching materials produced by the lecturers were collected, including course syllabuses, classroom presentations, worksheets, and the templates for logbooks and interim reports designed to guide students' work. Students' productions were also analysed, including group logbooks, interim reports, and final presentations produced at the end of each SRP. Finally, semi-structured interviews were conducted with the participating lecturers before and after the implementation. For the analysis, in the a priori phase, we progressively identified the core SRP infrastructures that have been developed through previous implementations, namely the semi-structured phases of study, the logbook and report templates, and the mechanisms used to manage the question-answer dialectic. These elements were used as reference criteria to analyse the two new cases. In the a posteriori phase, each implementation was analysed separately first, and then compared across cases according to the three dimensions that characterise the management of an SRP: chronogenesis, by tracing how questions evolved across phases, when new knowledge and tools were introduced, and how the study progressed over time; mesogenesis, by examining how the study milieu was constructed, enriched, and reorganised in response to emerging questions and needs; and topogenesis, by identifying how responsibilities for questioning, searching, validating, and synthesising were distributed between lecturers and students. For each dimension, evidence was triangulated from the data sources. The analysis was conducted by two researchers with expertise in ATD and mathematics didactics, who jointly discussed and refined the emerging interpretations. The participating lecturers also contributed to the interpretive validation of the analysis, helping to clarify the rationale behind key instructional decisions, the difficulties they encountered during implementation, and the changes they perceived in their own teaching practice. This analytic procedure enabled us to examine both the sustainability of the SRP infrastructures and the evolution of the lecturers' practices following their collaboration with the researchers.

RESULTS

SRP1: Sorting algorithms

SRP1 is particularly relevant for our research question because it shows how, after initial support, lecturer L1 was able to redesign and recontextualise the device for a new disciplinary setting. L1 had participated in LABINQUIRY in 2023-24 through the implementation of the SRP on Low Emission Zones (LEZs) in a statistics course for agri-food engineering, with the researchers' support. In this context, the LEZ topic proved to be less engaging for students than in previous engineering implementations, resulting in a less fluid and more challenging inquiry process. This first experience prompted L1 to identify two limitations of the proposal: the weak thematic alignment of the generating question and the difficulty of monitoring students' inquiry through the existing logbook structure. During the next academic year, L1 launched a new SRP

in a compulsory statistics course for the computer engineering degree. The new generating question presented to the students was: "Q_0*: Of the sorting algorithms we know how to implement, which one is the best?*". The question was explicitly presented to the students at the beginning of the course as the central focus of the inquiry.

The inquiry unfolded through six phases. In Phase 1, the teams compiled an initial set of questions about the sorting algorithms, focusing on issues such as speed, efficiency, complexity, stability, memory usage, hardware influence, and the impact of pre-ordered data. The first key task was to produce a dataset that would enable them to empirically study these questions by running the algorithms under different conditions and measuring execution times. In Phase 2, the students conducted their initial analyses, organised the results in the form of tables and graphs, and began refining their questions. In Phase 3, they shared their initial results and progressively narrowed the focus of the inquiry, while the teaching team began introducing the theoretical elements required to deepen the analysis. Phases 4 and 5 saw the students reformulating their questions as testable hypotheses and using inferential tools such as confidence intervals, hypothesis tests and regression models to analyse the behaviour of the algorithms. Finally, in Phase 6, they synthesised their results, assessed the suitability of the models used and identified the limits of their validity. They also formulated reasoned conclusions that often opened up new questions for further study.

Our analysis of this case reveals more than a simple change of topic: it shows a clear initial form of the lecturer's appropriation of the SRP. L1 was able to preserve the open-question structure, the progression through derived questions, the collective organisation of the inquiry, and the functional introduction of statistical tools, while adapting the generating question and some monitoring instruments to a new disciplinary setting. SRP1 therefore suggests that, following an initial supported implementation, the lecturer begins to transition from using the SRP as an externally provided proposal to treating it as a teachable and redesignable tool within their own practice.

SRP2: Cardiovascular risk factors

SRP2 complements the previous case by showing that the SRP can also be sustained in a disciplinary context where statistical inquiry is closely intertwined with biomedical interpretation. The SRP was implemented in an elective biostatistics course for final-year nursing students, using real data from an epidemiological study on cardiovascular risk factors (the Framingham Heart Study [2]). Students had already completed an introductory statistics course during their second year. The generating question was co-designed by the lecturer and the research team during preparatory meetings held before the start of the course: "Q_0*: How are the different biomedical and behavioural variables related to the risk of developing hypertension or cardiovascular diseases?*" It was introduced in the first session together with an exploration of the dataset. This allowed students to familiarise themselves with the variables before being asked to formulate their own investigative questions.

In Phase 1, the students explored the dataset and formulated investigative questions relating to risk factors such as age, sex, diabetes, smoking, and cholesterol. This quickly revealed the need to mobilise both statistical notions and basic epidemiological concepts. In Phase 2, they progressed to their first empirical analyses, working with frequencies, comparisons of means, contingency tables, and chi-squared tests, while beginning to interpret their results in relation to the health-related significance of the variables involved. In Phase 3, the sharing and focalisation of results led the students to formulate precise hypotheses, select appropriate tests, justify their choice of inferential techniques and distinguish between statistically significant associations and clinically meaningful results. This phase was particularly important because apparently contradictory results, such as smoking appearing as a "protective factor", opened the door to discussions about selection bias, confounding variables, and the importance of contrasting results with the scientific literature. In Phases 4 and 5, students incorporated logistic regression and model comparison procedures in order to move from two-variable associations to multivariate interpretations. This process helped them to understand the meaning of adjusted odds ratios and the need to control for confounding variables within a broader explanatory framework. Finally, in Phase 6, they evaluated the predictive ability of their models using indicators such as sensitivity, specificity, as well as the area under the ROC curve. They also reflected on the limitations of the analysis and possible ways to improve the models.

ANALYSIS OF THE SRPS

Beyond the specific dynamics of each SRP, comparing both cases reveals a common evolution in lecturers' practices following their initial collaboration with researchers. This evolution can be analysed through the dialectics of chronogenesis, mesogenesis, and topogenesis.

Chronogenesis or question-answer dialectic. The lecturer provides a template for each phase, containing sections for the following: questions addressed, related variables, results, new questions and consulted material. Students' reports and logbooks follow this structure. A clear evolution in the questions formulated by the students can be identified (see Table 1). In the initial reports, the questions are very naïve, but gradually transform into more focused inquiries and eventually become precise testable hypotheses. The process does not close upon itself as, in the final reports, new questions emerge that could lead naturally to a continuation of the inquiry. This evolution is a structural feature of SRPs, with initial questions focusing on general aspects of the phenomenon, evolving into more specific inquiries that eventually take the form of testable hypotheses and evaluative questions regarding the adequacy of the models and decisions. This common pattern across both cases confirms the epistemological dynamic characteristic of the paradigm of questioning the world, in which the generative power of the initial question is articulated through an organised study process. For the lecturers, sustaining this dynamic implies a shift in focus from planning teaching as a linear sequence of content to managing it as a self-sustaining inquiry.

Phase	SRP1	SRP2	Evolution of Questions
Phase 1. Encounter with Q_0 and derived questions	*Which sorting algorithm is fastest with random / ordered / reversed data?*	*Is diabetes related to hypertension?*	Initial broad, intuitive and non-formalised questions that express spontaneous curiosity without yet identifying variables or criteria for the analysis.
Phase 2. First explorations	*What is the relationship between the vector size and the execution time of each algorithm?*	*Are sex, age, smoking and diabetes risk factors for hypertension?*	Questions begin to become more focused: variables, relevant conditions and possible relationships are identified. The need to obtain additional data and explore empirical patterns emerges.
Phase 3. Sharing and focalisation	*In which specific cases can we claim that one algorithm is faster than another?*	*Is the mean age of people with hypertension different from that of people without hypertension?*	Formulations become more precise and technical: specific comparisons are defined, leading to questions that require methodological justification in order to be answered.
Phase 4. Hypothesis formulation	H_0: $\mu_{QS} = \mu_{MS}$ H_1: $\mu_{QS} \neq \mu_{MS}$	H_0: $\mu_{HTA} = \mu_{noHTA}$ H_1: $\mu_{HTA} \neq \mu_{noHTA}$	Questions transform into statistically testable hypotheses. Assumptions, dependent and independent variables, and contrast criteria are explicitly formulated.
Phase 5. Inferential analysis	*Is QuickSort faster in terms of the median? Is MergeSort faster in terms of the mean?*	*Which of the X variables is the most significant in explaining hypertension?*	Questions evolve towards deeper analysis: the need to explain phenomena, compare models and assess significance or stability of results appears.
Phase 6. Conclusions and new questions	*Is the theoretical order of efficiency consistent with the results obtained?*	*Is our model adequate for the study? What predictive capacity does it have?*	Questions reach a global and reflective level: the adequacy of the model is evaluated, results are interpreted within a broader theoretical framework, and justified conclusions are sought.

Table 1: Evolution of questions in both SRPs for the 6 phases.

Mesogenesis or media-milieu dialectic. In the temporal organisation of the Moodle sites, the necessary theoretical elements are not introduced before the tasks. Rather, the teaching team introduces new content in response to emerging needs, such as regression, hypothesis testing, etc. In SRP2, the lecturer organises Moodle into two sections (SRP and new topics), uploading the theoretical content required by students as their questions evolve. Both SRPs display a mesogenesis characterised by the continual expansion and reorganisation of the available study milieu, and the way in which these media are activated in response to evolving needs. This reveals a change in the lecturers' work: rather than presenting theory in advance as a fixed body of knowledge, they now reorganise the study milieu according to emerging needs.

Topogenesis or individual-collective dialectic. The work is carried out by student teams of three or four people. Course schedules include moments of small-group work, as well as collective discussions and presentations, during which teams share their results. Comments from the teaching team on specific submissions are also made visible to the whole class. In both SRPs, a clear shift in the traditional roles of teachers and students is observed. Lecturers do not act solely as transmitters of knowledge or evaluators of results, but as mediators, guiding the inquiry, organising collective discussions, providing resources when needed, and regulating the management of questions and collective agreements. Students, in turn, take on a distinctly epistemic role, formulating questions, selecting variables, justifying decisions, articulating hypotheses, analysing data, and confronting answers across groups. This highlights a key characteristic of SRPs in both cases, showing how lecturers move from content transmission towards creating a study community in which students occupy a central position in managing knowledge, while lecturers sustain the organisation and continuity of the inquiry process.

CONCLUSIONS

Our analysis of these cases reveals that, with the right support, a teaching team with no prior experience in the ATD or SRPs can design and implement an SRP autonomously after only one year of supported implementation, while remaining fully consistent with its fundamental principles. A detailed analysis of the process shows that the lecturers were able to preserve key characteristics of the SRP with respect to its three defining dimensions: chronogenesis, mesogenesis and topogenesis. This empowerment cannot be explained solely by the initial didactic infrastructure, but is also linked to certain professional qualities of the teachers: openness to uncertainty, willingness to give students space to ask questions, readiness to reorganise content according to emerging needs, and epistemological sensitivity, which enables critical interpretation of results. Our analysis suggests that the goal should be to create conditions that allow teachers to experiment safely, rely on transferable materials, and receive timely support to help them recognise the deeper meaning of the device, rather than to train "experts in ATD". Within this framework, SRPs can become a sustainable and transformative teaching practice, capable of taking root in new contexts without requiring ongoing support from the research team.

Moreover, once the first supported implementation has been experienced, the lecturers' disciplinary expertise plays a key role in sustaining the device. For example, L1's knowledge of computational methods enabled them to design a generating question based on sorting algorithms that the students had already studied in previous programming courses. This ensured that the students had the necessary technical background to generate datasets and formulate meaningful questions about algorithmic performance. L2 drew on her expertise in epidemiological research to select the Framingham Heart Study as a data source, recognising that its clinical variables would enable students to develop and test biomedical hypotheses within a familiar professional context. Teachers can adapt the generating question to their students' specific training needs without altering the basic SRP infrastructures, which remain stable and preserve their essential features.

However, the interviews also reveal a common difficulty experienced by both lecturers: the considerable workload involved in reading, organising and synthesising the diaries, as well as the large number of questions generated by the students, particularly during the initial stages of the inquiry. To address this limitation, we propose building on the collaboration initiated with a team of Argentinian researchers, who are developing a web platform and AI-based tools that can automatically cluster questions into semantic themes. This would facilitate the monitoring of the SRP and significantly reduce the time required for this task (Parra et al., in press). This new infrastructure opens possibilities for scaling SRPs and making their implementation more feasible in university contexts with limited time and high teaching loads.

ACKNOWLEDGMENTS

This work has been partially funded by projects LABINQUIRY (PDC2022-133812-C21/C22, MICIU/AEI/EU/PRTR); EDU-Paradigms (PID2021-126717NB-C31, MCIN/AEI and FEDER A way to make Europe).

NOTES

1. https://www.ub.edu/labinquiry/

2. https://www.framinghamheartstudy.org/

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A Sequence of Active Learning Tasks to Facilitate De-Ritualisation in a First Year Transition Course: The Case of Injective Function

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The secondary-tertiary transition presents many challenges for undergraduate mathematics students. One way to meet these challenges has been to infuse different forms of active learning in lectures to try and address some of these well-reported challenges. In this paper, we report work from an inter-community partnership that redesigned a predominantly chalk and talk set of lectures to ones containing sequences of active learning tasks. We utilise the commognitive construct of de-ritualisation as a tool to analyse the tasks and students' responses to these. We demonstrate how the tasks gave students opportunities to engage in deritualisation. To this aim, we draw on an episode from a sequence of tasks concerning the definition of injective functions.

Keywords: Active learning; Secondary-tertiary transition; Theory of Commognition; Deritualisation.

INTRODUCTION

Over the last three decades, research has reported on student challenges with the secondary-tertiary transition (hereafter, STT) (Di Martino et al., 2023; Gueudet, 2008). In this paper, we report an intervention that resulted from the work of an inter-community partnership between a university mathematics lecturer and Mathematics Education researchers. The focus was on redesigning a predominantly chalk and talk set of lectures to one being infused with active learning. The aim of the redesign was to support the pedagogical development of a university mathematics lecturer who sought support for implementing alternative pedagogies. The lecturer's motivation was rooted in the challenges they identified within their first-year students, many of which align with challenges reported within the STT literature. To meet the challenges of the STT, we sought to hybridise the chalk and talk practices to give students the opportunity to engage with a sequence of active learning tasks. In both the redesign and data analysis, we networked elements from two theories, the Theory of Commognition (Sfard, 2008) and the Communities of Practice (Lave & Wenger, 1991; Wenger, 1998). In this paper, we focus on utilising de-ritualisation characteristics (a commognitive construct) as a tool to explore students' engagement with the sequence of tasks that comprised the redesigned module. We report on an episode which demonstrates the use of active learning tasks that facilitate student de-ritualisation. We argue that the tasks provide opportunities for students to shift from ritual to exploratory engagement, with some tasks amalgamating both – and that these tasks also present opportunities for students to overcome reported challenges within the STT. This paper aims to explore the following research question:

RQ1: Do, and if so how, active learning-infused sequences of tasks facilitate students' de-ritualisation and their engagement with university mathematics?

LITERATURE REVIEW

First, we briefly explore literature related to student challenges in the STT. Then we discuss the active learning literature which has influenced the redesign of this university mathematics module.

The STT has been of interest to researchers in mathematics educations for decades, with insightful literature reviews by Gueudet (2008) and Di Martino et al., Di Martino et al., (2023) highlighted four main themes from the research: the mathematical gap, the use of digital resources, factors affecting academic success (e.g. standardised testing, prerequisite knowledge) and issues related to failure in the STT. Some literature focusses on proofs and the associated challenges for first year students. Selden (2012) notes that "*understanding and using both formal definitions and previously established theorems, as well as considerable creativity and insight*" (p.393). This formal proof process is a new mathematical experience as they engage with tertiary mathematics. Despite a growing body of research on the STT, Gueudet (2023) reports a lack of research focussed on interventions to support the STT. In this paper, we present an episode from our intervention which aimed to address some of these reported challenges. Active learning pedagogies formed a key component of the intervention.

Active learning is an umbrella term that often captures many and varied approaches. We work with the definition of Active Learning presented by Freeman et al., (2014) as a pedagogy "*that engages students in the process of learning through activities and or discussions in class, as opposed to passively listening to an expert*" (p. 8413). The broadness of the definition supports the use of a range of pedagogical approaches and activities such as blended/ flipped classrooms (Cevikbas & Kaiser, 2023), inquiry-based mathematics education (Rasmussen & Kwon, 2007) and specific types of active learning tasks. For example, solution sharing (Biehler et al., 2022), example generation (Kinnear et al., 2023) and multiple-choice quizzes (Feudel & Unger, 2022). The redesign process took inspiration from Nardi (2016) inter-community partnership characteristics that proposed building relationships that were non-prescriptive, non-deficit and influenced by the mathematical context. This supported flexibility in the pedagogical approaches and tasks most appropriate to meet the aims of the lecture.

THEORETICAL FRAMEWORK

As in earlier reports of our project (Hooper et al., 2025), we network two key theories: the Theory of Commognition (Sfard, 2008) and Communities of Practice (Lave & Wenger, 1991; Wenger, 1998). In this paper, we focus on the commognitive construct known as de-ritualisation (navigating between ritualised and exploratory engagement) as a lens to analyse the active learning tasks and the student responses to them. Ritual are described as being process-orientated routines in which "the task and procedure are one" (Lavie et al., 2019, p. 166). Ritual tasks tend to have a set procedure in which the lecturer wants the students to engage with, they have little room for deviate from modelled routines or make decisions about how to approach tasks – there is an

expectation on how we should act (Lavie et al., 2019, p. 166). On the other hand, explorations are "outcome orientated" (Lavie et al., 2019, p. 165) and are concerned with the production of new mathematical objects (Lavie et al., 2019). Viirman and Jacobsson., (2023) situating it within a university mathematics context, introduce six characteristics of de-ritualisation:

Flexibility: The students' ability to perform tasks in different ways, recognising problems may have multiple routines to solve them. A ritually engaged student typically relies on modelled routines with minor deviations. For example, an inductive proof with several cases requires an exploration of the routines that deviate from the "base case -inductive step" modelled solution.

Bondedness: This relates to the product of a routine in which the outcome contributes to the proceeding steps, whereby the culmination of bonded tasks supports an outcome. The ritually engaged students will view these outcomes as independent routines.

Applicability: This concerns the range of tasks in which students identify a routine being useful for. Those exploratory learners will increasingly spot features of tasks that make a given routine applicable.

Agentivity: Here students take responsibility for decision making, the processes they use, what steps they take and when to stop. Ritualised engagement will rely on scaffolding from a significant other to support decision making. Agentive participation will rely less on scaffolding, and these decisions can be made independently.

Objectification: This relates to the students being able to formulate narratives with a focus on routines as objects of mathematical discourse rather than just processes to follow.

Substantiability: This concerns students providing coherent arguments to support their decision making. Ritualised engagement may involve simply recanting steps taken in a task. In contrast shifting to exploratory engagement involves justifying the steps taken by referring to objects of the mathematical discourse.

We utilised these six characteristics to analyse the lecture recordings, audio data and solution uploads. They served as a tool to identify potential opportunities for de-ritualisation (e.g. from the task design). It provided a reference point to compare the students' solutions for evidence of deritualisation. We make the case that these active learning tasks helped facilitate student de-ritualisation.

CONTEXT AND METHODOLOGY

The course redesign was implemented in a first-year mathematics module entitled "Mathematical Skills, Sets, Numbers and Proof" and was described as a transition course. The intervention spanned 12 weeks in the autumn semester with 42 students enrolled. The inter-community partnership was comprised of Hooper, Witty (a mathematics lecturer) and two supervisors who did not attend meetings (Elena Nardi and Irene Biza, members of the Research in Mathematics Education group at UEA) but were consulted as Hooper's doctoral supervisory team. The redesign involved 5

open ended, 90-minute design meetings in which Hooper and Witty had discussions facilitated by previous experiences and resources (e.g. previous lecture notes). This process resulted in the redesign of 22 sessions, delivered in a hybrid format of chalk and talk and active learning tasks. The design phase has been the focus of a previous conference contribution (Hooper et al., 2025). The intervention sessions were delivered by Witty and observed by Hooper. Several sources of data were collected including lecture recordings, students' uploads to Padlet and student-lecturer conversations during activities. In this paper, we report an episode from lecture 15 (Week 8 of the semester) which focussed on introducing students to the definitions of injective, surjective and bijective functions. In line with Viirman and Jacobsson (2023), we use the de-ritualisation characteristics to analyse first the potential opportunities for deritualisation in the tasks followed by the students' responses to these tasks. The lectures were transcribed fully (where possible). Hooper analysed the transcripts, recordings and students' uploads to build a comprehensive account of the intervention. The characteristics were used to analyse these lectures by employing a coding-based system. The characteristics of deritualisation were applied with a coding approach. We coded for both potential opportunities identified in the tasks and analyse the written and verbal exchanges for evidence of deritualisation. We note that the tasks were not designed with these characteristics in mind since this concept for the data analysis was introduced after the design of the course had been completed.

TASKS TO MOTIVATE LEARNING ABOUT THE INJECTIVE FUNTION

In this episode, we report on data from a sequence of tasks that motivated the definition of injective (the focus of this paper), surjective and bijective functions. We explore a variety of active learning tasks, first to support ritual engagement before presenting opportunities for de-ritualisation. These exploratory routines aid students' engagement with potentially challenging concepts related to the injective function (e.g. the restriction). For example, Thoma and Nardi, (2018) reported students not considering the set of numbers they are working within and how this impacts the injectivity and surjectivity of a function.

Task 1 utilised crowd sourcing with students asked to list similarities and differences between four familiar functions (from secondary mathematics): $f(x) = x$, $h(x) = x^2$, $g(x)= x^3-x$ and $k(x) = e^x$). The purpose of this comparison was not explicit to start which adds an exploratory component to it. The task relies on students recalling helpful processes (e.g. finding the roots of an equation, the y-intercept) related to functions from secondary level mathematics. The student uploads are in Fig. 1 below. To support objectification, the lecturer intends to utilise the students' realisations of the function (rooted in their school experiences of mathematics) to motivate the formal definition of injective thus giving them a new mathematical realisation.

To motivate the definition of injective, the lecturer uses the response "*x^3-x has multiple roots whereas the other linear functions only contain one*". The lecturer corrected the idea that all other functions were linear but noted the idea of multiple roots would be

useful. The definition of injective is then written on the visualiser with this accompanying explanation:

> "The definition is saying if I have two things which obtain the same value, they should have been the same point in the first place. So, h(-1) and h(0) are the same okay but 0 and -1 aren't the same"

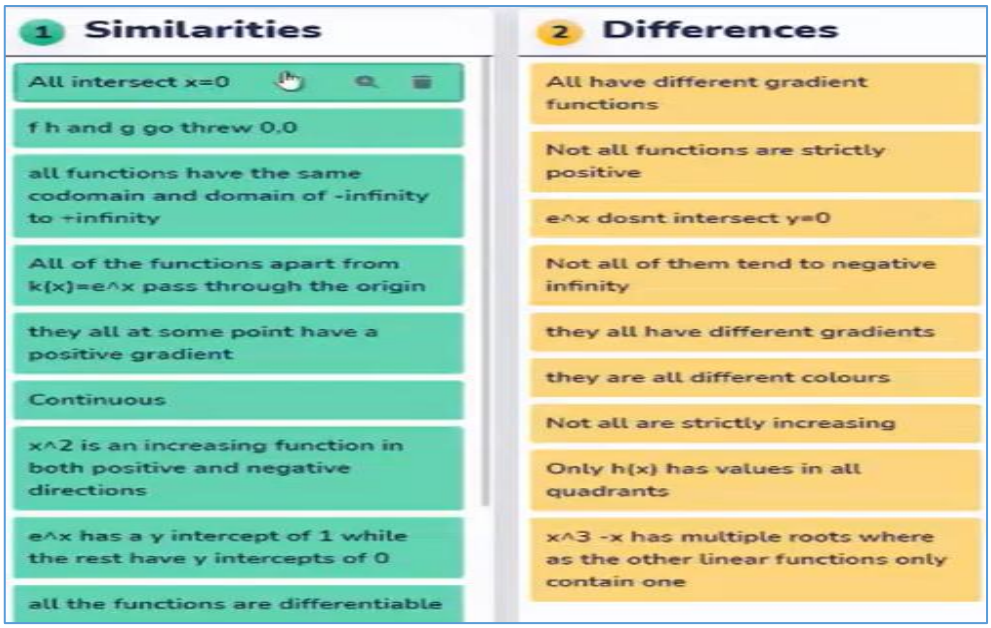

Fig. 1: Student Responses to Task 1.

The explanation by the lecturer utilised routines of finding multiple roots (e.g. h(-1) = h(0) = 0). This demonstrates ritual objectification since the lecturer formalises the definition of injective functions. The use of the crowd sourcing supported some initial exploratory agentivity since the students needed to decide what realisations of the functions from school would be appropriate to upload.

Task 2 was a multiple-choice Wooclap quiz. The students selected which of the four functions listed were injective. The motivation was to get students applying the new definition immediately. The quiz presents opportunities for ritual applicability and flexibility. The students apply the definition without deviations and can refer directly to the definition. The quiz prompts them to make a quick decision which does not require any formal substantiation and is thus a ritual orientated task. Although, we hope they have justified to themselves they are correct.

In Task 3, we begin to see evidence of de-ritualisation as students are asked to justify whether $k(x) = e^x$ is injective, surjective or bijective. A student's response is in Fig. 2. Before this, the lecturer went through a modelled example of another function. The task is designed to foster exploratory substantiation and agentivity, with students left to make the decision on how to prove if this function is injective, bijective or surjective. The addition of a justification element supports students in utilising previous mathematical routines to support substantiation and agentivity. However, applicability

and flexibility are potentially ritually fostered since there is a modelled example available.

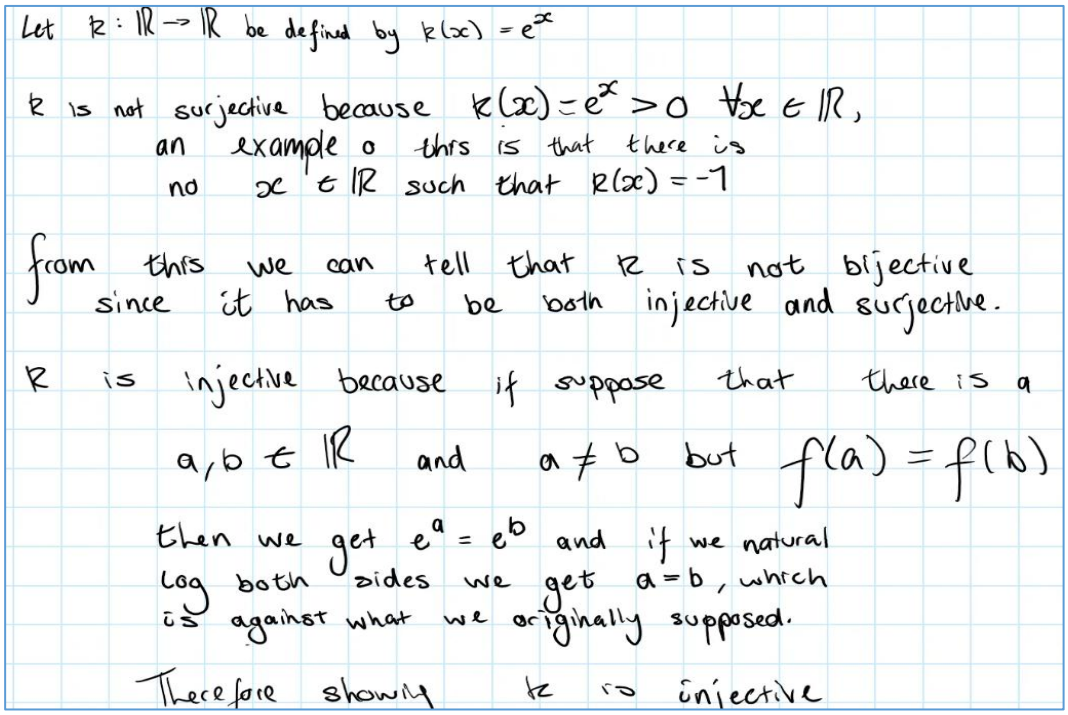

Fig. 2: A student's response to Task 3.

The student in Fig. 2 has recognised that a proof by contradiction is a useful routine to support the justification of the function being injective. The use of a previous proof routine suggests evidence of the student's response displaying exploratory applicability since they have spotted features of the proof by contradiction that will help them decide if the function is injective or not. However, the students can refer to the modelled solutions for inspiration for their proof. The task requires only a small deviation (the function being worked on) in the new realisation. Therefore, the task has given students an opportunity to ritually engage with the new realisation to consolidate their understanding whilst beginning to foster some explorations (substantiation: proof by contradiction) which supports other de-ritualisation characteristics such as student agentivity (e.g. making decisions regarding the injectivity of the function).

Before Task 4 began, Witty briefly added a new realisation to the students' narratives about functions: the definition of a restriction on a function. In Task 4, students were given a new set of questions, see Fig. 3.

Witty included the restriction function in part (i) and other functions that deviate from the familiar secondary mathematics functions used in previous examples and questions. These deviations help facilitate the opportunities for students to engage in exploration routines. For example, the restriction requires exploratory applicability and flexibility since they must appreciate how the restricted function maps differently.

Which of the following are injective/surjective/bijective?

$\varphi|_{\mathbb{N}}$ where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^2$.

$$f : \mathbb{N} \to \mathbb{N}, n \mapsto \begin{cases} 2n, & \text{if } 2 \mid n, \\ 8n, & \text{if } 2 \nmid n. \end{cases}$$

$$g : (0,1) \to (0,\infty), x \mapsto \frac{x}{1-x}$$

$$h : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x}{1+x^2}$$

Fig. 3: Task 4 questions set by the lecturer.

We focus on a student response to $\varphi(x)=x^2$, as in Fig 4 below. The addition of the restriction will give them opportunities for exploration as they do not have a modelled solution to refer to.

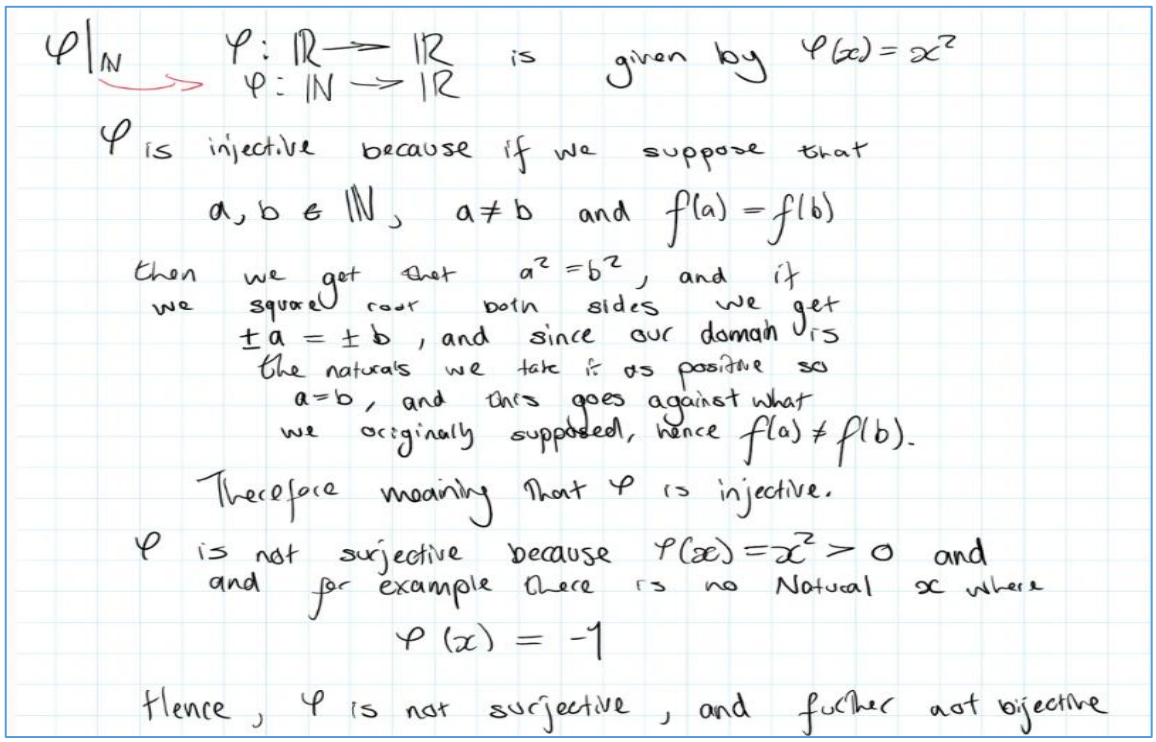

Fig 4: A student's response to Task 4.

Now, in the student's response above we begin to see evidence of the deritualisation in the fourth active learning task. Firstly, the student has been able to appreciate the deviation caused by the addition of the restriction function. Despite this they can still utilise a proof by contradiction and make specific reference to their not being a natural number x in $\varphi(x) = x^2 > 0$ where $\varphi(x) = -1$. This demonstrates exploratory applicability of proof routines. This goes hand in hand with flexibility: to perform this task, the students must recognise the domain is no longer $\mathbb{R}$ to $\mathbb{R}$ but $\mathbb{N}$ to the $\mathbb{R}$. The students must consider a different realisation of injective function so that it considers how the change in domain effects the routine they employ to complete the task. We expect a ritually engaged student to not consider this change of domain but follow the modelled routines previously made. The flexibility and applicability evidenced in the students'

work also implies a move towards exploratory substantiation. This is evident in the student's response as they directly reference the domain being $\mathbb{N}$ which justifies the student's response to not consider negative values which leads to a contradiction in their argument. The student correctly deduced the function was injective. This solution demonstrates a well justified argument going further than recanting steps, by considering the effect the restriction has and utilising this in their substantiation. Together, these three features in the student's response demonstrate the agentivity of the student who can make decisions and justify how to proceed with the deviation on the task, how to adapt the proof by contradiction and when to stop.

In this episode, we have seen how different tasks can support the shift from ritual to exploration by first consolidating the former to open opportunities for the latter.

DISCUSSION AND CONCLUSION

The analysis of this episode highlights how an active learning infused sequence of tasks has the potential to facilitate engagement with university mathematics through de-ritualisation. The episode demonstrates a gradual shift which starts with the consolidation of the ritual before adding in tasks that require a greater degree of exploration to be able to engage with them. For example, the lecturer only added deviations to the tasks after students had an opportunity to first engage with the definition of injective ritually first. The active learning tasks present opportunities for engagement both ritually and exploratory. These can be leveraged by design, for example Task 4 allows for exploration as the Lecturer infused it with unfamiliar functions which requires the students to engage with the new mathematical context that differs to previous modelled solutions. The lecturer in this episode utilised a variety of different active learning tasks such as crowd sourcing, multiple choice quizzes and proof questions. These different tasks required students to engage differently with each task, the first task used to encourage agentivity whilst tending to ritual objectifications of functions from secondary mathematics whereas the last task gave multiple opportunities at deritualisation. Across the tasks, we see evidence of agentivity, applicability, flexibility substantiation. The deritualisation is supported by a gradual gradual introduction of deviations within tasks, which require students to engage with the mathematical content in ways that go beyond questions that require similar solutions to modelled definitions, proofs and examples. The emergence of deritualisation has the potential to support students overcome challenges in the STT. For example, Thoma and Nardi (2018) reported commognitive conflicts amongst first year students working with functions not mapping from $\mathbb{R}$ to $\mathbb{R}$. This episode demonstrates a potential avenue to address STT challenges. In Task 4, the lecturer introduces a restricted function which encourages the students to consider how this affects the definition of injective, surjective and bijective. This allows students to consolidate the definitions ritually first before engaging in explorations and appreciate without becoming overwhelmed at the addition of new objects of mathematical discourse. This potentially alleviates some of the challenges reported by Thoma and Nardi (2018). The use of tasks that encourage substantiation (both ritually and

exploratory) potentially attend to challenges regarding the justification routines. Finally, the paper contributes to the development of commognitive theory in university mathematics, responding to calls by Gueudet (2023) for research that integrates design, implementation and analysis of interventions. We demonstrate how commognitive constructs such as deritualisation can be utilised as lens through which to analyse students' responses to tasks and therefore help evaluate the pedagogical impact of the module intervention that the tasks were part of (Nardi et al., 2021b).

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A university mathematics lecturer's experience designing and implementing active-learning teaching sessions with digital resources

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Keywords: Secondary-tertiary transition, active learning teaching design, digital resources, teacher-researcher partnerships, autoethnography.

INTRODUCTION

The transition from secondary to tertiary education has numerous associated difficulties for mathematics students (Gueudet, 2023) and, over the years, greater interest has been shown in interventions to support the transition to university mathematics. However, Gueudet (2023) notes that there are few studies related to the design, implementation and evaluation of interventions to support students through the transition to university. Gueudet (2023) also gives a summary of suggested interventions including the recommendation of support initiatives both before and when starting university to achieve a positive transition and prevent student drop out, as well as a call for more collaborative classroom-based mathematics work at university in order to create a community.

There has also been a growing trend to replace lectures with different methods of teaching. These include approaches involving active learning, which "engages students in the process of learning through activities and/or discussion in class" (Freeman et al., 2014, pp. 8413-8414), as well as a greater use of digital resources to enrich face-to-face learning environments (Hooper & Nardi, 2024).

In addition, research involving practice-oriented interventions in university mathematics has also grown (Biehler et al, 2022), in particular interventions taking the form of an inter-community partnership (ICP) where mathematicians and mathematics educators collaborate (Nardi, 2016). A "non-deficit, non-prescriptive, context-specific, example-centred and mathematically focused" approach is used to improve collaboration and communication between the two communities (Nardi, 2016, p. 372).

CONTEXT

When teaching a first-year mathematics course covering core definitions and concepts of pure mathematics, the first author (Witty) identified numerous issues associated with this transition, including issues linked to a lack of communication between students, low student engagement, students' attempts at mathematical writing as well as Witty experiencing a difficulty in conveying passion for mathematical thinking when consigned to a blackboard for teaching.

Witty attempted to overcome some of these difficulties by making changes to the teaching design in an ICP with mathematics education researchers Hooper, Biza and Nardi. Witty would share his personal experience with the researchers, highlighting challenges he had experienced in previous teaching, and the researchers would provide

research-informed suggestions of solutions. This led to the creation of a set of lesson plans for the course which involved active-learning elements such as group discussion, and digital resources such as quizzes and collaborative bulletin-boards using Padlet. These lesson plans were then implemented in the next iteration of the course.

RESEARCH QUESTIONS AND POSTER DESIGN

The poster will take a reflexive autoethnographic approach (Ellis et al., 2010), drawing on Witty's experiences of the course design and implementation. It will address the following questions: "How did the ICP impact Witty's teaching?", "How will the ICP impact Witty's future teaching?" and "What insights into ICPs between university mathematics lecturers and mathematics education researchers can be gleaned from Witty et al.'s experience?". The poster will outline the context of the course and issues identified in previous iterations of the teaching as well as offer reflections on the design and implementation of the course, supported by data from student submissions, with a focus on two areas identified as key: mathematical writing and logical thinking. The reflections will cover how Witty believed student participation and writing improved, and how greater interaction with students enhanced Witty's own experience. There will also be a brief discussion of limitations and challenges in the new teaching approach.

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Modelling mathematical thinking in university mathematics lectures: An analysis of pronoun use

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This study investigates how reflective mathematical thinking (Zolkower et al., 2015) is externalized during university mathematics lecturing. Drawing on a social semiotic perspective, we examine a lecturer's use of personal pronouns as semiotic resources that position students as interlocutors in a shared mathematical discourse. Through fine-grained functional grammatical analysis, we characterize pronoun use and their discourse functions. Our findings indicate that pronoun-mediated discourse functions constitute a structured form of reflective guidance, rendering aspects of mathematical thinking visible and pedagogically accessible. The study contributes to discussions on the language of university mathematics teaching and provides an analytical approach to understanding how lecturers externalize their mathematical thinking.

Keywords: linear algebra, reflective mathematical thinking, social semiotic perspective, personal pronouns, case study.

Although alternative teaching approaches—ranging from collaborative learning to inquiry-oriented and technology-mediated ones—have gained increasing attention, lecturing remains the most widely used and institutionally supported format in university mathematics education (Weber & Fukawa-Connelly, 2023). Curricular demands, time and institutional constraints often restrict the implementation of alternative teaching approaches, while several scholars have emphasized the enduring capacity of the lecture to model mathematical thinking. Weber and Fukawa-Connelly (2023) argued that lectures can demonstrate the lines of mathematical reasoning (big picture) in real time, thereby communicating mathematical activity that extends beyond content delivery. Similarly, Viirman (2021) considered lectures as models of mathematical discourse, through which students encounter characteristic ways of speaking, representing and reasoning mathematically. French and Kennedy (2017) further described the lecture as a setting in which the lecturer acts as an expert model, showcasing how a researcher approaches and structures problems. Collectively, these studies suggest that lectures offer an opportunity for students to experience a lecturer's mathematical thinking in real time.

The idea of making mathematical thinking visible resonates with Zolkower et al.'s (2015) conceptualization of reflective mathematical thinking. The act of reflecting on one's own experience has been described in the literature as "verbal thinking," "inner speaking," "covert self-talk," "internal monologue," "internal dialogue," and, most commonly, "inner speech" (Alderson-Day & Fernyhough, 2015). Drawing on a social

semiotic perspective, Zolkower et al. (2015) conceptualized reflective mathematical thinking as a dialogic process that unfolds both interpersonally, through interaction with others, and intrapersonally, through one's own inner speech. While this dialogic interplay is observable in school classrooms rich in whole-class discussion, tertiary mathematics lectures typically involve minimal interpersonal dialogue. This makes the lecturer's monologue central: when student input is limited, the lecturer must structure and articulate their mathematical activity in ways that externalize and communicate their underlying mathematical thinking.

This study examines this phenomenon in the context of university mathematics lecturing in Greece. We focus on a mathematician whose teaching practice is explicitly grounded in the intention to reveal his own mathematical thinking. As he stated in the interview: "I try to convey to students the way I think." Our aim is to understand and characterize how the externalization of reflective mathematical thinking emerges during his lectures. Specifically, we ask: How does a mathematician externalize his reflective mathematical thinking in a lecture setting at a Greek university? To investigate this, we analyze one specific semiotic resource—the lecturer's use of personal pronouns (e.g., I, you, we)—as a means through which he communicates his mathematical activity.

While Anna Sfard's commognitive approach foregrounds the central role of discourse in the teaching and learning of mathematics, it does not offer an in-depth analysis of personal pronouns as resources. Previous studies on pronoun use in mathematics and other scientific domains (e.g., Yeo & Ting, 2014; Zolkower et al., 2015) have provided valuable insights into the role of language in teaching and learning; however, they have not examined the use of pronouns by university mathematics lecturers, which constitutes the focus of the present study. By focusing on pronoun use in university mathematics lecturing, this study contributes to a relatively underexplored area: the semiotic mechanisms through which lecturers externalize inner mathematical speech and, in doing so, model expert mathematical thinking for students in lectures.

THEORETICAL UNDERPINNINGS

A social semiotic perspective conceptualizes reflective mathematical thinking as a dialogic and multi-semiotic process. Rooted in Dewey's (1938) account of inquiry, reflective thinking is understood as a form of conversation, initially with others and subsequently with oneself. Vygotsky (1978, 1986) argues that inner speech emerges through participation in socially organized, linguistically mediated activity. Within this perspective, reflective mathematical thinking is shaped by participation in lectures in which problems are set, mathematical results are used and various solution pathways/processes are considered. While students move from interpersonal discourse to intrapersonal meaning-making, the lecturer undergoes the inverse trajectory. The lecturer externalizes their reflective thinking so that students can join as interlocutors in a shared mathematical discourse.

In a social semiotic perspective, mathematizing (i.e., the structuring and re-structuring of problematic situations) (van den Heuvel-Panhuizen & Drijvers, 2020) is considered as a multi-modal semiotic activity. Mathematical thinking is accomplished through the coordinated use of verbal language, algebraic notation, diagrams, and other representational means that allow mathematizing. This involves projecting an imagined interlocutor—representative of the expectations and practices of the mathematical community—into the problematic situation and engaging in an intra-individual dialogue with that interlocutor.

Zolkower et al. (2015) developed a functional grammatical analysis of mathematical discourse in class. Within this analysis, personal pronouns are part of a broader set of semiotic devices that facilitate students' participation in the discourse through the modelling of mathematical thinking. In personal pronouns referents are determined contextually and multiple pronouns may share a single referent. In particular, speakers frequently alternate between specific and generic referents (Zolkower et al., 2015). This flexibility is common in mathematics classes, where lecturers address individual students, entire groups of students and imagined mathematicians engaged in mathematical activity.

Complementing these insights, Yeo and Ting (2014) examined the use of pronouns in higher education lectures across arts and science faculties. Their analysis distinguished between audience-directed and generalized uses of you, as well as inclusive versus exclusive forms of we, and proposed a taxonomy of pronoun-mediated discourse functions (e.g., activate students' prior knowledge, explain concepts). However, their work focused exclusively on lecture introductions and did not consider mathematics lectures or discourse in whole lectures.

In the present study, Yeo and Ting's (2014) taxonomy constitutes a central analytical construct. We adopt their categorization of pronoun-mediated functions, while extending it in two ways. Firstly, we follow Zolkower et al. (2015) to refine pronoun categories according to referent type. In particular, we consider you-generalized as you referring to an indefinite one, and we differentiate I into I-speaker (referring to the lecturer as an individual) and I for indefinite one. Secondly, we draw on Rowland's (1999) distinction between action and knowledge, aligning I with enacted processes and you with mathematical objects or truths. Building on these, we adapt Yeo and Ting's (2014) taxonomy to address the specificities of university mathematics lecturing, enabling a nuanced exploration of how pronouns facilitate the externalization of reflective mathematical thinking in this context.

METHODOLOGY

Context of the study and methods of data collection

The study was conducted at a Greek research university and the focal course was linear algebra for first-year students. The course structure comprised two weekly 1.5-hour lectures and one 2-hour laboratory session. Lectures were delivered in a conventional lecture format. Laboratory sessions required students to work on

assigned exercises with support from the lecturer and graduate assistants. The primary participant was the course lecturer, a research mathematician with nearly twenty years of teaching experience.

Data consisted of audio-recorded observations of six lectures and semi-structured interviews with the lecturer (two interviews), lasting for 20–30 minutes each. Lecture recordings were supplemented with detailed field notes capturing non-auditory features, including blackboard work and the coordination of verbal and symbolic representations. Lecturer interviews focused on his teaching intentions and the functions of specific linguistic choices, especially pronouns, within his exposition. Triangulation across data sources—lecture recordings, interviews and field notes—was used to enhance trustworthiness.

Methods of data analysis

In the first stage, we used the interview transcripts to identify the lecturer's articulated intentions of teaching the linear algebra content to first-year students. We then identified from the lecture transcripts all personal pronouns—I, we, you singular (sg), you plural (pl), and third-person neutral forms—followed by an initial identification of their referents. Drawing on Greek grammatical features, such as the encoding of person and number in verb morphology and the distinction between singular and plural forms of you, we were able to determine with precision whether a pronoun related to a specific interlocutor (e.g., the lecturer himself, an individual student, the student audience) or referred to a generic/imagined mathematical actor, following the distinctions highlighted in Zolkower et al. (2015).

In the second stage, we identified pronoun-mediated functions and connected them to the pronouns used. The categorization of functions followed an adaptation of Yeo and Ting's (2014) taxonomy, modified to accommodate the discursive demands of full mathematics lectures rather than only lecture introductions. For example, the category "explain concepts" was differentiated into: (a) state definitions or mathematical rules, (b) horizontally mathematize and (c) vertically mathematize (van den Heuvel-Panhuizen & Drijvers, 2020). Additional discourse functions—such as share personal experiences and views, direct students' attention and arouse interest, give instructions or make announcements, state aims and objectives, check students' understanding, use fillers, and engage in small talk—were likewise revisited or subdivided to enhance analytic clarity.

The final stage of analysis integrated the coded pronouns, their referents and their functions with the lecturer's teaching intentions. For each discourse function, we established anchor examples during the coding process and used them as references when analyzing the transcripts. In total, we found a preliminary set of sixteen pronoun-mediated functions corresponding to four lecturer's intentions of teaching the linear algebra content to first-year students. This interpretive synthesis enabled us to connect linguistic patterns with teaching intentions and thus to characterize how reflective mathematical thinking was externalized during the lectures.

FINDINGS

Pronoun use in the externalization of reflective mathematical thinking

We investigated an approach to teaching in which the lecturer systematically externalized his reflective mathematical thinking (Zolkower et al., 2015). Rather than presenting only polished mathematical results, he verbalized his internal flow of mathematical thought, enabling students to observe how mathematics was produced in real time.

A central semiotic mechanism supporting this externalization was the consistent variability in use of personal pronouns. While personal pronouns ("I", "we", "you" in singular and plural forms) played a prominent role in doing mathematics, third-person neutral pronouns (e.g., "it", "a system", "a solution") emerged systematically in moments where the lecturer stated mathematical knowledge. In particular, the lecturer's use of "I" (e.g., "if I have an equation, then I have infinitely many solutions") and singular (sg) "you" (e.g., "You will solve…, you find…, you must have…") frequently appeared in places where mathematical discourse would typically employ an impersonal or generic construction (e.g., "one"). These pronouns served to narrate mathematical thinking by positioning either the lecturer or an imagined mathematician as the agent performing the mathematical actions. In contrast, the use of third-person neutral pronouns (e.g., "a system doesn't have…", "there's the case that it doesn't have any solution") oriented discourse away from a mathematizing agent and toward the articulation of depersonalized mathematical knowledge.

The pronoun "we" fulfilled multiple roles. It was often used to construct a sense of collective engagement ("we can see that…", "we have already written…"), thereby framing the lecture as a shared mathematical discourse. Additionally, "we" was used retrospectively to invoke prior knowledge ("But here we have learned that…"), situating new content within a shared lecture history.

A noteworthy finding—made accessible by the Greek distinction between singular and plural second-person forms—was the differentiated use of singular (sg) "you" and plural (pl) "you". The "you (sg)" typically related to an imagined mathematician engaged in a thinking process. In contrast, the "you (pl)" had a dual role: at times it referred to an imagined mathematical actor ("when you (pl) take a vector…, when you (pl) have…"), and at other times addressed the student audience directly ("[you (pl)] don't sit and memorize things by heart"). For this distinction to be identified in languages where the same pronoun form is used for both singular and plural would necessitate an in-depth interview with the lecturer, enabling them to reflect on their linguistic choices and determine whether they were referring to a generic individual student or to multiple specific ones. In contrast, Greek and other languages that use different forms make this distinction more accessible.

The lecturer's shifting in pronoun use structured the lecture as a dialogic interplay between the lecturer-as-mathematician doing mathematics (e.g., use of "I", "you

(sg)") and the lecturer-as-mathematician teaching mathematics (e.g., use of "you (pl)", "we"). Through these shifts, the lecturer was "conversing with a mathematizing other" (Zolkower et al., 2015), whether that other was himself, the students or an imagined mathematician whose thinking he acted upon. Importantly, the shifting between personal pronouns and third-person neutral pronouns further contributed to this dialogic structure. While personal pronouns foregrounded mathematical activity as a human, agentive process, the neutral third-person forms marked transitions into institutionalized mathematical statements, such as rules and definitions. This interplay between agentive and neutral pronoun use thus served as a semiotic resource for coordinating reflective mathematical thinking with the formal demands of the lecture format.

Pronouns and discourse functions

The following transcript excerpts exemplify the connection between the pronouns used in a lecture and the pronoun-mediated functions they serve. That part of the lecture concerns a problem that introduces least squares optimisation. The excerpts focus on three pronoun-mediated functions: *state definitions or mathematical rules*, *vertically mathematize* and *horizontally mathematize*, which align with the lecturer's intention to present the mathematical content of the course. We chose the following transcript excerpts as they are paradigmatic of the pronoun use in these three functions.

Before the first excerpt, the lecturer had introduced the concept of the orthogonal projection. He then presented an example of the uses of the taught material "in an environment outside of formal mathematics" as he said in the interview; that is, a laboratory setting. The problem included a series of experiments that yielded linear equations in the process of determining certain properties of a material.

Lecturer: Let's say to… to take some measurements. If, let's say, you (sg) have x_1 to x_2, are the variables that correspond to these attributes and you (sg) want to measure, let's say, to calculate the quantity in the end. [The lecturer writes on the board: $\{x_1, ..., x_n\}$]

Lecturer: So you (sg) conduct an experiment and make an equation. [The lecturer writes on the board: $\alpha_{11} x_1 + \ldots + \alpha_{1n} x_n = b_1$]

You (sg) conduct a second experiment. [The lecturer writes on the board: $\alpha_{21} x_1 + \ldots + a_{2n} x_n = b_2$]

You (sg) continue your (sg) experiments. [The lecturer writes on the board: $\alpha_{m1} x_1 + \ldots + a_{mn} x_n = b_m$]

In these excerpts, the lecturer's horizontal mathematization consisted of translating the laboratory problem into mathematical symbols. The pronoun he used was "you (sg)". In Greek, it is evident that the lecturer did not address the students directly (you-audience), because he used the singular form of "you" rather than the plural form of "you". According to Zolkower et al. (2015), any personal pronoun that can be

replaced by “one” or by passive voice can be coded as having a generic referent. The lecturer’s use of “you (sg)” aligned with the formal generalised pronoun “one” as the scenario he narrated could happen to “anyone doing mathematics” [you for indefinite one]. Interpreted though the notion of generalized other, in particular, this pronoun choice positions the imagined mathematician not as a specific student but as a participant in the mathematical discourse: himself, the students or any other mathematician in a research setting (generic referent) [you for indefinite one].

The lecturer then said to students that the next step was to solve the linear system, so the quantities of the problem are found. However, unlike the linear systems the students had encountered before, the system the lecturer wrote on the board had no solution because of measurement errors, which are present in a laboratory setting.

> Lecturer: When you (sg) are in a laboratory, this is what you (sg) want to do. Meaning that you (sg) say, let’s say, let us see if I have one equation, then I have infinite solutions. If I have two equations, the solutions are infinite because there are a lot of free variables. But if I have, let’s say, my variables are 10, let’s say. If I make 10 equations, if they happen to be linearly independent columns, for example, it has a unique solution. I solve it and we are done.
>
> Lecturer: But here we learned that these are not the only cases, right? A system doesn’t have… either a unique solution, or infinite solutions. There’s the case that it doesn’t have any solution. And here what does it mean for it not to have a solution?

In these excerpts, the lecturer’s vertical mathematization included the connection of the solutions of a linear system to the process of solving it. His discourse shifted from the singular “you” to “I”. Thus, the prior reference to an imagined mathematician in a laboratory setting (a generic referent) shifted to the lecturer himself (I for indefinite one). This pronoun use resembled Rowland’s (1999) findings, whose analysis linked the use of “I” to reference to actions and the use of “you” to reference to mathematical knowledge (e.g., mathematical rules that applied). In Rowland’s (1999) study, however, the pupils made the reverse shift (from “I” to singular “you”) to pull back from the process and make it more impersonal. In these excerpts, we interpreted that the lecturer, by shifting to “I”, put himself further inside the process and let the students observe his action (signalling his own thought process). Within the social semiotic perspective, this shift was significant: the lecturer temporarily withdrew the generalized other and foregrounded his own subjective self, thereby modelling reflective mathematical thinking in its intrapersonal form. It is important, however, to distinguish the singular “you” from the third-person neutral pronouns observed elsewhere in the lecture. Whereas in our analysis the singular “you” served to narrate forms of mathematical thinking, the third-person neutral forms (e.g., “a system doesn’t have…”) marked transitions into authoritative, depersonalized mathematical knowledge, such as definitions or formal rules.

In the last excerpt, the lecturer used "we" to refer to content they had covered in previous lectures, but the singular pronouns were most prominent throughout the discourse. In the interview, the lecturer emphasized that he did not "want to put distance between himself and what is being discussed in the classroom". We interpreted his use of "we" contextually as an attempt to construct a shared mathematical stance, the "inclusive-we for you and I". In the interview the lecturer also said that his shifts between I, you (sg) and we were unconscious and he found them surprising in retrospect. This reflection further underscored the role of pronouns as semiotic resources that operate both intrapersonally and interpersonally, modelling reflective mathematical thinking in a way consistent with the appropriation of the generalized other.

Then, the lecturer stated that "A system doesn't have… either a unique solution, or infinite solutions. There's the case that it doesn't have any solution." In stating this mathematical rule, the lecturer used third person neutral pronouns. We interpreted that the pronoun-mediated function here was state definitions or mathematical rules. Within this function, pronouns were less oriented toward positioning an agent doing mathematics and more toward stating mathematical content knowledge for the audience, thereby supporting students' access to formalized mathematical content.

Zooming out, across two discourse functions associated with the presentation of mathematical content—horizontal and vertical mathematization—singular pronouns ("you (sg)" and "I") were the most common. Their predominance aligned with the institutional and disciplinary constraints of the course: the mathematical content is largely fixed by the curriculum and remains consistent regardless of audience composition. Generic pronouns, including third-person neutral forms used for stating definitions or rules, therefore provided an appropriate linguistic means of presenting content applicable to any participant in a mathematical learning environment. More importantly, within a social semiotic perspective, these pronouns mediated students' access to the multi-layered conversations—both interpersonal and intrapersonal—that constituted reflective mathematical thinking. While agentive pronouns ("you (sg)" and "I") positioned the lecturer or an imagined mathematician within the process of doing mathematics, third-person neutral pronouns foregrounded the generality of mathematical knowledge, thereby supporting students' access to formalized content and facilitating the appropriation of the generalized other of the mathematical community.

DISCUSSION

This study examined how a university mathematics lecturer externalized reflective mathematical thinking (Zolkower et al., 2015) through pronoun use during lectures. Pronouns functioned as salient semiotic markers through which the lecturer made his mathematical thinking explicit. The choice of pronouns was neither arbitrary nor stylistic; rather, it systematically aligned with distinct discourse functions. Indefinite referents—such as I for an indefinite actor and singular or plural you for an indefinite actor—appeared most frequently in relation to the presentation of mathematical

content. This suggests that the lecturer's discourse framed mathematical thinking as an activity accessible to anyone engaged in doing mathematics, rather than as idiosyncratic to himself. Additionally, third-person neutral pronouns were employed when stating definitions or mathematical rules, signalling that the content itself, rather than an agent, was the focus. In doing so, the lecturer foregrounded the generality and formal structure of mathematical knowledge.

The shifts between I, singular you, we, and plural you demonstrated how linguistic elements such as pronouns can offer shifting perspectives within the discourse of an expert in mathematics. In this case study, the lecture transcended a mere presentation of mathematical content, becoming instead a performance of mathematical thinking. Through these pronoun-mediated shifts, the lecturer created an interpersonal space (Zolkower et al., 2015) in which students could potentially follow his thinking and begin to adopt a reflective approach to problem solving. Pronouns thereby structured the lecture in a manner that transformed it from the transmission of content into an invitation for students to think alongside the lecturer.

The findings, although limited to a single case study design, indicated that even within the constraints of a conventional lecture format, linguistic forms function as resources for organising mathematical discourse. By differentiating between pronouns that position the lecturer or an imagined mathematician in the process of doing mathematics and those that signal mathematical content knowledge, the lecturer made reflective mathematical thinking observable while simultaneously facilitating students' access to formalised mathematical rules. These findings are consistent with Viirman (2021), who argued that lectures provide nuanced ways for modelling mathematical discourse.

Beyond the specific case examined, the study advances an analytical perspective on pronoun use by systematically linking personal pronouns to distinct discourse functions in university mathematics lecturing. In doing so, the study offers an analytical approach to pronouns and their discourse functions, providing insight into how reflective mathematical thinking is constituted and made observable in lecture-based teaching. At the same time, the single-case design necessarily constrains the scope of the findings. As the analysis is grounded in the linguistic forms of a single lecturer within a specific institutional and disciplinary context, the findings may be transferable to similar contexts.

Future research could therefore investigate how the externalization of reflective mathematical thinking might be implemented by other lecturers or adapted across different institutional contexts. In the present study, the lecturer reported that his shifts between I, singular you, and we were unconscious and surprising. This suggests that even experienced lecturers may not be fully aware of how their linguistic choices function pedagogically, despite these choices having implications for modelling mathematical thinking. In related research, Oliveira (2010) encouraged elementary teachers to reflect on their pronoun use during teaching, an approach that could be extended to university mathematics education to enhance lecturers' awareness of

their linguistic choices. Additionally, exploring students' perspectives in courses incorporating the externalization of reflective mathematical thinking would provide valuable insights into how this approach influences student learning outcomes and engagement.

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Supporting undergraduate tutors to notice students' mathematical thinking in first-year mathematics tutorials

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Keywords: noticing, mathematical thinking, undergraduate tutors, brief-but-vivid accounts, mathematics tutorials.

INTRODUCTION AND RESEARCH QUESTION

Peer-Assisted Tutoring in Mathematics is a course offered for credit to third- and fourth-year undergraduate mathematics and applied mathematics majors in University College Dublin. Colloquially referred to as "PATs" (Peer-Assisted Tutors), the students enrolled to the course support faculty in implementing active-learning tasks in first-year mathematics courses by facilitating one-to-one or small-group discussion in tutorials. The PAT model is similar to the Learning Assistant model from the U.S. (Otero et al., 2010) in which students, on a weekly basis, (1) facilitate activities in tutorials or support faculty in lectures; (2) meet with the course instructor; and (3) attend a course in STEM education. Points of difference with the PAT model are that the PAT course is offered for credit and has a specific aim of supporting PATs to notice students' mathematical thinking. Here we use the theoretical framework of Jacobs et al. (2010) which conceptualises noticing as attending to, interpreting, and deciding how to respond to, students' mathematical thinking. Although over 15 years old, the framework is one of the most cited in mathematics education and continues to underpin research in the field (Amador & Weston, 2024).

Having been offered since 2015, the *Peer-Assisted Tutoring in Mathematics* course was redesigned in 2025. The theoretical framework of noticing students' mathematical thinking (Jacobs et al., 2010) forms the spine from which the learning outcomes of the course, and hence the activities and assessments, stem from. The author, who is also the lecturer, undertook a study aimed at evaluating how the redesigned course supported the PATs in achieving the learning outcomes. Throughout the course, PATs were encouraged to reflect on mathematical interactions they had with first-year students that they considered significant in terms of the development of their noticing skills. Understanding the nature of the mathematical interactions they reflected on, and why they deemed them significant, is crucial in understanding the support they require from the course. This poster will address the following research question: *In attempting to elicit and notice students' mathematical thinking in a first-year mathematics tutorial, what type of PAT-student interactions hold significance for the PATs and why?*

One way in which the PATs captured mathematical interactions with students was by writing brief-but-vivid (BBV) accounts (Mason, 2002). Mason notes that "incidences which stay in memory are usually ones in which we have considerable emotional or intellectual commitment" (p. 39) and he advocates making a record of these so that one can reflect on them at later date or share them with others for discussion and

interpretation. He notes that for this to be effective, the incident must be recorded as impartially as possible, and to do this he proposes writing a BBV account, where one gives an "account-of" (p. 40), rather than an "account-for" (p. 40), an incident that holds significance for the writer. For Mason, reflecting on BBV accounts, and/or discussing them with colleagues, forms the basis of a professional development practice that can prepare one to be aware to act freshly in-the-moment. To address the research question, we will specifically focus on the BBVs recorded by the participants.

Eleven undergraduates who completed the course in 2025-26 agreed to participate in the study, resulting in 37 BBVs of mathematical interactions with students that they deemed significant. Content analysis is used to categorize the nature of the mathematical interactions, and the reasons for their significance. In addition, to examine the level of the PATs' skills of attending to, interpreting, and deciding to respond, each BBV is analysed using the Jacobs et al. (2010) theoretical framework.

PURPOSE OF POSTER

The purpose of the poster is to

(1) Summarise current research relating to the nature of undergraduate tutor-student interactions in active learning environments in mathematics and STEM contexts.
(2) Describe Mason's (2002) Discipline of Noticing, focusing on how writing and discussing BBV accounts can contribute to tutors' professional development.
(3) Elaborate on study participants, context, data collected, and methods of analysis.
(4) Present findings on the nature of the PAT-student mathematical interactions and the reasons they were significant to the PAT.
(5) Outline implications for practice, with a specific focus on how findings will inform the design of the PAT course.

This poster should be of interest to members of the INDRUM community who prepare undergraduate or postgraduate mathematics tutors for their roles, or to those looking for ways to better support active learning activities in their classrooms and tutorials.

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Collaborative problem solving in calculus exercise classes: Undergraduate teaching assistants' perspectives

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Keywords: calculus, case studies, collaborative inquiry, communities of practice.

The mathematics exercise classes (ECs-Übungsstunden) are interactive, practice-oriented classes that accompany mathematics lectures in mathematically intensive degree programs (Rios, 2024). In German-speaking countries, students enrolled in math EC are typically assigned a set of problems every week that they are expected to solve on their own. Then, they present their solutions on the board during class, and the instructor corrects the presented solutions as needed, or provides a sample solution. This approach can often lead to several difficulties and points of criticism: students do not know how to prepare effectively for the exercises at home, or they experience high levels of stress during the presentations (Del Chicca, 2025). Moreover, the explanation on the board may proceed too quickly for students who were unable to solve the problem independently, and the non-presenting students may revert to a passive role (Bauer, 2015), which can negatively impact their opportunities to learn.

We wanted our students to develop the necessary skills for productive mathematical inquiry (Bauer, 2015), while engaging them actively in the ECs. Informed by the notions of communities of practice and legitimate peripheral participation (Lave & Wenger, 2001), we centered collaborative peer learning while more knowledgeable others offered students regular support. In particular, we built on Mazur's concept of peer instruction (Mazur, 1997) and his 'ConceptTests,' and on Bauer's adaptation for calculus exercise classes (Bauer, 2019), to design new instructional activities for the ECs of an undergraduate analysis course in the third semester of a mathematics teacher training program. The class covered real and complex numbers, sequences, series, real functions and derivatives. We prioritized in-class groupwork and out-of-class extension of that work, instead of independent work and in-class presentations.

During the last two winter semesters, students worked in small, heterogeneous groups on 'guided' exercises, while the primary instructors (PIs; instructors with an advanced degree in mathematics) and undergraduate student assistants (USTAs) oversaw and provided help as needed. The USTAs were students who had successfully completed the class before and acted as near-peer mentors (Tenenbaum et al., 2014). Each week, USTAs attended a two-hour preparation course and solved the exercises that PIs had prepared for the students for the following week. Then, together with the PIs, USTAs discussed problems students might face and how to support them in-class and with the instructional materials. Given the importance of their role as near-peer mentors facilitating new students' move from peripheral to central participation in mathematical

work, we wanted to better understand (a) the challenges USTAs faced in their new role, (b) how well prepared they felt for it, and (c) their perspectives as experienced students who had encountered the traditional ECs format.

We have interviewed five USTAs so far, with four more being scheduled. Interviews last approximately 30 minutes, are audio recorded, and follow a semi-structured protocol covering the three topics above. Preliminary inductive thematic analysis of interview transcripts suggests that the USTAs had an overall positive view of the new format and that most of the difficulties lay in finding ways to help students without giving away the solutions to the problems. Better understanding how USTAs view and experience their role as near-peer instructors in reform efforts in undergraduate mathematics has implications for how universities prepare and support USTAs in this work, the sustainability of reform efforts in undergraduate mathematics education, and the impact USTAs can have in both peer learning and reform efforts. Our poster will include more details about the format of the ECs and USTAs' role and preparation for them, as well as examples of the 'guided' exercises PIs created. We will provide a clear and descriptive analysis of USTAs' views and experiences and look forward to discussing some of the implications of our work during the INDRUM 2026 conference.

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The Impact of Professional Development for Mathematics Professors Teaching Mathematics Courses for Future Teachers

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Teaching future teachers how to engage with mathematics from a teaching perspective is a nuanced pedagogical domain. Framed through situated cognition, this study examines how mathematics faculty describe changes in confidence to implement specific teaching strategies, as well as crossover of such strategies to other mathematics courses after participating in a professional development (PD) program. Surveys indicated that over time most faculty experienced positive shifts in confidence to implement equity-based teaching practices, engage students in Standards for Mathematical Practice, and apply specific pedagogical strategies. Findings suggest that PD for mathematics teacher educators can impact instruction across many mathematics courses, boosting confidence to engage in meaningful teaching practices.

Keywords: Instructor Professional Development, Mathematics for Teachers, Situated Cognition, Mixed Methods

Mathematics teacher educators (MTEs) consider the preparation and development of future teachers, and in context of this study, specifically primary school teachers (hereby labelled PTs or preservice teachers). In the United States, four-year institutions have predominantly prepared students for teaching and provided certification/licensure for primary school teachers through credential programs (Masingila et al., 2012). Another type of public post-secondary institution is a community college (CC), offering courses for the first two years of an undergraduate degree. With 43% of college students enrolling in CCs (Blair et al., 2018), CCs target a diverse, nontraditional group of students, offering flexible schedules, and lower costs to attend (Mali et al., 2019). In recent years, CCs have begun to offer more courses that specifically prepare students for the teaching profession, yet it is unclear how practicing mathematics faculty with no prior educational training are supported as MTEs (Tzur, 2001). Due to severe teacher shortages in the U.S., improving and expanding the pipeline of well-trained, STEM-ready PTs is of *critical* importance. This requires a greater number of well-prepared MTEs to provide adequate instructional opportunities for PTs, motivating the need to broaden pathways to primary school teacher education to those at CCs.

Teaching mathematics courses for future teachers entails a very different approach than teaching other mathematics courses. Just as teachers of mathematics require a knowledge of mathematics different than those not engaged in teaching (Ball et al., 2008), MTEs require knowledge of teaching mathematics that is developed and held in a way different than how teachers know it (Beswick & Goos, 2018). Mathematics taught in primary school is deceptively complex, especially when considered conceptually at both the level of the PT and the future primary school student.

Masingila et al. (2012) argue that "instructors teaching mathematics content courses designed for [PTs] may not be prepared to teach those courses in ways that will provide the type of mathematical support needed by [PTs]" (p. 355). Eighty percent of full-time CC mathematics faculty have master's degrees in mathematics or statistics, with 13% of faculty holding a degree in mathematics education (Blair et al., 2018). This indicates that a majority of faculty hold strong mathematical knowledge yet may not have extensive training as an MTE. Therefore, providing training and experience for mathematics faculty around important mathematics coursework for teacher preparation is paramount, especially as MTEs work to address the enduring challenges experienced in preparing mathematically strong and confident future teachers.

This paper describes a project which aimed to help address the challenges of teacher education, specifically regarding strengthening and expanding upon MTEs' capacity to develop mathematically prepared primary school teachers. We focus on the following research questions: 1) How do current practicing mathematics faculty describe changes in their confidence to implement pedagogical strategies and practices specific to PTs? and 2) In what ways does this training cross over to other university level mathematics courses? In this paper, we discuss a professional development program that aimed to help CC mathematics faculty transition to becoming MTEs, specifically around a first mathematics course of numbers and operations (N&O).

THEORETICAL FRAMEWORK

To frame the present work, we wanted to consider both the breadth of mathematical knowledge and the teaching experiences of CC mathematics faculty when thinking about their development towards becoming an MTE. We believe that practicing math instructors know "what to teach" (e.g., content knowledge; Ball et al., 2008), but that they may have varying levels of "how to teach" college level mathematics (e.g., teaching strategies). The mathematical content discussed in a N&O course is often subtle and nuanced. Therefore, professional development (PD) for MTEs needs to be context-bound and goal-oriented (Njenga, 2023).

We frame this work through the view of *situated cognition*, where "learning is a process of enculturation…in socially organised practices, through which specialised local knowledge, rituals, practices, and vocabulary are developed" (Hennessy, 1993, p. 1-2). While situated cognition has often been utilized to consider how students learn in the classroom, it provides also excellent framing for the PD of mathematics faculty training to become MTEs. In the context of situated cognition, Brown et al. (1989) describe *cognitive apprenticeship* in which learners acquire, develop, and use cognitive tools in authentic domain-specific activities—in the case of our work, this domain would be understanding how to teach others to teach mathematics. "Knowing what" (in this case, mathematics content) is very different from "knowing how" (in this case, how to train someone to teach another person to understand mathematics). This framing also makes visible the professional identity shifts required of faculty learning to become MTEs. The work involves new ways of attending to children's thinking and pedagogical purposes that emerge only through guided participation in authentic practices. Thus,

this "knowing how" is as much about becoming a different kind of practitioner as it is learning new strategies. Palincsar (1989) summarized Brown et al.'s (1989) proposed theory of learning to three approaches: 1) learners need to "think like practitioners, who act on ill-defined problems in a flexible manner", 2) situated cognition should aid learners "to form a sound epistemological framework … rather than simply knowing something, it is important that a [learner] know where to find information and how to relate it to a broader body of knowledge", and 3) "situated cognition is important because it can give [learners] a sense of agency in their learning" (p. 6).

We utilise this framework in nested ways, drawing on it as an interpretive lens to make sense of participants' engagement. Not only do we believe that the faculty participants transitioning to their new roles as MTEs need to see their learning as situated in the experience of making specific mathematics and pedagogical practices explicit for their students, they also need to be able to transfer this information to their students. We believe that meaningful learning happens both within social interactions and through development of community (Brown et al., 1989), and therefore learning in a professional learning community (PLC; Stoll et al., 2006) provides the opportunity for knowledge to move from being individually held "to being shared through engagement in socially shared activity and discourse" (Hennessey, 1993, p. 2).

METHODS

Given the complexity of the work and identified challenges, providing high-quality PD for MTEs is critical. The first two authors of this paper developed a PD program to support current CC mathematics faculty with 1) longitudinal PD on how to teach a N&O course for PTs and 2) assistance in building a network of support consisting of colleagues both within and outside of their institutions. The PD program contained three components, starting in August 2023 and ending August 2024: an opening institute (OI), year-long PLC, and a closing institute (CI). Throughout the PD, several themes were central. The first was a dual focus on the development of mathematical knowledge for teaching and the practices that support this learning. The second was a focus on equity-based practices, important for PTs who often experience lack of belonging in mathematics (Malinsky et al., 2006). Finally, the PD built on developing a sense of community, with the goal of continued collaboration amongst a typically isolated group of instructors for this type of course. Each day of the OI and CI was separated into two main modules, with each module focusing on a different content- or pedagogically- related topic. Faculty had ample opportunities to discuss and share ideas in groups around specific tasks, teaching practices, and decision-making.

A total of 17 CC faculty across four campuses in California participated in the PD. For purposes of this paper, we focus on 14 instructors who completed all PD trainings. Ten of these faculty had never taught a N&O course and four had taught the course at least once without training. To understand participants' engagement with the material, we collected five surveys (one pre-survey, three surveys throughout the year-long PD, and one a year after completing the PD) asking them to reflect on their perceptions of N&O, as well as their motivation and confidence in teaching the course.

In this paper, we report findings from three surveys collected: a Pre-Survey (Pre) prior to the OI (May 2023), Post-Survey 1 (PS1) following the OI (August 2023), and Post-Survey 3 (PS3) following the CI (August 2024). Although there were differences in the surveys, a core set of questions remained the same, including both open-ended and closed-ended responses asking about participants' overall perceptions of the PD and the N&O course. In this paper, we focus on three specific survey questions related to instructor confidence to implement or frequency of use of a specific teaching practice. Depending on the longitudinal data available for each question, we conducted either one-way repeated-measures ANOVA across the three survey time points or paired sample t-tests. These analyses assessed within-subject change over time and allowed us to examine how participants' use of specific practices shifted throughout the PD.

The first survey question asked, "To what extent do you enact the following equity-based teaching practices in your courses?" (EBTPs; provided in Table 1). In the Pre, faculty were asked to indicate which EBTPs they used. However, we realized that this question did not capture frequency or quality. We adjusted PS1 and PS3 to reflect frequency of use instead, with faculty responding to a 4-item Likert scale listing frequency of use as never (1), rarely (2), sometimes (3), and regularly (4). The second survey question (present on all surveys) asked instructors to reflect on their confidence to engage students in Standards for Mathematical Practice (SMPs) (NGA, 2010), responding to a 4-item Likert scale listed as not at all confident (1), somewhat confident (2), fairly confident (3), and extremely confident (4). Listed in Table 2, these standards emphasize mathematical habits of mind rather than content knowledge. Because the N&O course is focused on future teachers, making these practices explicit supports preservice teachers' pedagogical development. The final survey question analysed asked faculty to rate their confidence to apply or use given pedagogical strategies (hereby labelled PeS; listed in Table 3) in their instruction, specifically having students engage in such strategies, using the same 4-point Likert scale for confidence as before.

Most faculty in the PD program were not scheduled to teach the N&O course during the time of the PD series because we had intentional overrepresentation of faculty from each participating mathematics department (around 40% taught N&O at least once during the timeframe). Therefore, in PS1 and PS3, we asked participants whether they felt what they learned in the PD could be used in other courses they taught and to select one new SMP and PeS they planned to implement in the subsequent academic year. In both cases, we asked faculty to identify the mathematics course they would try it in and to describe how they plan to implement the SMP and PeS. We analysed these open-ended responses thematically to identify potential transfer to other courses.

FINDINGS

In this section, we discuss the impacts of the PD on participants' confidence and implementation of pedagogical strategies. We first discuss participants' frequency of use of EBPTs, then instructors' ratings of confidence in implementing SMPs and PeSs into their instruction. Then we discuss the ways that instructors described utilizing these strategies in the mathematics courses that they taught.

Incorporating Equitable Teaching Practices

Table 1 shows the percentage of faculty who indicated utilizing these practices in the Pre. Around two-thirds of these practices were utilized by more than 75% of the instructors. However, EBTP 9 and EBTP 11 (both related to equitable assessment practices) were reported less often. In the follow-up surveys, participants were asked to indicate the frequency of their use of the given practice in their instruction. The average value across all participants for each practice is provided in Table 1, where a value closer to 4 indicates regular use and a value closer to 1 indicates never using.

Equity-Based Teaching Practice (EBTP)	Pre %	PS 1 M (SD)	PS 3 M (SD)
1. Knowing my students' names and using them regularly in class	100%	3.93 (0.26)	4.00 (0)
2. Assigning competence to students	71%	3.57 (0.49)	3.64 (0.48)
3. Using and highlighting multiple students' problem-solving strategies	93%	3.36 (0.61)	3.57 (0.49)
4. Making time in class for students to get to know one another	93%	3.57 (0.73)	3.79 (0.41)
5. Creating the norm that wrong answers are beneficial for learning	93%	3.79 (0.41)	3.93 (0.26)
6. Promoting a growth mindset in students	93%	4.00 (0.00)	3.71 (0.59)
7. Demonstrating multiple ways to engage in a mathematical problem or task	79%	3.57 (0.49)	3.71 (0.59)
8. Drawing on students' resources and/or experiences and incorporating them into the classroom	71%	3.21 (0.56)	3.43 (0.62)
9. Utilizing formative assessment on a regular basis (e.g., daily or weekly)	50%	3.29 (0.70)	3.14 (0.64)
10. Providing opportunities for students to reflect on summative assessments	79%	3.14 (0.83)	3.00 (0.65)
11. Using non-traditional summative assessments	36%	2.57 (0.90)	3.07 (0.80)
12. Monitoring student participation and engagement to ensure there is a balance in who is "taking space" in the class	71%	3.21 (0.56)	3.36 (0.48)
13. Meaningfully listen to and engage with student thinking/contributions in class, even if what students say is not what I expect to hear and/or know is not yet mathematically correct	100%	3.86 (0.35)	3.86 (0.35)

Table 1: Participants' self-reported frequency of use of equitable teaching practices

Faculty showed small improvement in overall equitable teaching practice measures between PS1 (*M=3.48, SD=0.69)* and PS3 (*M=3.54, SD=0.62*). A paired-samples t-test did not indicate a statistically significant change, *t(13)=-1.53, p=0.149*. However, the effect size was small-to-moderate (Cohen's *d=0.41)*. Nine of the fourteen participants increased their scores, suggesting that while changes were not statistically significant in the small sample, most faculty experienced positive shifts over time.

Incorporating Standards for Mathematical Practice into Instruction

Table 2 shows that participants' confidence increased across the surveys for all SMPs except SMP 6 and SMP 8. SMP 8 was the only standard that showed an overall decrease. To examine changes in faculty's confidence in engaging students in the SMPs, we conducted a series of repeated-measures ANOVA with time as a within-subjects factor on each SMP separately. As seen in Table 2, mean scores increased over time for most SMPs. The effect of time on confidence in having students engage in the SMPs was statistically significant for SMP 1 (*F(2,26)=4.41, p=0.0225,* η^2_p*=0.25*), SMP 2 (*F(2,26)=6.55, p=0.005,* η^2_p*=0.34*), SMP 3 (*F(2,26)=6.55, p=0.005,* η^2_p*=0.34*), SMP 4 (*F(2,26)=4.11, p=0.0281,* η^2_p*=0.24*), SMP 5 (*F(2,26)=5.02, p=0.0143,* η^2_p*=0.28*), and SMP 7 (*F(2,26)=5.35, p=0.0113,* η^2_p*=0.29*). Across all SMPs, effect sizes were consistently large, indicating that large amounts of variance were attributable to time. However, time effects for SMP 6 (*F(2,22)=1.00, p=0.3840,* η^2_p*=0.08*) and SMP 8 (*F(2,26)=1.44, p=0.2542,* η^2_p*=0.10*) were not statistically significant, consistent with the visually flat means for these two practices over time.

Standards for Mathematical Practice (SMP)	**Pre M (SD)**	**PS 1 M (SD)**	**PS3 M (SD)**
1. Make sense of problems and persevere in solving them	2.93 (0.80)	3.29 (0.35)	3.43 (0.62)
2. Reason abstractly and quantitatively	2.64 (0.61)	3.00 (0.53)	3.29 (0.70)
3. Construct viable arguments and critique the reasoning of others	2.57 (0.73)	2.86 (0.74)	3.21 (0.77)
4. Model with mathematics	2.86 (0.83)	3.14 (0.74)	3.36 (0.81)
5. Use appropriate tools strategically	2.79 (0.77)	3.14 (0.74)	3.29 (0.80)
6. Attend to precision	2.92 (0.86)	3.07 (0.59)	3.07 (0.46)
7. Look for and make use of structure	2.71 (0.80)	3.07 (0.70)	3.29 (0.59)
8. Look for and express regularity in repeated reasoning	2.64 (0.89)	3.00 (0.57)	2.93 (0.80)

Table 2: Participants' self-reported confidence in having students engage with standards for mathematical practice

In addition to examining within subjects change over time in the PD program, we also analysed faculty's overall confidence in enacting SMPs and how this changed across

the three time points. On average across all SMPs, faculty's reported level of confidence increased from the Pre (M=2.75, SD=0.80) to PS 1 (M=3.07, SD=0.69), and further to PS 3 (M=3.23, SD=0.72). The effect of time was large and statistically significant, $F(2,14)=25.54$, $p<.001$, $\eta^2_p=0.88$, suggesting that the PD was associated with substantial overall gains in faculty's confidence to engage students in the SMPs.

Incorporating Specific Pedagogical Strategies into Instruction

Given that they are teaching PTs, we asked the faculty to rate their confidence in applying pedagogical strategies in the classroom in PS1 and PS3. The results from each survey are provided in Table 3. We notice that participants reported higher confidence in applying all PeSs from multiple time points, with reported scores increasing from the Pre to PS1, and from PS1 to PS3, for all PeSs.

Pedagogical Strategy (PeS)	Pre M (SD)	PS 1 M (SD)	PS3 M (SD)
1. Use more than one way to explain a topic	3.07 (0.80)	3.29 (0.59)	3.43 (0.62)
2. Present their mathematical ideas orally	3.21 (0.67)	3.36 (0.72)	3.64 (0.61)
3. Present their mathematical ideas in writing	3.07 (0.80)	3.29 (0.80)	3.64 (0.48)
4. Present their mathematical ideas through models, drawings, or pictures	3.00 (0.76)	3.29 (0.70)	3.43 (0.73)
5. Explain a concept to a peer	3.29 (0.59)	3.50 (0.73)	3.79 (0.56)
6. Work in small groups	3.36 (0.61)	3.71 (0.45)	3.86 (0.35)
7. Examine children's work and mathematical thinking	2.57 (0.62)	3.07 (0.59)	3.36 (0.72)
8. Listen meaningfully to other students' contributions, even when they may not understand their thinking.	3.36 (0.61)	3.64 (0.48)	3.79 (0.41)

Table 3: Participants' self-reported confidence in using specific pedagogical practices

A repeated-measures ANOVA using participants as the unit of analysis and reported confidence in applying PeSs as outcome revealed a statistically significant effect of time, $F(2,26)=10.13$, $p<0.001$, $\eta^2_p=0.44$. Mean ratings over all PeSs increased from Pre (M=3.12, SD=0.73) to PS1 (M=3.39, SD=0.68), and further to PS3 (M=3.62, SD=0.60), indicating participants felt more confident in applying the strategies in their classrooms as time passed. An additional repeated-measures ANOVA treating *pedagogical strategies* as the unit of analysis again found a strong effect of time, $F(2,14)=79.84$, $p<0.001$, $\eta^2_p=0.92$. This indicated that all PeSs showed strong and consistent growth from Pre, to PS1, to PS3.

Crossover into Other College Courses and Implementing SMPs and PeSs

All faculty indicated in both PS1 and PS3 that the material they learned and experiences that they had in the PD was beneficial to their instruction in other mathematics courses. One faculty indicated that they felt what they learned in the PD could be implemented into any class, stating, "We learned how to use manipulatives, to be aware of limited ways of thinking (whether it be procedural, wording of questions we ask, etc.), how to guide students' thinking without correcting or criticizing, and more." Faculty indicated that they felt that what they learned in the PD to become an MTE had value to incorporate into other classes that they teach, listing courses such as trigonometry, college algebra, precalculus, business calculus, survey of math (terminal math course), and statistics. Another participant indicated that they were already utilizing what they learned in their college algebra classroom sharing,

> "Just today…I had a student ask why we 'don't do anything' with the denominators when adding fractions (with like denominators). I was able to talk about what it means to be a fraction, draw an example on the board, and also a counterexample with different denominators to show the "why" we need same denominator (sized pieces)."

Here, the instructor was able to connect to conceptual explanations of fraction addition, something that many noted was a challenge even for students in their calculus courses. The instructor felt more capable to explain operations on fractions beyond simply pointing to the algorithm.

Faculty were asked to share one SMP they planned to implement in their teaching of any mathematics course upon completion of the OI and the CI. In PS1, the two most selected SMPs were SMP 1 (28%) and SMP 4 (28%). Faculty indicated that they had learned more about the anxiety that most preservice primary school teachers experience with mathematics (Malinsky et al., 2006), recognizing that through the various activities and non-traditional problems they engaged with in the PD, they understood the benefit of building perseverance with problem-solving and using manipulatives to model mathematical meaning for *all* students. In PS3, more faculty shifted to selecting the practice of constructing viable arguments and critiquing the reasoning of others (SMP 3, 28%), while also looking for and making use of structure (SMP 7, ~22%). Faculty reasoned that it takes time to help students learn how to attentively hear other students when they explain their thinking and for students to learn to critique others' work, thus allowing student voice to be more prominent. Faculty believed that an important skill for students was to see the structure within mathematical concepts.

When asked what PeS faculty planned to implement in PS1, faculty highlighted strategies related to students' communication of mathematics. Approximately half of the faculty mentioned asking students to explain ideas in more than one way, with other faculty focusing also on asking students to explain ideas orally or in written form. While faculty acknowledged the challenges inherent in these practices (such as requiring additional class time), there was an overall willingness and value placed on communication-based strategies where students not only make their own reasoning

more visible but also support a classroom culture of attending to more than one way of understanding and communicating mathematical ideas. In PS3, six faculty (43%) indicated that they already implement all these strategies in their teaching, whereas only 1 had indicated this in PS1. Most faculty wanted to have students examine mathematical work and mathematical thinking (36%), with a focus of having their students engage with "real experiences" of what students do and how they think.

DISCUSSION

Throughout the PD, faculty engaged in rich and meaningful tasks with the facilitators and with each other, using manipulatives, analysing children's thinking, and working through non-traditional problems. Our findings emphasise the powerful learning that emerged through participation in this authentic and domain-specific setting of learning to teach teachers, rather than only the transmission of discrete content or practices (Brown et al., 1989). The significant growth in confidence using both the SMPs and PeSs may be interpreted as fundamental shifts in faculty's participation in these disciplinary routines, and the evidence of transfer of these ideas to other mathematics courses indicate that participants had not only learned new content but had taken them up as part of a new shared professional identity. This is supported by a situated cognition view of learning as enculturation (Hennessy, 1993), where we saw participants take up new vocabulary, new practices, and new ways of being teachers of mathematics not only in N&O, but in other spaces as well. These patterns also suggest that the PD functioned not only as skill development, but as an evolving community in which new norms and ways of being as a mathematics teacher educator flourished.

It is important to note that there was not as discernible a shift in the use of EBTPs during the PD. This may be for a few reasons. Equitable teaching often requires highly relational and context-specific work in the classroom, settings that are less readily approximated in PD settings. While tasks that focused on SMPs or PeSs, such as those requiring multiple explanations, could be meaningfully targeted during PD sessions, it is much more difficult to provide learners with the socially embedded context of engaging diverse learners. It is also possible that the survey questions did not offer sufficient insight into not only whether participants leveraged EBTPs, but perhaps *how* their usage may have changed over time. Future research could examine more situated or classroom-embedded ways of exploring equitable teaching practices to understand how faculty take up these complex practices in their own instructional contexts.

Our work is limited by the small sample size and inherent challenges in interpreting Likert style items meaningfully, and statistical results should therefore be interpreted cautiously. Even so, our findings suggest that situating community college faculty in the authentic practices of mathematics teacher education and supporting them in the "knowing how" of teaching teachers supports meaningful development of broad knowledge that extends past the focal N&O course. Faculty not only learned new isolated pedagogical strategies, but applied them to diverse instructional contexts, highlighting the generative potential of situated learning within an MTE community.

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Aspects of Approximations of Practice that Support Calculus Instructors' Enactment of Equitable Active Learning Teaching Practices

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We led a professional development workshop series, training Calculus I and II instructors to implement active learning and equitable teaching practices. The workshops actively engaged instructors in approximations of practice (AoPs; Grossman et al., 2009) in which they learned teaching practices like designing tasks; eliciting, noticing, and responding to evidence of student thinking; orchestrating class discussions; fostering inclusive social norms; and equitably managing student participation. After the workshops, we interviewed instructors about their decisions to implement the teaching practices from the AoPs in their teaching. We identified the AoP features that they viewed as supporting their implementation of those practices.

Keywords: Approximations of Practice, Professional Development, Active Learning, Calculus, Qualitative Methods.

Despite the benefits of active learning and equitable teaching practices (Bartell et al., 2017; Freeman et al., 2014; Theobald et al., 2020), university faculty have been slow to adopt them (Rasmussen et al., 2019; Stains et al., 2018), often due to a lack of professional development (PD; Michael, 2007). There is a need for more PD on these practices. School teacher preparation programs and PDs often use Approximations of Practice (AoPs; Grossman et al., 2009) to give teachers opportunities to develop and enact specific teaching practices. AoPs are activities with hypothetical scenarios in which instructors can simulate a teaching practice without the complexity of being in a real-time classroom. AoPs have been effective in helping mathematics teachers develop active learning practices (e.g., Amador et al., 2021). Their use in PD programs for university faculty has not been commonly documented, however. We investigate how engaging in AoPs can support university instructors in implementing active and equitable teaching practices. We report on a study where Calculus I and II instructors participated in a PD program focused on these teaching practices through the use of AoPs. We address the research question: *What aspects of AoPs do mathematics instructors view as supporting their implementation of active learning and equitable teaching practices?*

LITERATURE REVIEW

Instructors can use many teaching practices to equitably implement active learning (e.g., Bartell et al., 2017; European University Association, 2019; Freeman et al., 2014; Gutiérrez, 2010; Mathematical Association of America, MAA, 2018; Reinholz, 2023). These align with two characteristics of successful calculus programs: the use of "student-

centered pedagogies and active-learning strategies" (Bressoud & Rasmussen, 2015, p. 145) and diversity, equity, and inclusion practices (Hagman, 2021). While some mathematics departments across the US have incorporated more active learning in their calculus sequences (Williams et al., 2022), many college calculus courses are still taught primarily through lecture (Stains et al., 2018). Several factors influence instructors' choices to enact active learning, including time constraints, content coverage expectations, class size, classroom setup, concerns about teaching evaluations, and accessibility of PD (e.g., Johnson et al., 2019). Instructors' participation in PD can support them in implementing active learning in their classes (Conference Board of the Mathematical Sciences, CBMS, 2016). In the US, PD designers have created programs, such as MAA's Project NExT, to support instructors in using active learning. Such PD has helped active learning gain momentum (CBMS, 2016), but there is a need for more research on PD design for tertiary instructors.

PD programs can be implemented in various formats and informed by various design principles. Soto and Marzocchi (2021) suggested PD should emphasize community among participants, create opportunities for collaboration, encourage faculty reflection and persistence in improvement, and build faculty's confidence and motivation. Researchers have emphasized the importance of actively engaging participants in PD to model active learning approaches (e.g., Akin et al., 2023). One way to do this is to provide opportunities to participate in AoPs. While AoPs have been used extensively in schoolteacher PD, the use of AoPs at the tertiary level has yet to be carefully examined.

THEORETICAL FRAMEWORK

AoPs are "opportunities to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p. 2056). These activities simulate a teaching scenario or practice, which provides an opportunity for teachers to rehearse and develop teaching practices without the complexity of a real-time class setting or risk of affecting students (Howell & Mikeska, 2021). AoPs can simulate various practices, such as noticing students' thinking (Jacobs et al., 2010) or responding to students in discussions (Jackson et al., 2024; Kaka et al., 2021). The design of AoPs can vary by the type of teaching practice approximated, the medium of the scenario (e.g., classroom videos or written student work), the degree of authenticity of the approximated teaching practice, and the scenario content (Sapkota & Max, 2024). AoPs informed the design of our PD.

METHODS

The first author led a PD workshop series on using active learning while attending to equity and inclusion. The program consisted of two full-day sessions before the semester, with six two-hour sessions during the semester. The PDs focused on: fostering inclusive class environments; designing group-worthy, cognitively demanding mathematical tasks (Stein & Smith, 1998); eliciting, noticing, and responding to evidence of student thinking

(Jacobs et al., 2010); orchestrating class discussions by anticipating, monitoring, selecting, sequencing, and connecting (Smith & Stein, 2018); and fostering equitable participation (MAA, 2018). Participants practiced these through 15 AoPs. Instructors were given a hypothetical scenario involving Calculus I or II students and responded to prompts that varied by teaching practice. Scenarios involved varied media like videos of students creating graphs with specific limits and discontinuities; scripts of class discussions about differentiation strategies; student quotes about their calculus courses; and hypothetical student work on optimization, continuity, and series convergence. The participants reflected individually and shared ideas in small and whole-group discussions.

The 11 participants included part-time and full-time lecturers and tenured professors who taught Calculus I or II, eight from a large public university in the southern US and three from a community college. Their experience ranged from 1 to 35 years. In the first PD session, participants shared their prior experience with using active learning. Most had primarily lectured, and some had used group work and think-pair-share activities. We conducted hour-long semi-structured interviews (Drever, 1995) with each instructor after the PD. Participants were asked to discuss their background and what motivated them to participate in the workshop. They were provided a list of teaching practices from the AoPs (see above) and asked to discuss which ones they had enacted. Additional questions focused on what participants remembered about the AoPs and what they found helpful. Interviews were conducted and transcribed over Zoom by the second author and another research team member. We chose 3 participants to examine for our case study (Yin, 2009) who gave detailed examples of how they engaged in a certain AoP and enacted the teaching practices in class. We used the teaching practices from the AoPs as a priori codes to deductively code the data (Miles et al., 2013). We also inductively coded (Miles et al., 2013) transcripts to identify aspects of the AoPs that they claimed supported their enactment of the approximated practice. Authors coded independently and met regularly to discuss and resolve discrepancies. Lastly, we observed each participant teach towards the end of the semester, taking field notes and collecting video recordings, as a form of data triangulation to verify that the practices instructors claimed to use were enacted.

RESULTS

Participants identified several aspects of the AoPs that supported their implementation of active and equitable teaching practices. Instructors claimed that the authenticity of the AoPs (i.e., representing realistic class scenarios and being calculus content-specific) was helpful. The research-based frameworks from the PD gave them specific terminology to discuss their teaching. AoPs provided them with opportunities to actively engage in those practices and helped instructors become more mindful and reflective. The equity-focused AoPs brought awareness of participatory inequities by race and gender, which helped instructors attend to equity during class. Lastly, observing the PD leader's facilitation of the AoPs, along with having the opportunity to work with and receive advice from other

faculty, was helpful. Below, we use three cases to provide specific examples of some of these aspects and how they supported instructors' implementation of active and equitable teaching practices. We provide context on the participants' teaching experiences, describe what practices they enacted from AoPs, and discuss their experience with specific AoPs.

Case 1: Sumaya

Sumaya was a Senior Lecturer who taught university mathematics for 17 years. While participating in the PD, she taught Calculus II in a class of approximately 60 students. Before participating in the PD, Sumaya claimed to implement some active learning activities, like facilitating group work and calling on students to answer her questions in interactive lectures. After the PD, Sumaya claimed she implemented these teaching practices from the AoPs: fostering inclusive social norms, designing group-worthy mathematical tasks, eliciting evidence of student thinking, orchestrating class discussions, and fostering equitable student participation. When asked which AoP impacted her teaching the most, Sumaya brought up the AoP on using the five practices for orchestrating mathematics discussions. Sumaya's group was given five samples of hypothetical students' work on the task, "Determine if this series converges or diverges $\sum_{n=1}^{\infty} 1/2^n$." They were given these directions: (1) Monitor and Select: Look through the student work samples and decide which ideas you would like to discuss as a class. (2) Sequence and Connect: Decide how to sequence the students' ideas and how to connect them as a teacher. (3) Perform: Act out the class discussion with one group member as the instructor and the rest as the students. Sumaya reflected on her experience, "I wasn't aware that some of the practices I had would fall into this definition of anticipating, monitoring, selecting, sequencing, and connecting." She went on, "After the workshop … I learned these are actually a sequence of different things we can do for the class discussions, and so I became more mindful of doing them." Sumaya claimed that, before the workshop, she would anticipate student responses and monitor students during class, but because of "time constraints," she would often just take "the first answer I would get. I would just say either yes or no and continue with that." After the workshop, she would "wait for more responses and then build upon those responses." Sumaya discussed how her participation in AoPs helped her reflect on her teaching and find ways to improve:

> Some things, for example, anticipation, selecting, and sequencing, when I heard it, my first reaction was, "Oh, yeah, I'm doing this." But then when … we talked about it, and then when I went to the class, and I reflected on it, I realized there is a lot of improvement I can make.

Participating in the AoP gave Sumaya the language to name the teaching practices she implemented and helped her become "more mindful" on how she could improve her practice of orchestrating class discussions, which led her to be more intentional about selecting, sequencing, and connecting student responses.

Besides helping Sumaya become more reflective about her teaching, there were other aspects of AoPs that she found helpful, like the facilitator "modeling, walking around with iPad and taking pictures of different students' work and then projecting it, so as a whole workshop, we could discuss it." Seeing these teaching practices modeled gave Sumaya a point of reference she could refer to as she tried them in her own classroom. Sumaya described how the facilitator intentionally included all participants' responses during the discussion of the AoPs. Sumaya said the facilitator "did elicit answers from everybody … she had different posters with different scenarios ... and each group had time to discuss." Sumaya connected this to her own teaching by saying, "That's something I would like to do in my class, not only ask for the answers from the students but maybe present a solution and ask them to share what they think." Observing the PD facilitator's teaching practices during the AoP implementation helped Sumaya implement those practices in her teaching.

Case 2: Richard

Richard was a tenured Professor who taught university mathematics courses for 35 years, primarily Calculus II, which he taught during his participation in the PD in a class of approximately 40 students. Before the PD, Richard claimed to implement interactive lectures in which he asked the class questions as he lectured. After the PD, Richard claimed he implemented the practices of eliciting evidence of student thinking during class, noticing and responding to evidence of student thinking, fostering inclusive social norms, and some of the five practices for orchestrating class discussions. One AoP that stood out to Richard focused on fostering inclusion by mitigating marginalizing experiences for students. Participants read excerpts from Battey et al.'s (2022) study on students' reactions to marginalizing experiences in Calculus courses and were given these directions: Read and discuss the students' reactions to the given real scenarios that happened in college calculus courses in the US. How do the marginalizing experiences seem to affect the students? What should the instructor have done differently in the scenario? If you were the instructor, how could you respond more productively to the students? The instructors approximated the inclusive teaching practice of interacting with and responding to students in non-marginalizing ways. Richard discussed his experience:

> We were given some scripts of interactions between an instructor and students [with] some negative connotations to it. The activity was to reflect and give feedback based on how we were interpreting this interaction, and what maybe could have been done differently from the instructor's point of view. That's an activity that stuck with me. There were some discussions there that really made me more aware of how I need to be interacting with the students.

Richard recalled a scenario where a student asked about a computation:

> It was trivial, but the student was not understanding … the instructor simply replied, 'You're supposed to already know that.' Of course, that puts the students down ... I think I'm pretty good about not responding to students that way. That scenario reinforced how,

> when a student asks a question, it's important to take it seriously … Not get frustrated that they didn't understand something trivial … it just kind of reinforces that. That's one aspect that's important in interacting with students … to answer them as if they're serious mathematical questions and dig in to see where their misunderstanding was coming from.

The AoP helped Richard become more mindful and reflective on his practices. He described the discussions about the scenario in the AoP made him "more aware of how I need to be interacting with the students." The AoP seemed helpful for Richard by validating his prior teaching-related orientations, as evident in his description of how the AoP reinforced his view of the importance of taking students' questions seriously. These AoP aspects seemed to support Richard's implementation of that teaching practice in his classroom. Richard exemplified how he implemented this practice while simplifying the expression x^{n+1}/x^n: "The first time an expression comes up, I usually do it in slow motion and say, from now on, we'll just go to this." When a student asked about x^{n+1}/x^n,

> Instead of saying, 'we talked about this earlier' … I tried to explain it in a different way ... That connection with the scenario in the PD came in … Be patient, treat the question as a serious question, and let's see if we can figure out where the misunderstanding is.

The AoP helped Richard reflect on how he interacts with students, intentionally respond to students in non-dismissive ways, and elicit further evidence about their thinking. Richard appreciated how the AoP was representative of realistic scenarios. This helped him connect scenarios that "stuck with" him to his experiences interacting with students.

Case 3: Lucia

Lucia taught at the university level for 13 years. During the PD, she taught Calculus I in a class of approximately 40 students. Before the PD, Lucia claimed to use "group problem solving, class discussions, and think-pair-share, but not in a systematic way." Lucia claimed she implemented these practices that she learned from the AoPs: orchestrating class discussions, fostering inclusive classroom social norms, and designing group-worthy mathematical tasks. Lucia gave a specific example of how she used the five practices for orchestrating class discussions in her Calculus I course after engaging in the AoP:

> I monitored them when they were doing their exercises … I had a few students who insisted to go to the board to show their work, but most of the work I wanted to be presented, I selected, or I put them in a certain sequence ... I selected first a student who had some errors, but had some good parts … and then we can discuss and take as a learning opportunity by going over the errors. And then we got to the perfect one, and then made the connection between the first and the last, and using any mistakes made in between … as a learning opportunity.

Here, Lucia exemplified the specific steps of orchestrating class discussions, which she learned from the AoP. She claimed that this AoP impacted her teaching the most. During this AoP, Lucia's group was given six samples of hypothetical students' work on an

optimization problem. They were given the same directions as the AoP in Sumaya's case. Lucia reflected on her experience with this AoP and remarked, "The problems [the facilitator] chose were really fun." She went on to describe how her group "tried to use every single" student work sample to select and sequence "in a certain way." She concluded by saying, "The engagement [in the AoP] was very nice. We each had a different role, so I thought if it was engaging and fun for me, it would be for students, too." The AoP's requirement for the participants to actively engage in practicing the teaching practice seemed to support Lucia's implementation of the practice in her class.

Another aspect of AoPs Lucia found helpful was her experience observing the facilitation of the workshops. She described how the facilitator was able to monitor the participants while they were working and used a whiteboard app to display different people's work. Lucia connected this to her own class by saying, "I like how she [the facilitator] monitors everybody ... which I do in class, and I assign group work, but sometimes you have a lot of students, and you cannot really go through all [their work]." She discussed how she had learned how to use an iPad to take pictures of student work and display them. She stated, "I like to do that in class, because then students see also that someone else can have a mistake or they can use their critical thinking to add something to a certain problem." Overall, Lucia felt like she had "learned a lot" from the workshops and was able to "apply at least a few of the things that we learned" in her classes. She appreciated how engaging the workshops were and claimed she "had learned some new techniques," like how to select and sequence student work and share this work using technology.

DISCUSSION AND CONCLUSION

All three instructors described how different aspects of the AoPs supported their implementation of the approximated teaching practices in their own classrooms. Some aspects of the AoPs that participants found helpful were in reference to specific activities that they had engaged in during a particular AoP (e.g., how Richard found the scenario from the marginalizing experiences AoP to be realistic). Other aspects that participants discussed as helpful were broader and experienced across multiple AoPs (e.g., the opportunity to actively engage in the teaching practice and observe the facilitator model the teaching practice). The AoPs helped instructors become more mindful and reflective on their practices and often validated instructors' prior teaching-related orientations, which is a similar finding in the K-12 literature (Sapkota & Max, 2024). AoPs provided instructors with research-based terms to describe their teaching practices, allowing them to explicitly name what they were doing (or could improve on). The AoPs were also representative of realistic Calculus class scenarios, which helped them connect the approximated practices to their teaching. The AoPs required participants to collaborate and actively engage in practicing the teaching practice. This experience supported instructors' enactment of the practice in their classrooms. Lastly, the AoPs were guided by a facilitator who modeled active learning teaching practices. Having this role model to

follow supported the instructors in implementing the teaching practices from the AoPs.

Although participants were able to identify aspects of the AoPs that were helpful, there are a few limitations to this study. This work was conducted at two institutions with a small number of participants. The findings in this paper may reflect the unique context and culture of those specific institutions. Another limitation is that most of the instructors who participated in the PD were interested in implementing active learning and equitable teaching practices. Thus, their perspective on what they found helpful from the AoPs may differ from faculty who might be more reluctant to implement these teaching practices. More research is needed to better understand how AoPs can support the implementation of teaching practices and what aspects of AoPs may or may not be helpful for faculty.

Our findings contribute to the literature on PD for university mathematics faculty and provide implications for PD designers. This study helps fill the gap in research on PD for instructors at the tertiary level and shows how AoPs, which have been used in school mathematics contexts, can be used to train tertiary instructors. AoPs in our PD supported instructors in implementing active learning and equitable teaching practices. AoPs have great potential to train university instructors to implement active learning. They provide a low-stakes environment for instructors to develop new teaching practices (Howell & Mikeska, 2021). Our case studies showed how engaging in AoPs allowed instructors to reflect on their teaching practices, be more mindful of their interactions with students, and practice several new teaching practices. Thus, we encourage designers of PD programs for university faculty to use AoPs to actively engage their participants in teaching simulations to increase instructors' use of active learning and equitable teaching practices.

ACKNOWLEDGEMENTS

This work is supported by the National Science Foundation (DUE-2421228).

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Principles for effective collaboration between undergraduate mathematics education researchers and mathematicians

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Keywords: Active learning; Change theory; Design research; Linear algebra.

POSTER SUMMARY

In the Fall of 2023, our research team formed a collaboration with four mathematicians who were teaching a proof-based linear algebra course for third- and fourth-year students. The goal of this collaboration was for mathematicians to implement effective active-learning strategies that both aligned with best practices in mathematics education and met the needs and values of the mathematicians. Elsewhere, we presented evidence that our collaboration was effective (Johnson et al., 2025). Our research question is what design principles allowed our collaboration to be successful. Our hope is that these principles can be useful to other RUME scholars interested in forming collaborations with mathematicians to change their teaching. The content of our poster will:

(i) present the motivation and needs for building these collaborations to change mathematical instruction at scale. We will document how although active learning leads to better learning outcomes in university classrooms, most university mathematics are taught by lecture (Melhuish et al., 2022). Artigue (2016) called for collaborations where mathematicians and mathematics educators worked together to find shared solutions that would meet the needs of both groups.

(ii) delineate Henderson et al.'s (2011) theory of instructional change at the university level. Henderson et al. argued for (i) semester-long interventions that (ii) simultaneously target instructors' beliefs and pedagogical practices that (iii) proceed in an incremental fashion and (iv) are compatible with the affordances and constraints of the complex departmental and university system in which the professors work.

(iii) describe our methodology: we audio recorded weekly meetings with the mathematicians, where we offered active learning pedagogical suggestions. They could reject any suggestion if they provided a reason and we would negotiate other forms of pedagogy that might meet their needs.

We generated these principle by identifying instances where mathematicians rejected a suggestion, and using an open coding scheme to identify the reasons they provided (see Johnson et al, 2025, for details) and why they chose to identify alternative pedagogical methods. The principles were:

(1) *Time is of the essence*: Time is one of the ultimate bottlenecks in developing a productive collaboration in three respects. First, mathematicians are unwilling to

allocate to much class time for active learning activities because this will limit their ability to cover content. Second, generating specific active learning activities requires time outside of class that mathematicians sometimes cannot allocate. Third, mathematicians need activities they can use in a *timely* fashion, several days in advance, so they can incorporate them into their lesson plan.

(2) *Understand mathematicians' obligations*: Mathematicians sometimes have obligations that mathematics educators may not share. For instance, our mathematicians felt a proper justification must be given to each question posed in class, which meant that even quick clicker questions can be time consuming, since the mathematicians felt the onus to prove the answer to the clicker question is correct.

(3) *Be flexible*. It is important to set goals that are feasible and desired by both mathematics educators and mathematicians. Our research team had an *a priori* sequence of tasks, many involving student-to-student interactions. Our mathematicians were against having students talk to each other for a variety of reasons. While the mathematics educators did not always agree with these reasons, they found other active learning tasks that all parties could agree to.

(4) *Go with the flow*. It is important that the activities that the mathematics educators propose be integrated within the lecture, both in terms of content and in terms of the lecturer's style.

On the poster, we will present qualitative data from transcripts from our meetings illustrating the need to follow these principles, and how they were followed to produce productive outcomes of mathematicians using active learnings strategies in their classrooms.

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How proofs are organized: a Theme analysis

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Systemic functional linguistics (SFL) posits that there are three metafunctions of language. One of these is the textual metafunction, which organizes text and which can be analyzed with a Theme analysis. In this paper, we use an SFL Theme analysis to study the textual organization of 20 mathematical proofs from popular undergraduate textbooks. The analysis pays attention to Theme status (to see which (a)typical grammatical structures proof Themes follow) and Theme composition (to learn about the constituents of proof Themes). The analysis highlights the dataset's proof authors' practice of using less common grammatical constructions to further their arguments—particularly by leading with dependent clauses and Adjuncts—as well as how they used textual and interpersonal resources to guide readers through their proofs.

Keywords: proof, systemic functional linguistics, textual metafunction, qualitative, Theme analysis.

INTRODUCTION

Reading (and comprehending) proofs is challenging (Hodds et al., 2014). Mathematical texts are dense, and a reader will typically have to jump between lines to follow the logical progression of the argument. To facilitate the reading of proofs, the organization of proof texts is critical. In this paper, by drawing on systemic functional linguistics (SFL), we seek to study how proof authors linguistically organize their proof texts.

A key idea from SFL is that what we place at the start of a sentence matters as it "serves as the point of departure" (Halliday & Matthiessen, 2014, p. 89). SFL calls this starting point the *Theme*, and it is one of the key ways in which authors can organize their message—ideally to help the reader. (In line with SFL tradition, we capitalize functional terms like Theme, Adjunct, Subject, and Complement.)

For example, the following two logically equivalent sentences read rather differently:

- If $a = b$, then for every real number $\varepsilon > 0$ it follows that $|a - b| < \varepsilon$.
- For every real number $\varepsilon > 0$ it follows that $|a - b| < \varepsilon$ if $a = b$.

Whereas in the first sentence the condition $a = b$ is immediately provided, in the second sentence the reader must wait for it until the end. One may think that the second is merely a pathological example, but it in fact conforms to the typical sentence structure of English: dependent clauses tend to come last (Thompson, 2014).

Our goal is to study the Themes of proof texts to learn what Thematic choices mathematicians make, and thus our research question is: How are Themes realized in proofs from undergraduate mathematics textbooks? As part of answering this question, we pay attention both to Theme status (roughly, how (un)usual Thematic choices are)

and Theme composition (roughly, how complex Themes are). We hope to thereby make explicit some common grammatical patterns in mathematical proofs, patterns which may seem unusual to newcomers to the field.

LITERATURE REVIEW

Systemic functional linguistics (SFL) originates with linguist Michael Halliday. It studies the grammatical choices humans make within linguistic systems and the functionality of these choices (Schleppegrell, 2012), and for over 50 years, SFL has crossed paths with mathematics education. In 1974, Halliday attended the "Symposium on Interactions Between Linguistics and Mathematical Education," after which he published a chapter about the mathematics register (Herbel-Eisenmann et al., 2017)—a register being "the constellation of lexical and grammatical features that realizes a particular situational context" (Schleppegrell, 2004, p. 18). Key to understanding registers and SFL are the three metafunctions—ideational, interpersonal, and textual—which are three types of meanings realized in every clause "to talk about the world [...] to interact with other people [...] [and to] organiz[e] language to fit in its context" (Thompson, 2014, p. 30). Below, we describe the textual metafunction, which is the metafunction most relevant to our work.

The Textual Metafunction (in Mathematics)

Two types of features realize the textual metafunction, that is, how messages are organized: structural features (i.e., thematic structure, information structure and focus) and cohesive features (i.e., reference, ellipsis and substitution, conjunction, lexical cohesion) (Halliday, 1994). Whereas the structural features are internal to a clause (or clause complex), the cohesive features focus on how clauses (or clause complexes) relate to one another (Halliday, 1994). For our purposes, a clause is "(potentially) any stretch of language centred around a verbal group" (Thompson, 2014, p. 17) whereas a clause complex is "a combination of two or more clauses into a larger unit" (Thompson, 2014, p. 186).

In this paper, we focus on textual organization via thematic structure, in particular, the concept of Theme. Theme, which we discuss in greater depth in our Conceptual Framework section, refers to "the element that serves as the point of departure of the message" (Halliday & Matthiessen, 2014, p. 89). An author's choice of Theme affects how the communication is taken up and is thus a key tool for maximizing the effectiveness of a message for its purpose and context (Eggins, 2004). Our choice to focus on thematic structure is not a rejection of other structural and cohesive features; it is merely our starting point to the study of cohesion.

In mathematics education, several scholars have studied aspects of the textual metafunction. For example, O'Halloran (2005) made several observations about mathematics texts with respect to Theme: (a) marked (i.e., nontypical, not the default) Theme constructions are used to emphasize important content (e.g., "From Equation 3", "When the derivative is small"); (b) dependent clauses as marked Themes are common (e.g., "If $x = 4$"), and (c) conjunctions and conjunctive Adjuncts as textual

elements of (multiple) Themes are common (e.g., if, then, therefore)—Adjuncts are clause constituents that provide information about the circumstances of a situation (e.g., when, where, how, why) (Thompson, 2014). Further, Morgan (1998) postulated that in deductive texts, it is likely that textual elements (e.g., hence, therefore) and reason-providing Adjuncts (e.g., "By Theorem 1") appear in the Theme. She also suspected that unlike scientific explanations, mathematical deductive texts may favor leading with the cause and then providing the result. On the other end of the textual metafunction spectrum, González and Herbst (2013) and González (2015) have focused on cohesive features by studying conjunctions, and conjunctions and lexical cohesion, respectively.

Last, we wish to acknowledge that there are also studies in mathematics education that focus on the other two metafunctions or combinations of the three metafunctions. For instance, there is work focused on the ideational metafunction (e.g., DeJarnette & González, 2016; Herbel-Eisenmann & Otten, 2011), the interpersonal metafunction (e.g., DeJarnette, 2018; Herbel-Eisenmann & Wagner, 2010; Mesa & Chang, 2010; Zolkower & Shreyar, 2007), both of these (e.g., Burton & Morgan, 2000), or all three metafunctions (e.g., Herbel-Eisenmann, 2007; Morgan, 1998; O'Halloran, 2005). Finally, not all SFL-related work foregrounds the metafunctions. For example, there is also work that draws on SFL's writing on genre (e.g., Marks & Mousley, 1990; Solomon & O'Neill, 1998).

Some Limitations of SFL

SFL often pays greater attention to "organizational and linguistic properties of genres than of the contexts where they carry out their social processes" (Hyon, 2018, p. 21). Also, since SFL is a theory of grammar focused mostly on clauses and their constituents, discourse analysts have to do work to connect the clause-level tools to text-level observations. That said, SFL offers some tools that extend beyond the clause (e.g., thematic progression analysis) as we note in our discussion.

Other Approaches to Studying Proof

Proof has been a focus of mathematics education research for decades and has been studied using a variety of approaches (e.g., cognitive, linguistic). We briefly highlight a few recent findings of different grain size about the language of proof. At the level of words and clauses, it has been shown that imperatives (Tanswell & Inglis, 2024) and masculine forms of communication (Melhuish et al., 2025) are prevalent in proof texts. Zooming out, Dawkins and Weber (2017) formulated a framework about norms of mathematical proof-writing and the mathematical values they uphold, and Lew and Mejía-Ramos (2019) identified linguistic conventions of proof. Recently, Weber et al. (2023) applied Grice's maxims to articulate several proof norms.

CONCEPTUAL FRAMEWORK

This report focuses on how the textual metafunction is realized in proofs, and SFL offers multiple analytic tools for analyzing the textual metafunction (e.g., Theme

analysis, conjunction analysis). The guiding concept for this paper is the concept of *Theme*: "the element that serves as the point of departure of the message; it is that which locates and orients the clause within its context" (Halliday & Matthiessen, 2014, p. 89). In English, placing something at the start signals that it is the Theme. Some other languages (e.g., Japanese) explicitly mark the Theme with words (Eggins, 2004). The Theme typically contains given (i.e., already present) information, and the remainder of the clause, called the *Rheme*, typically contains the unfamiliar or new information (Eggins, 2004; Fries, 1994).

For each of the three basic grammatical moods in English—which SFL terms *declarative* (e.g., "Melinda runs a marathon."), *imperative* ("Melinda, run a marathon!"), and *interrogative* ("Is Melinda running a marathon?")—SFL proposes that there are *unmarked* (i.e., usual) and *marked* (i.e., less usual) Themes. For example, it is natural to start a declarative clause with a Subject (e.g., "The weather is gorgeous."). But it is less common to start a declarative clause with a *Complement*—a "nominal or adjectival group that refers to the same entity as the Subject, or describes the Subject" (Thompson, 2014, p. 20)—(e.g., "Gorgeous the weather is."). In short, for declarative clauses, a Subject as Theme is considered an unmarked Theme, whereas other thematic choices are considered marked Themes.

In our analysis of Theme, we follow Thompson's (2014) deductive coding scheme, and there are a few key points about his conceptualization of Theme that merit mention:

1. When a clause complex starts with a dependent clause, Thompson (2014) suggests treating the entire dependent clause as the Theme (e.g., "**If it rains**, I'll stay home"; Theme in bold).
2. Textual and interpersonal elements that precede the first experiential (or topical) element are part of the Theme, which is then called a *multiple Theme* (e.g., "**Thus, surely, you** agree."). Note that a multiple Theme is not multiple Themes, but one Theme consisting of multiple elements that correspond to different metafunctions.
3. Clauses with "there" plus the existential process (usually realized by "be") include the existential process in the Theme (e.g., "**There is/was** a bird outside").

After our initial Theme–Rheme coding, we realized that proofs are full of (projecting) structures that signal to the reader the logical status of an upcoming statement (e.g., "Assume that […]", "We want to prove that […]"). After consulting with SFL expert Mary J. Schleppegrell (personal communication, September 23, 2025), we agreed that these can be considered interpersonal elements since they realize the interpersonal dimension of modality, that is, they express "an assessment of the validity of what is being said" (Halliday & Matthiessen, 2014, p. 144).

METHODS

Our dataset consists of 20 proofs from 10 popular undergraduate textbooks (i.e., Abbott, 2015; Axler, 2024; Dummit & Foote, 2004; Fraleigh, 2003; Ireland &

Rosen, 1990; Mendelson, 1990; Munkres, 2000; Rudin, 1976; Silverman, 2012; Strang, 2016) that cover typical undergraduate mathematics content: real analysis, abstract algebra, linear algebra, number theory, and topology. Within each textbook, two proofs were randomly selected with a random number generator.

All proofs were separated into T-units: "independent clause[s] together with all the clauses that are dependent on it" (Thompson, 2014, p. 161). T-unit boundaries that occurred within sentences were indicated with a double slash (//). Proofs with less than 5 T-units or more than 50 T-units were replaced to ensure that there were meaningful proof texts to analyze, roughly within the same order of magnitude. The 20 proofs totalled 316 T-units.

After the 20 proof texts were segmented into T-units, the Theme (and Rheme) of each T-unit were identified, following Thompson (2014). Next, the Theme was studied to: (a) determine whether it was marked, and (b) identify any textual and interpersonal elements (highlighted in orange and blue, respectively) as part of a multiple Theme. For an example of the T-unit separation and coding of a sentence from a proof by Abbott (2015), please see Table 1.

Theme	Rheme	Marked?
If $a = b$,	then $\|a - b\| = 0$ //	marked (dependent clause)
and so certainly $\|a - b\|$	$<\varepsilon$ no matter what $\varepsilon>0$ is chosen.	unmarked (declarative)

Table 1. Example Theme analysis of a sentence

10 proofs were coded by the first author and at least one other author, and all coding conflicts were resolved—our early coding can be found in Küchle et al. (in press). Thereafter, all four authors convened to discuss the coding, and the first author proceeded to code the remaining 10 proofs. (The authors are currently in the process of double coding the remaining 10 proofs.) The coding was imported into MAXQDA 2024, which allowed for the further study of the Themes—in particular what the common marked Themes were and what the textual and interpersonal elements were with which authors complemented their experiential Theme constituents.

RESULTS

Theme Status

Theme status describes whether a Theme is marked or unmarked. In our dataset, slightly over 30% of the Themes were marked (96 out of 316) and close to 70% were unmarked (220 out of 316). We first discuss the marked Themes.

The most common marked Theme was beginning a T-unit with a dependent clause (66 out of 96, 68.8%). Dependent clauses were mainly used to introduce conditions (32 if-clauses, 1 when-clause), to introduce warrants (15 since-clauses, 2 because-clauses), and to introduce warrants with conditions (5 "For if"-clauses). Lastly, the dependent clauses sometimes began with infinitives, which pointed readers towards a

goal or direction (e.g., “To show/check/see/prove […]”) (6 times), and sometimes with participles (e.g., “Cancelling p_1 […]”) (5 times).

Second-most common were marked Themes with Adjuncts: 26 declarative T-units and 3 imperative T-units began with Adjuncts. (It may be helpful to recall that Adjuncts are clause constituents that provide information about the circumstances of a situation, such as when, where, how, and why, Thompson, 2014.) These Adjuncts most commonly (13 times) began with “By” and explained why (e.g., “By Theorem 2.12 […]”) or how (e.g., “By inspection […]) something followed. Second-most common (8 times) were Adjuncts beginning with “For,” which qualified (e.g., “For the largest value t_{max} […]) and quantified statements (e.g., “For each x […]”). Adjuncts beginning with “from,” “in,” “as regards,” or “given” each occurred less than a handful of times and appeared to orient the reader and frame the Rheme.

Apart from dependent clauses and Adjuncts, there was only one other marked construction, which appeared only once: a declarative T-unit beginning with a Complement (i.e., “This we can do because […]”). No other marked constructions (e.g., marked interrogatives, marked imperatives with non-Adjuncts) appeared in the dataset.

Regarding unmarked Themes, it can be observed that most were declaratives (178 out of 220, 80.9%) and some were imperatives (42 out of 220, 19.1%). Interrogatives did not appear. (Although our dataset did not contain a single interrogative, we know from experience that they are merely very rare and unconventional rather than non-existent.) Among the unmarked declaratives, there were few instances of the four constructions that Thompson (2014) calls *enhanced Themes*. In particular, there were only four instances of existential ‘there’s (i.e., “There exist(s)/are […]”), one instance of a thematic equative (i.e., “What we must verify is that […]”), and no instances of preposed or predicated Themes. In short, the proof texts made use of a very limited palette of Themes.

Theme Composition

Theme composition describes whether a Theme is simple (i.e., it contains only an experiential element) or multiple (i.e., it also contains textual and/or interpersonal elements). Around 65.2% of Themes were multiple (206 out of 316) and roughly 34.8% were simple (110 out of 316).

Most occurrences of multiple Theme were conjunctive (141 out of 206, 68.4%), that is, they consisted of textual and experiential elements (e.g., “Hence $d\lambda_{\mathbf{F}}$ […]”, “Thus (a) […]”). Next, 18.9% of multiple Theme occurrences (39 out of 206) were attitudinal, that is, they combined interpersonal and experiential elements in the Themes (e.g., “It remains to show that (γ,ρ) […]”). Last, 12.6% of multiple Theme occurrences (26 out of 206) were attitudinal *and* conjunctive, thereby combining textual, interpersonal, and experiential elements in the Themes (e.g., “Thus assume that A […]”).

Looking across the total of 195 textual Theme elements, there were 35 distinct textual elements with the most common being: if (37), and (29), since (15), then (15), thus (14), so (13), hence (9), but (6), now (6), therefore (5), and for (5). Looking across these (and the less frequently used textual Theme elements), one can see that they allow authors to assume (i.e., if), to add (i.e., and, also, again), to justify (i.e., since, for), to deduce (i.e., then, thus, so, hence, therefore), to stage (i.e., now, then, but, first), and to contrast (i.e., but, however). At times, multiple textual elements were combined, mostly to organize the text *and* do mathematical work (e.g., "Now if", "Again by", "And hence", "But then").

Looking across the total of 75 interpersonal Theme elements, there appeared to be three categories. Most common (62) were interpersonal elements that signaled to the reader the validity of the upcoming statement: (a) what follows is assumed or supposed (e.g., "Assume that […]"), (b) what follows remains to be proved ("We prove that […]"), (c) what follows is true (e.g., "Note that […]"), or (d) what follows is chosen (e.g., "Let […]"). Less common were interpersonal elements that modified the previously discussed interpersonal elements (e.g., "we *must* verify that") (9 times) and interpersonal adverbs (i.e., clearly, easily, simply, certainly) (4 times).

The experiential elements of Themes were: Subjects in unmarked declarative clauses (e.g., "**The *i,j* entry of AB** is […]") (178 times), dependent clauses in marked clause complex constructions (e.g., "**If *r* is itself irreducible**, then […]") (66 times), Predicators (i.e., the verb that the Subject "does") in unmarked imperative clauses (e.g., "**Let** n = dim V") (42 times), Adjuncts in marked declarative clauses (e.g., "**By Theorem 2.12**, $g^{-1}(F)$ is […]") (26 times), Adjuncts in marked imperative clauses (e.g., "**For each** α, choose […]") (3 times), and a Complement in a marked declarative clause (i.e., "**This** we can do […]") (1 time). We also observed that unmarked declarative Subject-Themes sometimes included Pre- and Postmodifiers. For example: "**The very definition of φ_a applied to a constant polynomial $a \in F[x]$, where $a \in F$** gives $\varphi_\alpha(a)=a$", where the Pre- and Postmodifier are underlined. Thompson (2014) refers to such Subjects as *'heavy' Subjects*.

DISCUSSION

Our results align with and extend O'Halloran's (2005) observations and Morgan's (1998) conjectures about Themes in (deductive) mathematics texts. In particular, our results suggest that mathematicians typically write declarative and imperative sentences and do not use many different types of (marked) Thematic constructions. Indeed, the authors of our dataset appeared to mainly rely on two marked constructions: leading with a dependent clause and leading with an Adjunct. Both of these marked constructions appear very functional for communicating mathematics as they allow authors to lead with conditions and warrants, which aligns well with the flow of logic, and enables authors to manage the flow of the argument and the links between claims.

Regarding the Theme composition, it stands out that the authors in our dataset heavily relied on the use of multiple Theme. Textual elements were used in common, everyday

ways such as adding (e.g., “and”) and staging (e.g., “Now”), but also in ways that supported the flow of mathematics (e.g., if, since, thus). Interpersonal elements were mostly used to signal to the reader the logical status of upcoming statements, that is, whether they were assumptions, goals, observations, or choices. Surprising to the authors of this paper was that adverbs such as “clearly,” whose use is questionable due to their problematic positioning, appeared in only 4 out of 316 Themes.

Regarding next steps: We are currently in the process of building on this work by conducting a Thematic Progression analysis (Thompson, 2014), a type of analysis that studies how Themes connect and build on previous Themes and Rhemes. Studying thematic progression gives insight into “the skeleton of the plot” (Daneš, 1974, p. 114). Other fruitful avenues might be to complement this Theme analysis (and Thematic Progression Analysis) with a cohesion analysis, which focuses on the use of the cohesive linguistic resources of reference, ellipsis, conjunction, and lexical cohesion.

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Teachers' reflections on the causes of students' errors in university mathematics: epistemological obstacle as a potential analysis tool

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Abstract. This presentation addresses the potential of the epistemological obstacle as a referent in Mathematics Education that contributes to understanding the possible causes of students' errors in university mathematics. Through the analysis of a group of students' solutions to tasks on limits, a group of in-service university mathematics teachers discuss and reflect on the possible causes of the errors present in these solutions. The paper specifically discusses the incorporation or non-incorporation of the epistemological obstacle in the teachers' analyses of the causes of the errors. These reflections take place within the framework of a Mathematics Education mini-course conducted as part of the author's doctoral research.

Keywords: learning of university mathematics, causes of errors, epistemological obstacle, change of teachers' conceptions, qualitative research.

THE RESEARCH PROBLEM

The way in which university mathematics teachers approach their students' errors in their professional practice is significantly conditioned by their conceptions and beliefs, which frequently differ from the theoretical explanations originating from Mathematics Education. As several studies suggest, teachers tend to attribute the causes of errors to factors that completely diverge from the theoretical frameworks of Mathematics Education that help to understand their possible causes (Gagatsis & Kyriakides, 2000; Ramírez, 2012). Naive approaches to the causes of errors can negatively influence how teachers approach this aspect of mathematical learning. On the other hand, research in Mathematics Education offers references that help to critically understand the possible causes of errors, such as epistemological and didactic obstacles (Brousseau, 1976), misconceptions (Schoenfeld, 1985) and semiotic transformations (Duval, 2017).

The doctoral research was based on the premise that integrating these references from Mathematics Education into teachers' analyses could induce a positive transformation in their conceptions about the causes of errors. We consider that, by overcoming naive explanations, these analyses would be significantly enriched, in accordance with what Ball et al. (2008) pointed out regarding the fact that expert teaching in mathematics requires that the teacher has the ability to judge in a timely manner the possible causes of the errors of his students. The main objective of the doctoral research was to examine in depth the possible changes in the participating university teachers' conceptions regarding the causes of their students' errors in mathematics. The study answered the research question: What changes in the conceptions of a group of teachers (in service) about the causes of errors in mathematics, of university students, occur from reflection, discussion and socialization, in specific focus group meetings, based on the analysis of theoretical referents of Mathematics Education? This presentation focuses specifically

on how teachers incorporated the concept of epistemological obstacle into their analyses and reflected on its relevance.

THEORETICAL FRAMEWORK

Brousseau (1976) establishes a connection between mathematical learning problems and the historical evolution of concepts through epistemological obstacles. Analyzing errors from this perspective reveals them not as deficiencies, but as prior knowledge that was functional in another context yet now offers resistance. According to D'Amore and Fandiño Pinilla (2012), the discontinuities, fractures, and radical changes during a concept's historical evolution constitute these obstacles, manifesting as barriers to learning. This framework acknowledges the student's prior cognitive constructions as limited but necessary stages in developing complex concepts like the limit (Sierpinska, 1987) [1]. These obstacles persist over time and recur among students, explaining the underlying causes of persistent errors in university mathematics. While Brousseau (1976) also identifies didactic and ontogenetic obstacles [2], this study focuses specifically on the epistemological dimension.

METHOD AND CONTEXT

The study was developed in two phases. In the first phase, which was exploratory, 25 in-service university mathematics teachers with vast teaching experience were invited to participate. In this stage, they were asked to report frequent student errors, based on their experience, and to propose possible explanations for the causes of these errors. The second phase consisted of a Mathematics Education mini-course with twelve teachers, organized into four focus groups with 60 to 90-minute sessions per referent (epistemological obstacle, didactic obstacle, semiotics, and misconceptions). Focus groups 1, 2, and 3 participated in the analysis session on epistemological obstacles [3]. The participants are in-service university mathematics teachers from universities in Bogotá and nearby municipalities, with diverse backgrounds and experience. Most have taught Basic Mathematics (Precalculus), Differential Calculus and Integral Calculus; some have also taught Vector Calculus and Differential Equations.

Each session was structured in four moments: 1) initial individual reflection on the causes of errors present in students' solutions to specific tasks, 2) initial socialization and discussion in each focus group based on the individual reflections, 3) presentation by the researcher of the theoretical reference corresponding to the session, and 4) final discussion on the relevance of this referent for analyzing the causes of errors. The tasks presented to teachers in the epistemological obstacle session were selected based on two criteria: a) they corresponded to real classroom activities from first-semester university mathematics courses, and b) they required students to mobilize multiple facets of the limit concept (numerical, algebraic, and conceptual). Teachers' conceptions regarding the causes of errors were characterized through an inductive-deductive coding process (Fereday & Muir-Cochrane, 2006; Navarrete, 2011). Five a priori categories were established based on the objectives and the specialized literature (Gagatsis & Kyriakides, 2000; Peng & Luo, 2009): 1) identification of the error, 2) explanation of

the error, 3) disposition to incorporate Mathematics Education referents in the analysis of the causes of errors, 4) conceptions about the causes of errors, and 5) other factors influencing the analysis of the causes of errors. Descriptors (subcategories) emerged inductively from the data.

To establish changes in participants' conceptions, we used the criterion of comparing conceptions (before and after the study program) as employed by D'Amore and Fandiño Pinilla (2004) and Alsina (2012). In our case, the comparison was made for each participant, for each focus group, and at the end of each meeting. For the purposes of this research, we adopt Llinares' (1996) definition of conception as "encompass[ing] a whole range of teacher cognitions, including beliefs, knowledge gained from experience, etc." (p. 9). Following Wilson and Cooney (2002), we define changes in conceptions as the development or modification of conceptions over time. The doctoral research also assumed that changes in teachers' conceptions contribute to improving teachers' teaching (Bobis et al., 2016) and to improving student performance (Wilkins & Brand, 2004).

Development of the session on epistemological obstacle

This session explored how epistemological obstacles contribute to understanding the causes of students' errors in the process of learning limits. The limit of functions of a real variable was selected as a representative example of the existence of these obstacles in learning, as evidenced by various studies (Sierpinska, 1987; Dubinsky, 2000). Conversely, research by Carlson et al. (2010) and Oehrtman (2009) offers an asset-based perspective that moves beyond identifying deficiencies. These authors identify students' use of creative metaphors and their ability to reason about changing quantities as productive starting points for successfully mastering calculus. The session analyzed some milestones in the historical evolution of limits.

First activity: individual analysis of possible causes of errors

In the first activity, the teachers performed two tasks consisting of reviewing and analyzing some student solutions to problems on limits. In both cases, the participants answered three questions aimed at identifying the errors present and explaining their possible causes. The first task consisted of analyzing the responses of three students to a question included in a Differential Calculus assessment, solved individually.

x	1,9	1,94	1,98	1,998	1,9998	2	2,001	2,012	2,03	2,08	2,1
y	-10	-100	-1004	-300007	-48901235	3	3,0005	3,02	3,08	3,2	3,5

Table 1: values of x, y for task 1 (epistemological obstacle session). Author's own elaboration.

The statement proposed to the students was the following: for a certain function $f(x)$, the following values of x and y are given. Based on the values in the table 1, determine (if possible) $\lim_{x\to 2} f(x)$. Explain your answer. The following table shows the solutions of three students to this task:

Solution of Eduardo	Solution of Diego	Solution of Lucía
This is possible because it tells us that when x is 2, y is 3. Therefore, it is possible to determine the limit $\lim_{x\to 2} f(x) = 3$	*x tends to 2 on both sides, so it is clear that the value x=1.9998 must be chosen since it is the closest value to 2 without being 2.*	*It is not possible to determine because* $[\lim_{x\to 2^-} f(x) = -48901235] \neq [\lim_{x\to 2^+} f(x) = 3{,}0005]$

Table 2: Student solutions to task 1. Author's own elaboration.

In the second task, participants analyzed the solutions of two students to an activity in which they were asked to calculate the following limit, indicating the procedure performed: $\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$. Table 3 shows the solutions of two students to this activity.

Solution of Carolina	Solution of Arturo
$\lim_{x\to 0} 1(e^x-1)^{-1} - 1x^{-1} \quad \lim_{x\to 0} -1(e^x - 1)^{-2}(e^x) + 1x^{-2}$ $\lim_{x\to 0} \frac{e^x}{(-e^x-1)^2} + \frac{1}{x^2} = 0$ *It does not exist, it cannot be done, not even by the rule of l'hopital*	$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-(e^x-1)}{(e^x-1)x} = \lim_{x\to 0}\frac{x-e^x+1}{(e^x-1)x} = \frac{0-1+1}{(1-1)(0)} = \frac{0}{0}$ *L'Hopital* $\frac{f'(x)}{g'(x)} = \frac{1-e^x}{(e^x)(x)+(e^x)(1)} = \frac{1-e^0}{(e^0)(0)+(e^0)(1)} = \frac{1-1}{(1)(0)+1\cdot 1} = \frac{0}{1} = 0$

Table 3: Student solutions to task 2. Author's own elaboration.

Socialization of participants before the researcher's presentation

The following summarizes participant ideas from the socialization session before the introduction of the epistemological obstacle concept.

Focus group 1 (Ernesto, César, Carlos)

Ernesto views limits as inherently difficult to learn, teach, and communicate. While he and César both categorize error levels, Ernesto attributes them to prerequisite deficiencies, whereas César focuses on conceptual hurdles requiring cognitive maturity. For Ernesto, conceptual aspects pose greater challenges than algebraic ones. Furthermore, Ernesto and Carlos identify the understanding of infinity and the dynamic nature of limits as critical factors. César argues that bypassing the formal $\epsilon - \delta$ definition can impede comprehension, while Carlos emphasizes that the concept's complex historical consolidation significantly impacts the learning process.

Focus group 2 (Diego, Nicolás, Miguel)

Diego presents a detailed analysis of several problematic aspects: the predominance of algebraic approaches among students, the methods of teaching limits at the university level, and the treatment of functions in textbooks. He asserts that the limit is difficult

to learn and acknowledges the historical challenges in its consolidation as a mathematical object. Nicolás also acknowledges difficulties in learning limits, linking them to weaknesses in students' prior concepts and the emphasis placed on algebra over geometry. He focuses on teaching and learning issues, although he occasionally strays from the central discussion of error causality. Finally, Miguel points to two problematic factors: the technical language, which is often incomprehensible to students, and the inherent difficulties in understanding infinity.

Focus group 3 (José, Hernán, Manuel)

José attributes some errors to the teacher (e.g., lack of clear explanation) and others to student-related factors such as lack of understanding or insufficient study. In task 2, he notes that Carolina misapplies L'Hôpital's rule due to unclear conceptual understanding, while other errors are algebraic in nature. He suggests that practice helps students familiarize themselves with such situations. Hernán identifies and explains students' errors in detail, considering them conceptual, of different types, and with diverse causes (e.g., lack of student understanding or failure to perform timely geometric analysis). He acknowledges that the limit is difficult to learn and specifically highlights how problematic the concept of infinity is for students. Manuel describes errors in the analyzed tasks in detail. He points out that the limit is difficult to learn (particularly the idea of "approaching") and suggests that some errors arise from students wanting things to be easy and becoming too mechanical. He also notes that understanding infinity is problematic, identifies issues in students' prior knowledge, and acknowledges some teacher responsibility in the presence of these errors.

Researcher's presentation of the epistemological obstacle concept

The researcher introduced epistemological obstacles by analyzing students' solutions to the following continuity problem: find a and b such that the function is continuous over the entire set of real numbers

$$f(x) = \begin{cases} x + a & si \ \ x \leq -2 \\ x^2 & si \ \ -2 < x < 1 \\ bx - 1 & si \ \ x \geq 1 \end{cases}$$

To solve this problem, we use the definition of continuity of a real-variable function at $x = -2$ and $x = 1$. In his intervention, the researcher presented the solutions of three students (Verónica, Hernán, and Juan Carlos), identifying different errors. In all three cases, the limit is almost entirely absent as a tool for solving the problem: Verónica omits it completely; Hernán attempts to use limits but does not include lateral limits; Juan Carlos makes an apparent reference to lateral limits and continuity without using the notation explicitly, and he also equates expressions without justification. The students' procedures reveal different types of difficulties not exclusively related to the conceptualization of limits; nonetheless, they attempt to apply knowledge implicitly linked to rules for calculating limits.

The researcher presents problematic aspects of learning limits documented in the literature (Sierpinska, 1987; Tall, 1993) and discusses the complex historical

development of the concept, emphasizing that its formal definition—the work of Weierstrass—was consolidated only after more than twenty centuries. Based on this, he proposes a reflective question: What does all of this have to do with the difficulties of learning limits? This issue is directly linked to the learning process, as limits are typically studied for only two to four weeks in Differential Calculus. Consequently, a fundamental question emerges: how can a student cognitively construct, in such a short period, a concept that required more than twenty centuries to consolidate as a formal mathematical object? The historical development of limits involved multiple setbacks, ruptures, and difficulties—characteristic of any mathematical argument (D'Amore et al., 2010). According to these authors, the nature of mathematical objects and the challenges in their historical evolution serve as indicators of epistemological obstacles. In the learning process, these obstacles manifest as recurring, typical errors that persist year after year, as observed with the concept of limits. The researcher concludes by discussing the defining characteristics of epistemological obstacles (Brousseau, 1976; D'Amore et al., 2010).

Teachers' Analysis After the Researcher's Intervention

Focus group 1 (Ernesto, César, Carlos)

Ernesto points out that he was not familiar with the theoretical concept of epistemological obstacles and believes that it should help explain the causes of students' errors in some way; although he presents an example of a possible epistemological obstacle, he does not incorporate this theoretical concept into his analysis of the proposed tasks. For César, although the epistemological obstacle could help explain the causes of errors in activities involving limits, it could also lead to the stigmatization of this concept. César identifies the possible presence of an epistemological obstacle in the concept of infinity. Carlos does not explicitly incorporate the epistemological obstacle into his analysis of the causes of errors in tasks involving limits, although he draws attention to the complexity of its historical development.

Focus group 2 (Diego, Nicolás, Miguel)

Diego mentions some historical aspects in the evolution of the limit, but his analysis strays from the central focus of the session. After the theoretical presentation, he refers tangentially to the epistemological obstacle, indicating that limits cannot be taught in two or three weeks and that learning this concept requires understanding, reviewing, reinforcing, practicing, and reflecting. Nicolás points out that the researcher's presentation is interesting but does not specifically mention the epistemological obstacle as a reference point for analyzing the causes of students' errors. In contrast, Miguel refers directly and specifically to one of the problematic aspects of understanding limits: infinity. He presents as an example the case of students who approach infinity as if it were a number. However, Miguel does not explicitly refer to the epistemological obstacle as a resource for explaining the possible causes of errors.

Focus group 3 (José, Hernán, Manuel)

José acknowledges the possible existence of epistemological obstacles to learning a concept such as limits but does not delve into their potential for analyzing causes of errors; he focuses his discussion on the possible usefulness of limits for students' professional training: if students are not interested, this may be an even greater obstacle to their learning. Hernán states that the idea of epistemological obstacles could arise in all areas of knowledge and adds that teachers "learn against the clock" and that overcoming these obstacles is "part of the dynamic." For Hernán, learning derivatives, integrals, and series can present these obstacles, but not much time can be devoted to studying them in class. Hernán's justification for this statement is that the goal of students is not to become experts in mathematics. Hernán admits that the limit is a difficult concept to learn and refers specifically to how problematic the concept of infinity is for students; furthermore, he proposes an example of a possible epistemological obstacle. Manuel finds it somewhat difficult to "accept that students do not learn" due to the epistemological obstacle, a concept that could be an easy way out to explain the causes of students' errors: they do not understand because no one has ever understood, which he sees as a danger of generalization; however, he recognizes the work behind the consolidation of the concept of epistemological obstacle and points out that it should not be overlooked. Manuel proposes as an example of the possible presence of an epistemological obstacle the case of trigonometric identities, which are very difficult for students to learn.

Focus group	Teacher	Discusses epistemological obstacle as a useful referent?	Incorporates epistemological obstacle to analyze task errors?
1	Ernesto	Yes (finds it interesting, believes it should help explain errors)	No
1	César	Yes, but cautions it could stigmatize the concept	Partial (infinity)
1	Carlos	Yes, recognizes complexity of historical development	No
2	Diego	Tangentially (limits cannot be taught in 2-3 weeks)	No
2	Nicolás	Finds presentation interesting, but no specific mention	No
2	Miguel	Yes (refers to infinity as problematic)	Partial (infinity as number)
3	José	Yes (acknowledges possible existence of epistemological obstacle)	No
3	Hernán	Yes (suggests epistemological obstacle could arise in all areas of knowledge)	No
3	Manuel	Yes, but cautions it could be an "easy way out."	No

Table 4: Teachers' discussion and incorporation of the epistemological obstacle. Author's own elaboration.

The previous analysis reveals a consistent pattern, showing that while most teachers discussed the epistemological obstacle as a potentially useful theoretical referent, only two (César and Miguel) partially incorporated it into their error analysis, and both did

so exclusively in relation to the concept of infinity. No teacher fully incorporated the epistemological obstacle to analyze errors in the proposed tasks (see Table 4).

Results and Discussion

The epistemological obstacle was a novel concept for the participants, generating diverse reflections. While some (e.g., Ernesto) recognized its potential, others (César, Manuel) questioned its relevance, and a few (Miguel, José) described it as 'interesting' without further elaboration. Two recurring themes emerged: most participants acknowledged the difficulties in learning limits, and some partially attributed these difficulties to the understanding of infinity, aligning with the literature (Arrigo et al., 2011). Only a few teachers attempted to integrate the epistemological obstacle into their post-intervention analyses, citing the need for deeper study. Nevertheless, these reflections and the application of an academic approach to student errors provide a solid starting point for transforming their conceptions about the causes of errors.

Several limitations should be acknowledged. First, this study examined teacher reflection within a single session (per focus group) on epistemological obstacles; longer-term studies are needed to assess whether initial uptake translates into sustained changes in classroom practice. Second, the study relied on teachers' discourse within focus groups rather than direct classroom observation. Future research should examine the extent to which university mathematics teachers modify their error analysis practices when designing or responding to student errors in situ. Specifically, further studies could investigate how explicit recognition of epistemological obstacles influences the design of didactic situations, the management of higher education classroom interactions, and the support in overcoming learning obstacles. Moving from teacher reflection to actual university classroom practice remains a fundamental challenge for the field.

NOTES

1. Sierpinska (1987) points out that static visual images, excessive formalism, and certain intuitions hinder students' grasp of the dynamic and infinite nature of the limit concept. Identifying these specific forms of resistance shifts error analysis from simple correction to deep didactic diagnosis.

2. Didactic obstacles arise from the teacher's decisions and choices in teaching—for example, always drawing rectangles in the same position (with horizontal and vertical sides), representation that can be problematic in certain contexts while ontogenetic obstacles relate to the student's stage of development. Although focusing on epistemological obstacles regarding limits, this analysis remains sensitive to these other types of obstacles.

3. Focus group 4 did not participate in the epistemological obstacle session; due to scheduling constraints; therefore, it is not included in this analysis.

4. Students and teachers are identified with pseudonyms.

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Domain of Validity framework: A new instructional theory for addressing students' preconceptions

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Annually, during their final probability exam, the majority of first-year Mathematical Sciences students in ULB tend to apply numerical techniques to the limits of sequences of sets. We present the Domain of Validity (DoV) Framework, a new theoretical and operational approach contributing to research on conceptual change. This framework offers a new way to analyse and address specific learning difficulties that we define as preconceptions, understood as models associated with an overgeneralised domain of validity. We first introduce the core ideas of the framework and illustrate them through a case study conducted with 30 first-year mathematics students.The results confirm the presence of the targeted preconception and show a shift in the choice of knowledge students mobilised to solve the exercise.

Keywords: probability, conceptual change, mixed methods, collaborative action research.

Originally the Domain of Validity (DoV) Framework has been developed in engineering to analyse when students apply a model beyond its valid domain of applicability. The framework was formalised by Quinlan, Sommeillier & Robert (2020).

The aim of this research is to test if the DoV framework can identify and resolve the incorrect application of numerical techniques to the limits of sequences of sets among first-year students in mathematical sciences at ULB (Université Libre de Bruxelles). The DoV Framework defines knowledge as a pair : a model together with his domain of validity. A preconception is a model with an overgeneralised domain.

The instructional method includes three steps:

1. diagnostic analysis : identifying errors linked to an oversized domain;
2. translating the student's model into an explicit restricted form;
3. creating a paradoxical experience showing contradictions that arise when the model is misapplied.

We therefore designed a pre-test, a paradoxical experience and a post-test that is exactly the same convergence question that the pre-test question. The paradoxical experience involved presenting students with three sample written solutions without disclosing their respective grades. These sample written solutions consisted of :

- An incorrect solution (graded 0/10): this response relies solely on numerical techniques, which are valid only for sequences of numbers.
- An intermediate solution (graded 5/10): uses set techniques incorrectly.
- A correct formal solution (graded 10/10): uses set techniques correctly.

Students evaluated and graded these solutions independently, then compared their marks with the expert grading. This discrepancy triggered a short socio-constructivist discussion to highlight why numerical techniques fail for sequences of sets. Finally, students completed a post-test. Data were collected via Wooclap (a Student Response System) on smartphones. MCQs and evaluation tasks provided quantitative data, while justifications yielded qualitative data. Responses were anonymized and aggregate results were hidden until all students (n=30) had voted. In the pre-test, students were asked to evaluate a sample solution that was mathematically invalid (graded 0/10 by experts) because it relied on numerical techniques instead of set-theoretic tools (limsup/liminf). Despite its objective invalidity, the quantitative analysis shows that students awarded it a mean grade of 6.42/10 (median: 7, mode: 8). This significant discrepancy demonstrates that the incorrect model was not only accepted but highly valued. Similarly, qualitative data revealed that 0/30 students used limsup or liminf in their justifications. In the post-test, however, 8 students correctly used limsup and liminf in their justifications, marking a shift toward appropriate set-theoretic reasoning.

The results suggest that the DoV Framework may offer a productive lens for interpreting how students sometimes extend mathematical techniques beyond their appropriate domain of validity. In this sense, it can support teachers and researchers in analysing students' reasoning and in designing tasks that could help address such difficulties. In this case, the teaching sequence was designed and closely guided by the research team. Future work should explore how teachers engage with the DoV Framework when given greater freedom, and how such autonomous implementations influence students' reasoning.

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Exploring Peer-Led Team Learning from a Vygotskian Perspective with Mathematics Pre-service Teachers

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This paper explores how pre-service secondary mathematics teachers experience a peer-led team learning activity in a university Mathematics Education course, from a Vygotskian perspective. Eighteen Italian master's students took part in a three-hour small-group task where some acted as peer leaders, temporarily facilitating mathematical-didactical work. A qualitative analysis of post-activity questionnaires shows that peer teaching was perceived as a dialogic, collaborative and emotionally safe environment, in which the peer leader is seen as a slightly more knowledgeable peer. In addition, the study highlights how pre-service teachers interpret this experience not only as learners, but also as future teachers, reflecting on the conditions under which peer-mediated interaction may support learning within the Zone of Proximal Development in secondary school mathematics.

Keywords: peer education, peer-led team learning, pre-service teacher, vygotskian perpespective, undergraduate education

INTRODUCTION AND RATIONALE

Over the last few decades, the scholarly debate on innovative teaching methodologies has increasingly emphasized the role of cooperative learning and peer-based approaches. Within this broad family, peer education is widely recognized as a promising way to support learning, promote inclusion, and foster relational and metacognitive skills that are essential for professional life. Although the term "peer education" encompasses a heterogeneous set of practices, its roots can be traced back to the early work on cooperative learning in the 1980s (Slavin, 1980; 1987), which highlighted how structured collaboration among students can reshape participation patterns and opportunities to learn. Within this landscape, the Peer-Led Team Learning (PLTL) model emerged in the United States in the early 1990s, particularly in STEM (Science, Technology, Engineering and Mathematics) course. PLTL is not grounded in a single explicit theory, but it is consistent with sociocultural and social constructivist perspectives, especially Vygotsky's notion of the Zone of Proximal Development (ZPD) (Vygotsky, 1978). In PLTL, an "expert student" acts as a peer leader who facilitates small-group work, encourages dialogue, and supports problem solving, without taking on the formal role of instructor. Several studies document how PLTL can contribute to improved academic performance, reduced dropout rates, and narrowing of achievement gaps, particularly for underrepresented student groups (Snyder, Sloane, Dunk & Wiles, 2016); at the same time, leading peers may strengthen

the leaders' self-efficacy, communication skills and other transversal competences (Gafney & Nelson, 2007). Other studies have been carried out with the aim of increasing the number of STEM graduates, ensuring not only the attraction of new students, but also favorable conditions for their permanence and academic success (Reisel & Jablonski, 2013).

In this paper, we focus on a closely related variant, often referred to as peer teaching or peer-to-peer learning, in which students alternate between the role of learners and that of peer leaders. Different contributions (Laskey & Hetzel, 2011; Maheady, 1998) suggest that learning with and from peers who share the same educational context may offer specific advantages, due to both cognitive and social proximity. However, despite a rich international literature on peer tutoring and PLTL in higher education (Hidayat, et al., 2023), systematic implementations and empirical investigations remain relatively limited in the Italian context (Capone, 2022). This is particularly true when we consider university Mathematics Education and the preparation of future secondary mathematics teachers. In fact, when we narrow the focus to Mathematics Education, the literature appears to offer relatively few studies (Capone, 2022) that examine how pre-service secondary mathematics teachers experience peer teaching within university didactics courses, and how they perceive its potential for their future classroom practice. In particular, there is limited evidence about how these students position themselves with respect to the role of peer leader, which opportunities and challenges they recognize in this methodology, and under which conditions they consider it feasible for secondary school mathematics lessons. The present contribution addresses this gap through a small-scale qualitative exploratory study conducted in an Italian master's degree course in Mathematics Education. In this context, a group of pre-service secondary mathematics teachers was introduced to a peer teaching activity in which some participants temporarily assumed the role of peer leaders. Rather than aiming to demonstrate the effectiveness of the approach in terms of learning outcomes, the study investigates how participants experience and interpret peer teaching interaction in relation to key Vygotskian construct, such as mediation, participation and the Zone of Proximal Development (Vygotsky, 1978). Particular attention is given to how these pre-service teachers, positioning themselves both as learners and future teachers, reflect on the conditions, constraints, and possibilities for implementing peer-mediated learning in secondary school mathematics.

RESEARCH QUESTIONS

To achieve our aim, we address the following research questions:

RQ1. How do pre-service secondary mathematics teachers experience and interpret peer teaching interactions, in terms of participation, roles, and mediation, within a Vygotskian perspective?

RQ2. How do they conceptualise the potential of peer teaching for secondary school mathematics, particularly in relation to the conditions for activating the Zone of Proximal Development?

By focusing on participants' perspectives in a specific Italian university setting, this exploratory study aims to provide an initial, context-sensitive understanding of how peer teaching interactions are experienced and interpreted by pre-service mathematics teachers, particularly in terms of participation, roles, and mediation within a Vygotskian perspective, and to outline some conditions, limitations, and open questions related to the activation of the Zone of Proximal Development in secondary school mathematics for its possible implementation in secondary school mathematics.

THEORETICAL BACKGROUND

The theoretical background of our experimentation is mainly rooted in a Vygotskian perspective on learning. In this view, cognitive development is not a purely individual process but grows out of social interaction and participation in shared cultural practices. Vygotsky (1978) underlines how working together with others, talking, and solving problems in a community are fundamental for the construction of meaning and for the development of new ways of thinking. Two ideas from Vygotsky are particularly relevant for peer teaching and for our activity. The first is that interaction with someone who is "more competent", such as a teacher or a peer who has already mastered certain contents or strategies, can strongly support learning. In a peer teaching setting, students who temporarily take on the role of leader can offer this kind of support: they do not replace the teacher, but act as a resource for their classmates, supporting them in interpreting the problem, selecting appropriate strategies, and evaluating their reasoning. The second idea is the Zone of Proximal Development (ZPD), defined as the gap between what a learner can do alone and what they can do with the help of a more expert other. While the more competent other refers to the social role of a participant who provides support, the ZPD refers to the relational space in which such support enables learning. For Vygotsky, teaching is effective when it is positioned inside this zone: tasks should not be too easy, but also not so difficult as to be inaccessible without support. In this sense, we can interpret the role of the peer leader as a form of scaffolding in a Vygotskian sense: a temporary support that allows classmates to work within their ZPD and to progressively gain autonomy. Wood (1996) highlights three key features of this type of support: (1) it is built through a collaborative interaction between learner and "expert"; (2) it requires continuous attention to the learner's current level, so that the proposed activities remain challenging but manageable; and (3) it is gradually reduced as the learner becomes more competent. These principles guided the design of our peer teaching activity, in which peer leaders were asked not to "explain the solution", but to help their group engage with the mathematical tasks, negotiate strategies, and slowly take charge of their own learning. While the study is primarily framed within a Vygotskian perspective, we acknowledge that peer-led interactions can also be interpreted through dialogic and participation-based frameworks (Lave & Wenger, 1991). These perspectives offer complementary lenses to examine how knowledge is co-constructed through discourse, interaction, and participation in shared practices.

DESCRIPTION AND METHODOLOGY OF THE PROJECT

The experimentation was conducted with 18 pre-service secondary mathematics teachers attending the final year of a master's degree. Participants were divided into four groups, each with a peer leader selected from those who had previously participated in a seminar on Design-Based Research (DBR), as they were expected to have preliminary familiarity with the content. The activity, lasting three hours, was organised into several phases. The task consisted of a group activity in which the peer leader introduced the main ideas of Design-Based Research (DBR) through a presentation, while the other participants engaged in discussion and collaborative work aimed at reconstructing the underlying theoretical principles. This process was supported by examples connected to mathematical tasks, which were used to ground the discussion and relate DBR to concrete teaching situations. At the end, participants completed a questionnaire (with a version adapted for peer leaders), consisting of open-ended prompts aimed at exploring their perceptions of peer interaction in relation to key constructs from a Vygotskian perspective. Several questions addressed the role of the peer leader, which can be interpreted in terms of the "more knowledgeable other" providing mediation within the learning process. Other prompts focused on group interaction, capturing the collective dimension of meaning-making and the idea of learning as a socially mediated activity. Questions concerning the differences with traditional lectures were intended to explore the socio-emotional and relational conditions that may enable or constrain participation within the Zone of Proximal Development. Finally, prompts related to the potential use of peer teaching in secondary school contexts aimed to elicit participants' reflections on the conditions under which such interactions could support or hinder the activation of the Zone of Proximal Development in classroom settings.

The data were analysed through an inductive thematic analysis (Braun & Clarke, 2006). Responses were first open-coded and then grouped into broader categories, which were progressively refined into four main themes. While the analysis was primarily inductive, themes were subsequently interpreted through a Vygotskian lens, particularly in relation to mediation, participation, and the role of the more knowledgeable other within the ZPD. Coding decisions were iteratively discussed among the authors to enhance the credibility and consistency of the interpretation. Given that participants were pre-service teachers, the analysis also considered their reflections on the potential transfer of peer teaching to secondary school contexts, focusing on perceived constraints and conditions for implementation.

RESULTS

The qualitative analysis of the questionnaires offers a coherent picture of how pre-service secondary mathematics teachers experienced the university peer teaching activity in terms of mediation, participation, and the socio-relational conditions for learning, and how they interpreted its potential use in secondary school mathematics through a Vygotskian lens. Four intertwined themes emerge from the data.

The first theme concerns the perception of the activity as a dialogic and collaborative learning environment. As reported as follows:

> Peer Leader 1: “the group asked many questions during the presentation, and they were always closely connected to the ongoing discussion”.
>
> Peer Leader 2: “it was helpful to hear the doubts raised by other colleagues, because we could listen to the leader’s answer before having the same doubt ourselves”.

participants repeatedly highlight the central role of questions, discussion and shared problem solving, suggesting that understanding did not arise from a single explanation, but from a sequence of clarifications and reformulations within the group. In this sense, the peer teaching setting is described as a space where meanings are progressively negotiated and understanding emerges through mediated interaction within the ZPD, rather than through the transmission of fixed explanations.

A second theme concerns the specific position of the peer leader within the group. The leaders are generally evaluated positively, but they are not perceived as “mini-teachers”. Instead, as highlighted in the next sentences, they are seen as slightly more competent peers who help to make the content accessible while still consolidating their own understanding.

> Peer Student 1: “the leader immediately captured my attention and was very clear and well prepared, even though she had only recently learned the topic”.
>
> Peer Student 2: “the leader managed to clarify our doubts effectively, trying each time to explain the idea with different words so that we could really internalize it”.

At the same time, leaders themselves underline that the work of explanation was constructed together with the group.

> Peer Leader 3: “the group raised doubts that I also had, so working together pushed me beyond my personal boundaries”.
>
> Peer Leader 4: “in some moments I did not feel very adequate, because I was not yet sufficiently expert in the wider area in which the topic is situated”.

Overall, the peer leader emerges as a relatively more knowledgeable other in a dynamic and evolving position, where mediation is not unidirectional but co-constructed through interaction with the group.

Furthermore, a third theme regards the emotional climate and the perceived reduction of hierarchy, which participants explicitly link to their willingness to take part in the activity. Many students report feeling freer to intervene and to expose their doubts in the small-group setting than in whole-class lectures.

> Peer Student 3: “within the group we were all on the same level, so especially from an emotional point of view I felt free to ask questions or speak”.

Peer Student 4: “with the team leader it felt more like a dialogue than a lesson; there was no fear in asking questions, the desire to understand was stronger than the fear of speaking up”.

In contrast, the figure of the university teacher is often associated with authority and potential judgement.

Peer Student 5: “the atmosphere was not the usual school one, where the teacher represents an authority figure; it was more colloquial and friendly”.

Peer Student 6: “when I am in front of the professor, I feel judged and I think I am asking ‘stupid’ questions”.

The data therefore suggest that the reduced hierarchical distance appears not only as a social condition, but as a key factor enabling participation within the ZPD, allowing students to externalise partial or uncertain understandings.

Finally, a fourth theme concerns the perceived potential and constraints of peer teaching for secondary school mathematics classrooms in relation to the conditions for activating the ZPD. Participants not only evaluate the effectiveness of peer teaching, but position themselves as future teachers, reflecting on the pedagogical conditions required to sustain meaningful peer-mediated learning.

Overall, participants express a positive attitude towards the methodology, especially in terms of participation, attention and dialogue among students, as highlighted as follows.

Peer Leader 2: “the main advantage, from a didactic point of view, is definitely the higher level of active participation, which is linked to the dynamic nature of the activity and leads to more solid learning”.

Peer Student 7: “it is a very good methodology to keep students’ attention more active, since questions and contributions prevent them from switching off. Being with people at the same level and focusing on one’s own group encourages greater commitment”.

Several responses also suggest that peer interaction may support students who feel intimidated by the teacher, allowing them to express doubts more freely within a less hierarchical environment.

Peer Leader 1: “it would have many positive effects on those who are interested but feel intimidated by the teacher, because they might feel more at ease in a group of peers and express doubts and insecurities more easily”.

At the same time, the same participants identify important constraints and conditions for implementation in real school contexts. In particular, some students emphasise the role of class maturity and social dynamics, noting that group interactions may be affected by issues such as immaturity, exclusion, or conflict.

Peer Student 8: “I think it may work better or worse depending on the class; upper secondary school is a particular moment, where there is still a lot of immaturity among students and this could create management difficulties”.

Some responses highlight the risk that the peer leader may be negatively labelled or marginalised, which could undermine the collaborative process.

Peer Student 4: “on the one hand this kind of activity allows students to interact and may reduce the problem of closed ‘cliques’; on the other hand, it could generate more conflict if the team leader is seen as a ‘geek’ to be rejected or bullied”.

For this reason, future teachers insist that peer teaching cannot replace the teacher but requires to be integrated within a clear structure of supervision and epistemic guidance.

Peer Student 2: “it could work as a support to the teacher, not as a replacement. There is a risk of losing important information that a teacher considers crucial but a student does not”.

Accordingly, participants emphasise the importance of specific preparation for peer leaders, thoughtful group composition, and the presence of a clear instructional framework. As one peer leader notes:

Peer Leader 3: “this methodology can work in upper secondary classrooms, with some precautions: a basic introduction to the topic for everyone, and careful group formation to create conditions in which all students can participate. Choosing the team leader is crucial: they do not have to be the best student, but someone who is willing to step forward and engage”.

These elements can be interpreted as attempts to create the conditions under which peer-mediated interaction can effectively support students’ engagement within the Zone of Proximal Development. Several respondents finally note that, unlike in the university context of the experiment, in a school class not all students may be equally motivated, which could limit the impact of the activity if these conditions are not met.

Taken together, these four themes depict peer teaching as an experience that pre-service mathematics teachers perceive as dialogic, collaborative and emotionally safer than traditional lectures, and as a potentially valuable tool for secondary school mathematics, provided that the teacher maintains a clear supervisory role and that social and organizational conditions are carefully designed.

DISCUSSION

The findings of this exploratory study can be read coherently within the Vygotskian framework that guided our design. The four themes emerging from the analysis, dialogic and collaborative work, the “semi-expert” role of the peer leader, the emotional climate and hierarchy, and the perceived conditions for transfer to secondary school, all speak, in different ways, to how the ZPD is (or is not) activated in concrete teaching–learning situations. While these themes resonate with established findings on peer learning, the specificity of this study lies in how these processes are experienced

and interpreted by pre-service mathematics teachers, who position themselves simultaneously as learners and future teachers. This aligns with research on exploratory talk and dialogic learning, which emphasises the role of interaction and discourse in supporting meaning-making (Mercer, 2002). The first point concerns the nature of the learning environment described by the participants. The peer teaching activity is consistently represented as a space where understanding is constructed through dialogue, shared questions and progressive clarification within the small group. This picture aligns closely with a sociocultural view of learning, in which cognitive development is rooted in social interaction and in the collective negotiation of meaning rather than in the individual processing of ready-made explanations. From a Vygotskian perspective, the dense circulation of questions and the reformulation of ideas observed in the data can be interpreted as indicators of a ZPD that is "distributed" across the group: it is not only the peer leader who supports the others, but the interaction itself that becomes a mediating structure, allowing students to move from a partial, intuitive grasp of DBR to more stable and articulated understandings. In this sense, our results seem to confirm that peer teaching, even in the "light" version proposed in this course, can function as a dialogic learning environment in which language effectively plays the role of psychological tool that Vygotsky attributes to collective discourse. Within this framework, the specific role of the peer leader deserves more fine-grained reading. The data do not portray a "little teacher", but rather a slightly more expert peer, who has just accessed the content through the DBR seminar and revisits it together with the group. This intermediate position closely mirrors the figure of the "more competent other" in the ZPD: someone who is not qualitatively different from the learner, but who is one step ahead in the process of conceptual appropriation. Interestingly, the leaders explicitly state in their answers that they themselves are moving within a zone of proximal development: facing the group's doubts forces them to clarify what was not yet stable in their own understanding. This suggests a dynamic of "crossed ZPDs", in which mediation is not one-directional but produces simultaneous advances for both leaders and peers. This aspect goes beyond a traditional interpretation of the ZPD as a unidirectional relation between expert and novice, suggesting instead a more dynamic and reciprocal form of mediation. From the perspective of mathematics teacher education, this is a relevant element: participants experience first-hand what it means to provide contingent scaffolding (reformulating, adjusting the level of detail, supporting without replacing) and, at the same time, what it means to learn through such scaffolding. A further relevant element concerns the emotional climate and the reduction of hierarchical distance, which participants associate with a greater willingness to participate. This finding is consistent with dialogic approaches that highlight how symmetrical interaction and a sense of epistemic safety can support students in expressing partial or tentative ideas (Mercer, 2002). In contrast, more traditional lecture settings are often perceived as limiting participation due to the presence of evaluative authority. Finally, participants' reflections on the potential transfer of peer teaching to secondary school highlight both opportunities and constraints. On the one hand, increased participation and engagement

can be interpreted as forms of more active involvement in shared practices (Lave & Wenger, 1991). On the other hand, concerns related to classroom management, student maturity, and group dynamics suggest that such approaches require careful orchestration of social and organisational conditions (Cohen et al., 2018). In this respect, the teacher's role remains central, not as the sole source of knowledge, but as an organiser and epistemic guide within a system of interacting learning processes.

While the results are broadly consistent with existing literature on peer-led learning, the contribution of this study lies in its focus on pre-service secondary mathematics teachers and on how they interpret such experiences from a dual perspective, as both learners and future teachers. In particular, participants not only describe the benefits of peer teaching, but also critically reflect on its feasibility and conditions for implementation in school contexts. This dual perspective offers a more nuanced understanding of peer-led learning within mathematics teacher education and highlights the importance of considering both pedagogical potential and contextual constraints.

CONCLUSIONS

This exploratory study investigated how pre-service secondary mathematics teachers experience a peer teaching activity and how they envision its use in school contexts. The findings suggest that peer-led interactions can create dialogic and collaborative learning environments, in which both peer leaders and group members engage in shared processes of meaning-making. While consistent with existing literature, the study contributes by highlighting how pre-service teachers interpret such experiences from a dual perspective, as learners and future practitioners, and by pointing to the emergence of interconnected zones of proximal development within peer interaction. At the same time, the results underline that the effectiveness of peer teaching cannot be taken for granted, but depends on carefully designed social and organisational conditions, as well as on the active role of the teacher as epistemic guide. Given the small-scale and exploratory nature of the study, further research is needed to examine how such approaches can be adapted and sustained in diverse mathematics classroom settings.

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TWG6: Equity and affect in university mathematics

Students' motivation in a course design with a focus on formative assessment

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This study investigates how three newly designed learning opportunities with a special focus on formative assessment affect engineering students' motivation in introductory university mathematics courses. Rooted in self-determination theory, the research examines effects of formative assessment throughout the semester on autonomy, competence and social relatedness. Semi-structured interviews with students of varying performance levels and usage intensities reveal that most perceive the opportunities positively, valuing the weekly structure, feedback, and low-stakes performance checks. Results also suggest increased intrinsic motivation for some of the students, while others report they are only using the learning opportunities for extrinsic reasons.

Keywords: formative assessment, student motivation, tertiary education, teaching innovation.

THEORETICAL BACKGROUND

Mathematics teaching at colleges and universities is often criticized for a lack of student participation in traditional courses (Artemeva & Fox, 2011). Particularly in lectures, but also to some extent in tutorials, most students are often passive, especially when tutorials are designed in such a way that solutions prepared at home are presented by the lecturer or very few students (Wlassak, 2026). This is problematic because students can only refer to model solutions or subjective perceptions of their own understanding of the course as feedback on their own performance.

In mathematics courses for engineers, summative assessment dominates. At the end of a course, assessments, often in the form of high-stakes tests, are carried out to determine learning success (Alpers et al., 2013). In this assessment cycle, students tend to memorize known solutions rather than trying to understand the concepts involved (Gueudet, 2008; Iannone & Simpson, 2011; Thurston, 1994). Formative assessment, on the other hand, aims to improve the learning process by providing targeted feedback (Ni Fhloinn & Carr, 2017). Since positive effects of formative assessment have been reported for school mathematics in terms of both student performance and motivation, a transfer to tertiary education appears promising (Moyosore, 2015). However, there are also findings that attribute the positive effect of formative assessment not only to the concept itself but also to its specific implementation (Boström & Palm, 2023). Therefore, we will present the design of the learning program – as well as the formative assessment therein – created for this study.

In engineering degree programs, many dropouts are attributed to a combination of cognitive and motivational variables (Heublein, 2014). This study reports on a teaching experiment with engineering students in their first year of study, in which regular teaching was designed in such a way that it both enabled a large amount of student activity in the tutorials and incorporated formative assessment into several course elements to particularly stimulate and simultaneously reward student activity.

To analyze the influence of the additional learning opportunities on student motivation, self-determination theory (SDT) (Ryan & Deci, 2020) was used. This theory enables us to understand motivation in an educational context. SDT distinguishes between two types of motivation: intrinsic motivation, which occurs when the content itself is perceived as interesting or challenging or the engagement with the content is perceived as meaningful, and extrinsic motivation, which occurs when the content is engaged with primarily to avoid negative consequences or generate positive consequences (Deci and Ryan, 2020).

In the context of SDT, motivation can arise when the three basic psychological needs for autonomy, competence, and social relatedness are met. Autonomy refers to the feeling of being able to explicitly control one's own actions. In the literature, this basic need is also related to choices for one's own learning (Deci & Ryan, 1985). Competence refers to feelings that arise when one's own abilities can be successfully demonstrated. Success does not necessarily have to be measured against a specific standard; instead, one's own development and the achievements of other learners can also be used as a reference (Elliot, McGregor, & Thrash, 2002). The need for social relatedness is fulfilled when learners perceive themselves as part of a social group.

SDT has been used to measure motivation for mathematics at university in various studies, e.g., by Liebendörfer (2018) and Wang et al. (2022). In contrast to these studies, which focus on motivation in regular mathematics courses that feature summative assessment, the present study explicitly examines the effects of elements of formative assessment on student motivation.

This raises the question of the extent to which the designed learning opportunities are also viewed positively by students especially towards their motivation towards mathematics. The following two questions will therefore be answered below:

1. How do engineering students describe their motivation towards mathematics in a course design with elements of formative assessment?
2. What obstacles prevent students from using (non-compulsory) activities with formative assessment and which alternative design options can be identified based on this?

To answer these questions, three additions to a regular mathematics course were developed and tested during the winter term 2025/26 and the summer term 2026 in the two courses "Engineering Mathematics I" and "Engineering Mathematics II" at Nordhausen University of Applied Sciences, a smaller German university with a

technical focus. Feedback from students was obtained using a qualitative research approach in the form of interviews.

DESCRIPTION OF THE COURSE

Traditional elements of teaching in the introductory phase of studies include lectures, tutorials, exams, and homework assignments (Kyle, 2015). These are usually set and marked on a weekly basis and determine admission to the exam. Correction is a form of formative assessment, as students receive individual feedback on their own solutions and can use this in their learning process (Haenze et al., 2013). From a theoretical perspective, this can be interpreted as enabling students to experience competence, especially when corrected solutions are perceived as a review of their own performance.

At Nordhausen University of Applied Sciences, there are no admission requirements for taking exams in mathematics courses, so submitting homework is optional. Instead of the traditional approach where homework task solutions are just presented in the tutorials, we rewarded the students in such a way that reaching certain point thresholds resulted in a bonus for the exam of up to 5% of the total score. Thus, an external incentive to take advantage of this opportunity was deliberately created, which from a theoretical standpoint might be a necessary design choice, because engineering students might interact with mathematics for more pragmatic reasons than mathematics students (Kolbe, 2025). To receive the full bonus, three-quarters of all tasks had to be solved correctly, which meant that students had a certain degree of freedom in choosing which tasks to complete. This, in turn, could have a positive effect on students' sense of autonomy. Furthermore, the use of such tasks contributes to networking among students in learning groups (Haak et al., 2021). Below, the tasks in this additional learning opportunity are referred to as homework.

The tutorials were designed so that students could ask questions about the lecture content at the beginning and subsequently solve tasks independently or in small groups. Typically, students had around 60 minutes to solve about four tasks. As with the homework assignments, there were more computational than proof-based tasks. After the working time, students could present their solutions. In contrast to a traditional approach, where students might only receive feedback, this was also rewarded with a bonus of up to 5% in the final exam, whereby students had to present their solutions at least three times in the semester for the full credit. The opportunity to work together and the freedom to choose which tasks to work on or present are expected to have positive effects on the basic needs of social relatedness and autonomy. The corresponding tasks are referred to below as classroom assignments.

As a final newly created opportunity for students to receive feedback, two midterm exams were held in the first semester after approximately one-third and two-thirds of the semester had passed. In the second semester, this took place halfway through the semester. The midterm exams were 45-minute tests that covered the content in the course so far. Apart from time restrictions, the tasks were comparable to those in the

final exam, which was also communicated to the students. These midterm exams were graded, allowing students to see which content they had already mastered well and where there was still room for improvement. Depending on the proportion of tasks solved correctly, students could earn a bonus of up to 5% for the final exam for each midterm exam. The midterm exams are expected to have a positive effect on students' experience of competence, as they can demonstrate their own abilities in a low-stakes test.

Overall, students were able to achieve a bonus of up to 20% in the first semester and up to 15% of the total points in the respective final exams in the second semester. The achieved bonuses averaged 13.2% (n=53) in the first semester and 11.4% (n=38) in the second semester. The difference in the average bonus stems from the higher number of midterm exams in the winter semester. In the final exam, 50% of the total points had to be achieved to pass, with the bonus helping to exceed the pass mark (see Table 1).

Number of students passing the exam	1st semester	2nd semester
Passed (including bonus)	41	32
Passed (not including bonus)	33	25

Table 1: Exam results with and without the bonus points

The bonus points achieved by the students can be seen as an initial indication of the fruitful use of the homework, classroom assignments, and midterm exams. The extent to which this is also perceived as such by the students and whether the additional learning opportunities only generate extrinsic motivation through the bonus will be clarified below.

DATA COLLECTION

To record the students' opinions, the lead author conducted semi-structured interviews (n=6) lasting between 25 and 45 minutes using a qualitative research approach (Galletta, 2013). To obtain as broad a picture as possible of the use of the learning opportunities, students were selected based on two characteristics. First, they were divided into three groups according to their performance. An entrance exam was conducted at the beginning of the semester, and the results of the two final exams (without bonus) were considered. Students in the top 20% in all three aspects were classified as high-performing students, those in the middle 40% in all three aspects were classified as average students, and those who were in the bottom 20% in all three aspects were classified as low-performing students. Although not all students could be classified this way, for those who could, their level of performance was quite certain.

The second characteristic was the intensity of use of the midterm exams, homework, and classroom assignments. Students who did not use at least one of these opportunities in a semester, or who used at least two of the resources to no more than half their full extent (e.g., less than 6 of the 12 homework sheets), were classified as low-intensity users, while the other students were classified as high-intensity users. Based on these

characteristics, a matrix of six possible types of students was created, which were selected for the interviews based on availability. The interviews discussed the designed learning opportunities and the perceived effects of the formal assessment.

Qualitative content analysis was used to analyze the interview transcripts (Kuckartz & Rädiker, 2023), focusing on the use of formative assessment and their corresponding effects on student motivation. This was done using a deductive analysis that focused on intrinsic or extrinsic motivation as well as the three basic needs described in SDT.

RESULTS

Effects on intrinsic and extrinsic motivation

Overall, the students rate the three additional learning opportunities and the effects on their motivation positively. In doing so, they focus primarily on the weekly structure provided by homework and classroom assignments with their integrated feedback. However, it is noticeable that the learning opportunities create different types of motivation. Some students report the motivation tends to be primarily based on external incentives, i.e., that to avoid failing the final exam.

> Interviewer: What is your goal when you use the additional learning opportunities?
>
> Student 1: The goal is simply to earn these extra points. If I earn more points, I need fewer points on the exam and can pass more easily.

But this is not the case for all students, because there are also reports of positive effects of presenting solutions in the tutorials and getting feedback immediately.

> Student 2: I think it was precisely through the assignments that I wanted to learn to do mathematics myself.
>
> Interviewer: Assuming there was no bonus, would you use the learning opportunities differently?
>
> Student 2: I don't think it would make any difference to me; I would have done the assignments either way.

Students recognize the benefit of individual feedback on their own learning process, which seems to create intrinsic motivation for continuous learning in mathematics. These students report more lasting learning effects from using the additional course opportunities with integrated feedback than from simply studying before the exam.

> Student 3: In retrospect, I also found it better than other courses where we didn't have that, because you were really forced to look at it and understand it. In other modules, everything came at the end of the semester, and it was almost overwhelming. So, you basically had to repeat all the material again without having learned everything properly in between, and in math, the effort at the end was much more relaxed before the actual exam.
>
> Student 4: […] the fact that you have something to do every week on a continuous basis and not just once in a while or something like that, you're more likely to stick

with it. It helps you understand the whole thing a little better, that you don't have to do everything at once.

When analyzing the different performance levels of the students and their answers, low-performing students focus on the external incentive more than average- or high-performing students. This might also be interpreted as a pragmatic interaction with mathematics.

The value of formative assessment for the basic need of competence

Students emphasize the importance of the *corrected* homework. Especially in preparation for the final exam, they use the corrected solutions to identify which content was particularly important and to learn from their mistakes. This can be interpreted as a positive effect on the basic need for competence. Students use their corrected solutions as a focused starting point for improving on their abilities instead of just memorizing model solutions. Because of the feedback they see how to improve their performance.

Student 5: It's also easy to get feedback and be evaluated. Ah, the corrections. The corrections also make a big difference, because they are all points that you can work on, even for the exam. (...) These exercise sheets really helped me a lot.

For the midterm exams, both lower-performing students and students with average performance unanimously reported that although the bonus motivated them to prepare, they had little or no opportunity to do so due to time constraints and so did not view the midterm exams as a possibility to demonstrate their competence. The students in question used the midterm exams to get feedback on their current performance level and to find out what a university-level exam looks like.

Interviewer: How did the midterm exams benefit you?

Student 6: So I had a rough idea of what the actual exam would be like and how I would need to prepare. You knew what to expect and how it would work. That's really good because it takes away the fear of the exam for many [students]. At least for me it felt like it did.

High-achieving students, on the other hand, saw the midterm exams as practice for the final exam and prepared for them in a similar way to how they would prepare for the final exam. These students found the opportunity to demonstrate their own competence during the semester to be very motivating.

The interaction between formative assessment and social relatedness

High-intensity users unanimously describe a positive effect of homework and classroom assignments on social networking. Students feel motivated to work on tasks in a learning group and report synergy effects when working together due to their different educational backgrounds so that even if they received negative feedback they could help each other improve. They also suspect that networking in learning groups

would not have taken place without homework and classroom assignments. Networking is described as having the effect of ritualizing task completion, i.e., students develop typical working patterns, such as initially working on tasks alone, then reviewing the lecture content together, and finally solving the tasks that could not be solved alone and discussing the given feedback together with the heterogeneity of a learning group being perceived as particularly helpful in collaborative work.

> Student 5: We mostly did them [the assignments] in a study group at college and made really good progress there. […] We were able to help each other because I was in a group with [student X] and [student Y]. [Student X] came directly from high school. I had previously completed vocational training. [Student Y] had recently completed his technician training, and I think it's really good that everyone knew something different and we could help each other. That made everything a lot easier throughout the semester.

Other responses show that networking in groups and social relatedness may even be a necessary condition for the long-term, regular completion of the assignments. All the students interviewed reported that they were motivated at the beginning of the first semester to work on the assignments in detail and on their own. Not all students were able to maintain this motivation without a learning group throughout the semester especially when receiving negative feedback. They report that the overall workload for all courses is relatively high and that the voluntary nature of the homework means that they are not completed or at least not submitted.

Criticism from students

It is apparent that students who value that feedback for improving their abilities are in need of a quick return of their corrected solutions for the best use. If more than one exercise sheet is still held by the corrector students report the feedback loop is not designed optimally, because students want to improve their competence with the feedback from the homework.

> Student 3: The only thing I might criticize, which wasn't so good, is that, I think it was before two midterm exams, several exercise series had not yet been returned by the corrector. And those were just a few tasks that would have been quite important for understanding.

It also became clear that students did not present their classroom assignment solutions because they were worried that their solutions might be wrong or simply did not want to present solutions in front of their fellow students during the tutorial. This could also be a sign of lacking social relatedness. For the midterm exams, the main criticism is the amount of work involved in multiple performance reviews during the semester, which could have a negative effect due to competition with other modules. In general, the amount of time required to take advantage of *all* the opportunities is viewed critically in some cases and the autonomy of choosing to skip (some of) the optional parts of the course is not seen by these students.

SUMMARIZING DISCUSSION

This study presented a teaching innovation consisting of three optional course opportunities, each of which offered elements of formative assessment as well as a bonus for the final exam. Interviews were conducted with students to determine how and why exactly students did (not) take advantage of these opportunities and how they impact their motivation throughout the semester.

It was found that, in addition to the external motivation created by the bonus for the exam, students primarily perceive learning-related benefits from the courses, as they find pure memorization before exams in other modules to be detrimental to their understanding. Since the students report a meaningful interaction with the content through a feedback loop, there seems to be a positive effect on intrinsic motivation for most of the students. Positive influences on the experience of competence (Ryan & Deci, 2020) also appear to be noticeable. The correction and thus individual feedback for all three opportunities enables students (when using them) to work with the corrected documents afterwards and thus improve specifically on their abilities. High-achieving students find it motivating to be able to test their performance during the semester and thus assess their own competence.

Similarly, networking among students in learning groups contributes to their motivation to use the learning opportunities and to engage with mathematics in general. The results indicate that social relatedness and motivation throughout the semester are influencing each other. If students are organized in groups, they tend to use all opportunities whereas students that work on their own lose motivation (especially when receiving rather negative feedback). So, linking the designed learning opportunities with the management of study group formation seems promising. For example, homework could be submitted in groups, with each participating student having to write down at least one task solution themselves to ensure that all students contributed to the submission. To a certain extent this may encourage social relatedness for students (Deci & Ryan, 1985).

Although homework, classroom assignments as well as the midterm exams were all optional for students, they did not recognize the advantage of autonomously choosing to only engage in parts of the formatively assessed parts of the course.

Some students perceived the classroom assignments as demotivating, mainly because they were afraid of presenting in front of their fellow students or of making mistakes in their work, which would be particularly noticeable during the presentation. As an alternative, one could consider submitting classroom assignments to alleviate these concerns while still maintaining the basic idea.

Overall, it can be summarized that students perceive the teaching innovation presented as positive and recognize advantages over traditional course design, which relies less on student activity. The results suggest that high- and average-performing students are mostly intrinsically motivated because of the benefits formative assessment provides for their learning process, whereas the low-performing students are rather extrinsically

motivated by the bonus. Whether this is just a correlation or caused by a higher intrinsic motivation remains an open question. Given the small sample size and the limitation of the data set to one university location, it is not possible to generalize the results, but it could be an interesting question for further investigation, especially whether the results carry over to courses with a greater number of students.

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The role of learning strategies and performance feedback for dropout intention in mathematics-intensive study programmes

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Dropping out of mathematics-intensive programs is common and costly. This longitudinal study, involving 134 participants, investigates the interplay between mathematics-specific learning strategies, performance feedback, and dropout intentions throughout a semester using linear mixed models. Better exam results reduced dropout intention more steeply. Dropout intention was negatively associated with resisting frustration and building connections. High time investment was associated with an initial struggle, yet predicted a favourable post-exam decrease, while repeating significantly worsened the trajectory of dropout intention over time. These findings highlight the critical role of mathematics-specific deep-processing and self-regulatory strategies for dropout prevention.

Keywords: stochastics, dropout intention, learning strategies, linear mixed models.

INTRODUCTION

Dropping out of a university study programme entails significant personal, institutional, and societal costs (Faas et al., 2018; Schneider & Yin, 2011). Internationally, dropout rates are consistently and notably high in mathematics-intensive study programmes (Chen, 2013; Heublein et al., 2020; Hoed et al., 2018), a phenomenon largely driven by performance issues and motivational difficulties (Chen, 2013; Geisler et al., 2023; Messerer et al., 2024). This high attrition rate highlights the challenging transition phase from school to university in mathematics-intensive study programmes as a crucial period for research (Di Martino et al., 2023; Gueudet, 2023).

Most university mathematics contexts demand a high proportion of self-study. Consequently, the ability to engage in effective self-regulated learning becomes paramount for success (Göller et al., 2024). The specific learning strategies students employ can directly influence their academic performance and study satisfaction, which in turn are protective factors against dropout (Geisler et al., 2023; Liebendörfer et al., 2022).

Given the specialised demands of university mathematics – such as the central role of proof and rigorous argumentation – the explanatory power of general strategies is limited (Büchele, 2023; Griese, 2017). Therefore, this paper adopts a framework of mathematics-specific learning strategies (Liebendörfer et al., 2022) as its primary theoretical lens. Building on this foundation, the study investigates how these specific strategies relate to dropout intention. The following sections will first introduce dropout intention and this theoretical framework of mathematics-specific learning strategies, followed by an empirical analysis and discussion of how these strategies influence students' intentions to leave their studies.

THEORY

Dropout and Dropout Intention

As actual dropout is difficult to track precisely, many studies utilise dropout intention as an empirically validated indicator of potential study termination (Breetzke & Bohndick, 2024; López-Angulo et al., 2023; Véliz Palomino & Ortega, 2023). Dropout intention is a longer-term process characterised by the perception of an insufficient personal fit with the study program, thoughts of quitting studies, and deliberation toward a final decision (Bäulke et al., 2022). Hence, high dropout intention reflects perceived dissatisfaction and incompatibility with the study programme, and is known to predict actual dropout, though with a typically small effect (Findeisen et al., 2024).

Mathematics-Specific Learning Strategies in the LimSt questionnaire

Learning strategies are behaviours and thoughts used to influence knowledge encoding (Weinstein & Mayer, 1986). Such learning strategies are traditionally classified based on information processing theory into *cognitive strategies*, detailing different ways of processing information (such as *rehearsal, organisation, and elaboration*), and *resource management strategies* aiming at regulating *internal* (e.g., time and effort) and *external* (e.g., peer learning) resources (Wang et al., 2023). While such strategies generally show positive correlations with achievement (Credé & Phillips, 2011), these effects are often inconsistent in mathematics-intensive programs, underscoring the need for subject-specific approaches (Büchele, 2023; Griese, 2017).

The LiMSt questionnaire (Liebendörfer et al., 2022) builds on this traditional classification and integrates the unique cognitive demands of university mathematics – such as the focus on proof but also the necessity of acquiring efficient procedural routines – to tailor it to tertiary mathematics learning. As rehearsal strategies, *practising* (carrying out procedures to build routines) is differentiated from simple *repeating* (memorising rules). Organisation strategies include *using proof* and *simplifying* difficult content. Elaboration strategies include general *building connections* alongside mathematics-specific *using examples* (e.g., to illustrate definitions or propositions) and *connecting to practice* (to link mathematical concepts with non-mathematical content). Regarding internal resource management strategies, *resisting frustration* is distinguished from mere *time investment*. Finally, *peer learning*, a common practice in university mathematics learning (Gildehaus et al., 2025), is included as an external resource strategy. These mathematics-specific learning strategies have been shown to predict academic achievement in mathematics-intensive study programmes (Liebendörfer et al., 2022) and might therefore also affect dropout intention (Geisler et al., 2023).

RESEARCH QUESTIONS

The primary aims of this study are to longitudinally track students' mathematics-specific learning strategies and dropout intentions across a semester and to determine how these strategies relate to both the overall level of dropout intention and its change

over time. In this context, performance feedback (from achievement situations, such as examinations) may influence students' learning strategies and dropout intentions. To better understand this complex interplay of learning strategies, performance feedback, and dropout intentions, the following research questions will be analysed:

RQ1: How does students' dropout intention change over the course of a semester in mathematics-intensive study programmes, and how does performance feedback influence dropout intention over time?

RQ2: Which mathematics-specific learning strategies are associated with students' overall level of dropout intention?

RQ3: Which learning strategies help to reduce or, conversely, worsen the trajectory of dropout intention during a semester?

METHODS

Study Design and Sample

This study employed a longitudinal design to track students' learning strategies and dropout intentions at two measurement points: the midpoint (T1) and the end (T2) of a semester, with T1 occurring approximately 1 week before a midterm examination. The sample consisted of students enrolled in mathematics-intensive study programmes at an Austrian university. The raw data comprised a total of 196 observations across $N = 134$ participants, of whom 114 (64 female, 46 male, 2 diverse, 2 without specification) participated at T1 and 86 (35 female, 51 male) at T2. The study programmes participants were enrolled are: International Business and Economics ($n_{T1} = 39$, $n_{T2} = 45$); Applied Informatics ($n_{T1} = 32$, $n_{T2}=17$); Robotics and Artificial Intelligence ($n_{T1} = 19$, $n_{T2} = 8$); Mathematics Teacher Education ($n_{T1} = 12$, $n_{T2} = 9$); Technical Mathematics ($n_{T1} = 5$, $n_{T2} = 4$); and Management, Economics & Data Sciences ($n_{T1} = 3$, $n_{T2} = 3$). The data were collected during definition-theorem-proof-based lectures that covered topics in stochastics and business-related mathematics.

Measures

At T1 and T2, participants completed a paper-based questionnaire that included items capturing dropout intention (Bäulke et al., 2022) and learning strategies (Liebendörfer et al., 2022), as listed in Table 1, on a 6-point Likert scale (1 = strongly disagree to 6 = strongly agree). Additionally, participants indicated their gender and provided informed consent, allowing their data to be matched with their midterm exam results and used for scientific analysis.

Statistical Analysis

A linear mixed model was employed in Jamovi to analyse the change in dropout intention over time. This method was selected to analyse the longitudinal change in dropout intention between two measurement points (T1 and T2), while explicitly accounting for the non-independence of repeated observations nested within individual students and within the several study programs. The analysis followed a hierarchical,

step-by-step procedure to build the fixed effects structure upon an established random structure. The baseline model established the basic random effects structure without any fixed predictors by including random intercepts for both individuals and study programmes to assess the variance partitioning. The variance components indicated high individual differences (Intra-Class Correlation (ICC) = .64) but very low study-programme differences (ICC = .00). Consequently, the random intercept for study programme was dropped from all subsequent models for model parsimony.

Variable	Items	α_{T1}	α_{T2}	Example Item
Dropout Intention	4	.76	.77	I'm considering quitting my studies.
Repeating	4	.74	.79	I try to retain the content better by repeating it several times.
Practising	3	.58	.71	I go through calculation paths again and again to get routine.
Building Connections	4	.81	.84	I try to understand how new content relates to what I have previously learned.
Using Examples	4	.70	.70	I create examples for propositions in order to understand the statement.
Connecting to Practice	3	.80	.80	With new content, I think about what it means in the real world.
Using Proofs	3	.71	.77	I try to understand proof methods in order to be able to use them in other tasks.
Simplifying	3	.66	.66	I try to simplify difficult content.
Time Investment	3	.58	.54	If I set myself a certain workload to study, I strive to get it done.
Resisting Frustration	3	.78	.78	Even if I get frustrated, I still keep on learning.
Peer Learning	2	.83	.80	I meet with other students to develop ideas for solutions together.

Table 1: Overview of the variables surveyed, with item counts and Cronbach's alphas

Building on this, the analysis followed a hierarchical, three-step nested procedure: Model 1 established the longitudinal structure with the variables time, gender, midterm, and the time-midterm interaction, to analyse RQ1. Model 2 added the main effects of all ten learning strategies to test RQ2. Model 3 added all time-learning strategy interaction terms (to test RQ3), predicting the change in dropout intention over time.

RESULTS

A summary of random components and model fit statistics is given in Table 2. The fixed and random effects of the analysed models explained between 64 % and 71 % of

the total variance in dropout intention, with the majority of variance existing between individuals. The baseline model established the random effects structure, confirming the necessity of a random intercept for individuals (persons) with a variance of 0.34 and a high intraclass correlation (ICC = .64). The subsequent introduction of time, gender, midterm examination, and time-midterm interaction as fixed effects (Model 1) explained 6 % of the total variance, while the introduction of the main effects of the mathematics-specific learning strategies (Model 2) significantly improved the explanatory power of the fixed effects to 21 %. The fixed effects of the final, most complex model (Model 3), which incorporated all strategy-time interactions, explained 23 % of the total variance in dropout intention.

	Baseline Model	Model 1	Model 2	Model 3
Between-person variance	0.34	0.30	0.21	0.24
Within-person variance	0.20	0.15	0.17	0.14
Intraclass Correlation (ICC)	.64	.66	.55	.63
Marginal R^2	.00	.06	.21	.23
Conditional R^2	.64	.68	.65	.71

Table 2: Summary of random components and model fit statistics for nested linear mixed models predicting dropout intention

The concrete results of the linear mixed model analyses are given in Table 3. Addressing RQ1, the average change in dropout intention between T1 and T2 was not statistically significant. However, this average trend was conditional on performance feedback, as the interaction between time and midterm was significant, indicating that for every standard deviation increase in midterm performance, the slope of dropout intention over time becomes 0.18 units more negative. As shown in Figure 1, students with better midterm results experienced a reduction in dropout intention, whereas the dropout intention of lower-performing students increased.

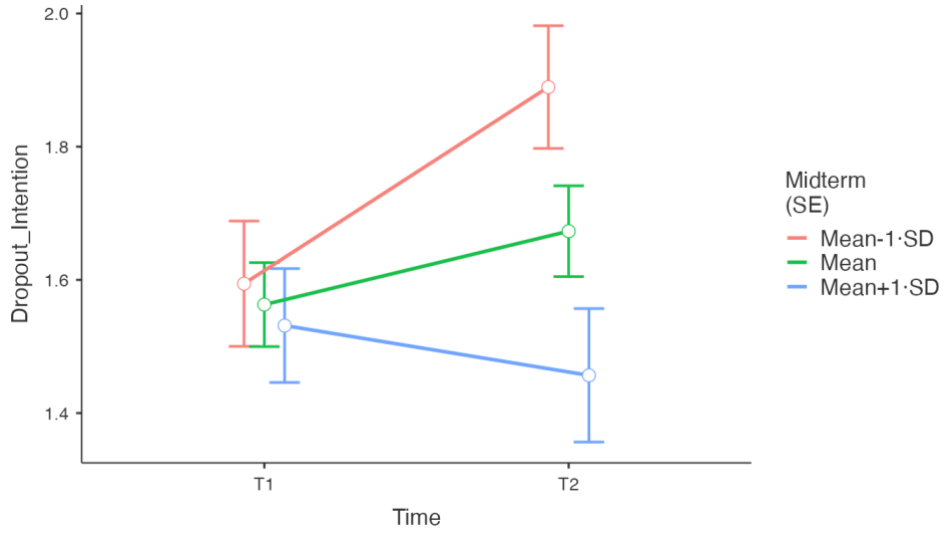

Figure 1: Visualisation of the Time × Midterm Exam interaction effect (Model 1)

	Model 1		Model 2		Model 3	
	β	SE	β	SE	β	SE
Intercept	1.47***	0.09	1.52***	0.08	1.49***	0.09
Time	0.11	0.07	0.07	0.07	0.05	0.07
Female	0.18	0.11	0.10	0.10	0.15	0.11
Midterm Exam	-0.03	0.06	-0.01	0.06	-0.03	0.06
Repeating			-0.13#	0.08	-0.22*	0.09
Practising			-0.04	0.07	-0.06	0.09
Building Connections			-0.15*	0.07	-0.11	0.08
Using Examples			-0.12#	0.07	-0.07	0.08
Connecting to Practice			0.08#	0.04	0.10*	0.05
Using Proofs			0.03	0.05	0.02	0.06
Simplifying			0.05	0.06	0.04	0.07
Time Investment			0.15*	0.06	0.23**	0.07
Resisting Frustration			-0.15*	0.06	-0.20**	0.07
Peer Learning			-0.06#	0.04	-0.01	0.04
Time × Midterm	-0.18*	0.07	-0.16*	0.07	-0.15*	0.07
Time × Repeating					0.31*	0.15
Time × Practising					-0.01	0.11
Time × Building Connections					-0.01	0.11
Time × Using Examples					-0.15	0.11
Time × Connecting to Practice					-0.03	0.06
Time × Using Proofs					0.02	0.08
Time × Simplifying					0.01	0.12
Time × Time Investment					-0.27*	0.13
Time × Resisting Frustration					0.13	0.10
Time × Peer Learning					-0.09#	0.05

Table 3: Overview of fixed effects (β) and standard errors (SE) across the nested linear mixed models predicting dropout intention. # $p < .10$, * $p < .05$, ** $p < .01$, *** $p < .001$.

Regarding RQ2, two resource management strategies showed significant associations: Resisting frustration was significantly associated with a lower overall level of dropout intention, indicating that every standard deviation increase in frustration resistance

predicts a 0.15-unit decrease in average dropout intention. Conversely, time investment showed a significant positive association with dropout intention, suggesting that students who invest more time might be struggling with the course demands (cf. Figure 2). The cognitive elaboration strategy building connections showed a significant negative association with dropout intention.

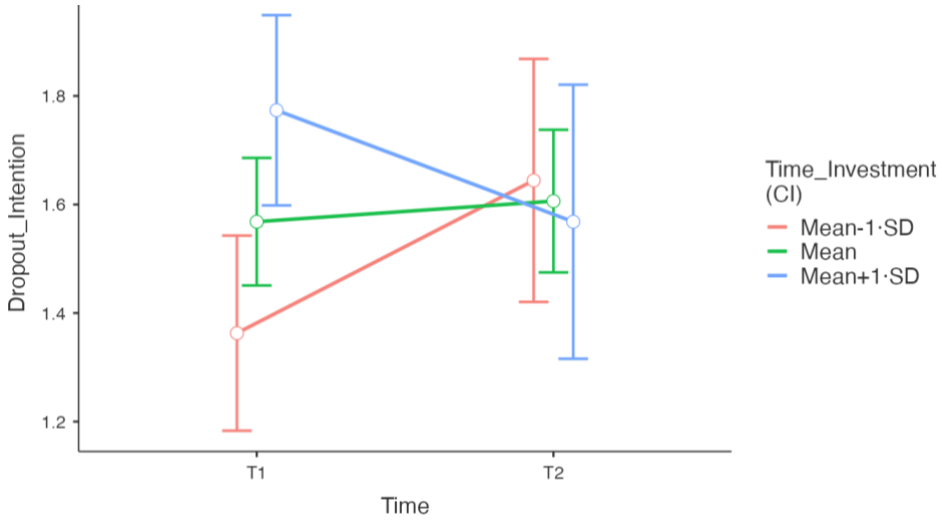

Figure 2: Visualisation of the Time × Time Investment interaction effect (Model 3)

Finally, concerning RQ3, two strategies were significantly associated with changes in dropout intention over time. First, the time-time investment interaction was significantly negative. While a higher time investment was associated with a higher dropout intention, every standard deviation increase in time investment predicted a 0.27-unit steeper decrease in dropout intention after the midterm (Figure 2). Second, time and repeating showed a significant positive interaction, indicating that every standard deviation increase in repeating was associated with a 0.31-unit steeper rise in dropout intention, resulting in a less favourable trajectory over the semester (Figure 3). The time-peer learning interaction was marginally significant, indicating that peer learning can contribute to a favourable trajectory in dropout intention.

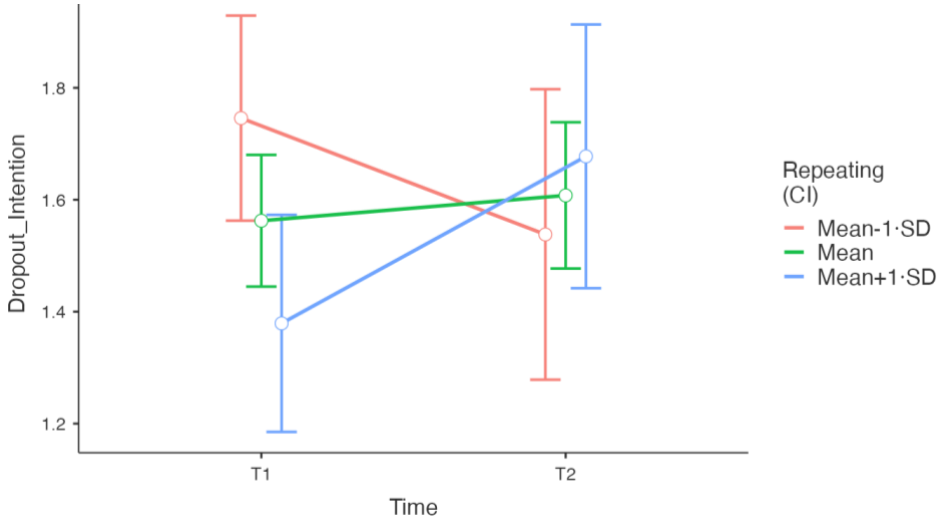

Figure 3: Visualisation of the Time × Repeating interaction effect (Model 3)

DISCUSSION

The primary objective of this longitudinal study was to analyse the role of mathematics-specific learning strategies and performance feedback in predicting the level and longitudinal change of dropout intention within mathematics-intensive study programs. The use of linear mixed models enabled the detailed decomposition of variance and the robust analysis of intra-individual change across two measurement points: T1 (pre-midterm) and T2 (post-midterm).

Addressing RQ1, the trajectory of dropout intention was influenced by the performance in the midterm exam. The significant time-midterm interaction suggests that academic success acts as a protective factor, leading to a greater reduction in post-examination dropout intention, which underscores the critical role of positive performance experiences in mitigating high attrition rates (Chen, 2013; Geisler et al., 2023).

In response to RQ2, the overall level of dropout intention was negatively associated with resisting frustration and building connections. This confirms the benefit of subject-specific strategies (Liebendörfer et al., 2022), possibly linking deeper engagement, persistence, and emotional regulation to perceived program compatibility (Bäulke et al., 2022; Büchele, 2023). High time investment was correlated positively with dropout intention, which may be attributed to an underlying initial struggle and higher costs (Breetzke & Bohndick, 2024).

For RQ3, specific strategies affected the change in dropout intention over time. The negative time-time investment interaction suggests that sustained effort benefits struggling students by resulting in a steeper post-midterm decrease in dropout intention. Crucially, the positive time-repeating interaction indicates that reliance on this surface-level rehearsal may not align with tertiary mathematics learning (Di Martino et al., 2023; Gueudet, 2023). Here, it significantly worsened the dropout trajectory. Conversely, peer learning showed a marginally significant, favourable effect, which aligns with existing qualitative findings (Gildehaus et al., 2025; Göller et al., 2024).

When interpreting the results, it should be considered that the study is limited by using dropout intention, not actual attrition (Findeisen et al., 2024), and relies on self-reported learning strategies and a single-university sample, restricting international generalizability. Nonetheless, the findings offer direct pedagogical insights, such as encouraging deep-processing strategies like building connections or peer learning and discouraging repeating. Future research should extend the two-timepoint design and seek replication in different institutional settings to enhance the generalizability of the findings.

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University students' attitude towards proof

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Keywords: Proof, Attitude, Emotions, Affect.

PROOF AND EMOTIONS

Proving has a central role in mathematics, and it has been studied extensively. However, prior research has focused heavily on the cognitive aspects of proof. The role of affective attributes of proof has been underrepresented in the literature even though affects play an important role in mathematics learning. This poster is part of a larger project concerning affective attributes of proof. Previously we have studied quantitatively how students' attitudes towards proof are related to their gender, prior skills and course performance (Häsä et al., 2023).

In this poster, we study qualitatively students' attitude towards proof and proving activities. We rely on the three-dimensional model for attitude (TMA; Di Martino & Zan, 2010), characterized by three dimensions: emotional disposition, perceived competence in mathematics, and vision of mathematics. With respect to the emotional disposition dimension, we are interested in both the emotional dispositions toward proof and the emotions experienced while engaging in proving activity. To address a wider spectrum of emotions than the TMA model allows, we use Pekrun's (2024) Generalized Control-Value Theory to expand the emotional disposition component. Pekrun divides emotions that occur in education into four dimensions. Achievement emotions are related to achievement activities or outcomes. Epistemic emotions are linked to knowledge acquisition and the cognitive properties of knowledge. Social emotions can be related to how others perceive the person or how the person feels about others. Finally, existential emotions concern one's existence, health and life.

Our research question is: How does a university student's attitude towards proof manifest when they recall their prior experiences or engage in proving activities?

DATA AND METHOD

Our data consists of interviews of students who took an introduction to proof course in a Finnish university. The data was collected in 2019, after the course had finished. Here we present a case study of one interviewee, Antti (a pseudonym). At this stage, our purpose is to investigate how the Generalized Control-Value Theory can shed light on the range of emotions students experience related to proving and how the theoretical tool may be combined with TMA. The interviews consisted of three parts. In part 1, the students recalled emotions they had encountered when constructing mathematical proofs, in part 2 they constructed a proof and discussed a proof that was shown to them,

and in part 3 the students described what had supported them in learning proving. We analysed the interview using deductive content analysis.

RESULTS AND FURTHER STEPS

In part 1 of the interview, we find the description of very different situations: Antti reports having experienced difficulties when working alone on proofs, while carrying successfully out proof-related activities with the help of peers. From his words, a generally medium level of perceived competence emerges: he is aware that he has difficulties carrying out the activity on his own but has no problem asking peers for help. This can be connected to a vision of proof as a process to be carried out, and if needed, in collaboration with others. More generally it indicates a view of learning mathematics as a social process. Concerning emotions, Antti recalled or showed mostly achievement emotions and epistemic emotions, but also social emotions were detected. Achievement and social emotions were present in the parts 1 and 3 of the interview when Antti talked about his past experiences. One example of a negative achievement emotion was frustration. Antti described how constructing a proof is difficult and takes time, especially when you are working alone. An example of a positive achievement emotion was contentment Antti expressed when recalling a proof he managed to complete. Social emotions appeared when Antti talked about working with other students and expressed gratitude to them for all the help he had received. Epistemic emotions were detected in part 2 of the interview when Antti interacted with proofs. Confusion was a typical epistemic emotion which emerged when Antti did not understand something or did not know what to do next.

This work constitutes a second step to our previous work in two ways: first, it aims to move from a quantitative analysis to a qualitative analysis. Second, it explores an integration between the TMA and Generalized Control-Value Theory. Additional integrations will be explored later on, for example to further characterize students' vision of proof. The next step in our study will be to analyse all the 33 interviews we have conducted and to give a more comprehensive description of students' attitudes towards proof.

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The role of basic needs satisfaction in mathematics tutorials for university exam performance

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University mathematics courses place high cognitive and emotional demands on students, making their motivational experiences critical for academic success. This study applies Self-Determination Theory to examine how competence, autonomy, and relatedness satisfaction within exercise tutorials predict exam performance. Longitudinal data from N = 130 students in three mathematics-intensive university courses were analysed using linear mixed models across two time points. Relatedness emerged with a consistent positive effect on exam performance while competence and autonomy showed no overall effects but displayed distinct, course-specific trajectories. Theoretical and practical implications are being discussed.

Keywords: dropout intention, learning strategies, university mathematics education.

INTRODUCTION

University mathematics courses are widely recognized as intellectually demanding learning environments that confront students with persistent cognitive challenges and frequent experiences of failure and uncertainty. These conditions often evoke intense negative emotions (Di Martino & Gregorio, 2019; Göller & Gildehaus, 2021) and can undermine students' confidence in their academic abilities (Gildehaus et al., 2024). Over time, such experiences may accumulate into broader patterns of disengagement, including feelings of not belonging, alienation from the mathematical community, or the perception of not being "a math person" (Lahdenperä & Nieminen, 2020; Solomon & Croft, 2016). Given the well-established link between affective variables and critical academic outcomes such as exam performance, study satisfaction, and dropout (Geisler et al., 2023; Göller & Gildehaus, 2024; Hailikari et al., 2008), understanding how to support students affectively in this demanding context is a pressing concern.

Typical university mathematics courses often rely on lectures, weekly homework assignments, and weekly exercise tutorials. (Gildehaus et al., 2025; Göller, 2021; Göller et al., 2024; Kim & Evans, 2025; Zakariya, 2021). In these tutorials, students solve mathematical problems aligned to the lecture and homework together with a tutor. Tutorials thus play a crucial role—not only for clarifying abstract content but also as a social and motivational hub in which students interact with peers and tutors, negotiate their understanding, and receive feedback that can affirm or undermine their sense of capability (Kim & Evans, 2025). Despite their centrality, we know surprisingly little about how the psychological experiences within tutorials shape students' longer-term academic development and exam performance.

To address this gap, the present study draws on Self-Determination Theory (SDT; Ryan & Deci, 2020), which posits that students' sustained engagement depends on the satisfaction of three basic psychological needs: competence, autonomy, and

relatedness. While SDT has been widely applied in educational settings, its role in the highly demanding and often emotionally charged context of university mathematics remains underexplored (cf. Liebendörfer & Hochmuth, 2015, 2017). In particular, it is unclear how students' moment-to-moment experiences of need satisfaction or frustration within tutorials contribute to their broader engagement and, ultimately, their exam performance.

The aim of this study is therefore twofold: (1) to examine the extent to which mathematics tutorials satisfy or frustrate students' basic psychological needs, and (2) to investigate how these need-related experiences predict students' engagement and subsequent exam performance in a challenging introductory mathematics course. By integrating SDT with the specific affordances and pressures of mathematics tutorials, this study offers new insights into the motivational mechanisms that underlie success and persistence in university mathematics. Such insights are crucial for designing learning environments that not only foster conceptual understanding but also support students' psychological well-being and academic success.

THEORY

Self-Determination Theory

SDT provides a comprehensive meta-theory of human motivation, positing that the quality of motivation (the degree to which it is autonomous or controlled) is the key determinant of optimal functioning, learning, and persistence (Ryan & Deci, 2020). The maintenance of high-quality, autonomous motivation is directly contingent upon the satisfaction of three innate and universal basic psychological needs: *Competence*, the feeling of mastery and successful growth; *Autonomy*, the sense of volition and ownership over one's learning; and *Relatedness*, the feeling of belonging and connection to others.

These basic psychological needs are considered to mediate the relationship between the learning situation and the student's motivational outcomes (Ryan & Deci, 2020). Empirical evidence in undergraduate mathematics repeatedly confirms that, especially, competence satisfaction is a dominant predictor of academic performance (Wang et al., 2022)

Exercise tutorials as a dynamic context for basic needs satisfaction or frustration

The exercise tutorial, as described earlier, may serve as a central social context where basic needs satisfaction or frustration is pronounced. Here, students actively engage with peers and teaching staff to discuss exercise solutions, receive critical feedback, and either affirm or lose confidence in their abilities (Kim & Evans, 2025). Furthermore, the use of learning groups, often organised around these tutorials, provides a context for cognitive exchange and emotional regulation (Gildehaus et al., 2025). From this perspective, the tutorial's social climate appears to be vital for relatedness, but also for students' perception of competence and autonomy. Accordingly, the weekly exercise tutorial is not merely a venue for discussing

solutions; it may be the primary motivational environment in which basic needs are challenged and affirmed.

Given the intense, feedback-driven structure of such mathematics tutorials, basic needs satisfaction is likely to be dynamic and context-dependent. Thereby, situational social experiences and performance in weekly assignments or midterm exams provide continuous feedback, which may serve as a significant input to students' basic needs. Therefore, the predictive power of basic needs satisfaction on subsequent exam performance may change significantly over the course of a semester. Furthermore, the instructional requirements of different mathematics courses may present varying levels of abstraction and rigour, which may pose different challenges to the basic psychological needs. These differing demands might moderate the temporal relationship between basic needs and exam achievement, making the motivational trajectory specific to the disciplinary context. We thus aim to establish different time points to investigate students basic needs satisfaction.

RESEARCH QUESTIONS

Hence, we longitudinally track university students' satisfaction of their basic psychological needs, with a focus on mathematics tutorials, and to determine how these needs relate to subsequent exam performance. As stated above, students' basic needs satisfaction may interact with the specific instructional context of their respective courses and change over time. Accordingly, this study aims to investigate this complex interplay of basic needs satisfaction, course context, and exam performance over the course of a semester. The following research questions will be analysed:

RQ1: How is the satisfaction of the three basic psychological needs (competence, autonomy, relatedness) related to exam performance in university mathematics courses?

RQ2: How does the influence of basic psychological needs satisfaction on exam performance change over the course of a semester?

RQ3: Is the change over time in the effect of basic psychological needs satisfaction on exam performance different across different courses?

METHODS

Study Design and Sample

The data for this study were collected during three mathematics courses at a small Austrian university. Course 1 covered topics in stochastics for engineering students, course 2 covered basic mathematics content for business students, and course 3 covered topics in stochastics for mathematics majors and higher-secondary preservice mathematics teachers. All these courses consisted of lectures, in which mathematical theory was presented on a virtual blackboard, weekly homework and exercise tutorials. For the exercise tutorials, students were required to prepare exercise solutions and select in an electronic system which exercises they had completed. In the exercise tutorials, the electronic system randomly chose students to present their solutions. The

quality of these presentations was graded (10 %) and contributed, together with the number of selected exercises (10 %), a midterm exam (30 %), and a final exam (50 %), to the final grade for the course.

Students' basic psychological needs satisfaction was tracked longitudinally at two measurement points at the middle (T1) and at the end (T2) of the semester. The raw data comprised a total of 194 observations, involving $N = 130$ individual students, of which 108 (61 female, 44 male, 2 divers, and 1 without specification) participated at T1, and 82 (34 female and 48 male) at T2. Study participants were enrolled in several mathematics-intensive study programs: International Business and Economics ($n_{T1} = 38$, $n_{T2} = 43$); Applied Informatics ($n_{T1} = 31$, $n_{T2}=17$); Robotics and Artificial Intelligence ($n_{T1} = 16$, $n_{T2} = 7$); Mathematics Teacher Education ($n_{T1} = 12$, $n_{T2} = 9$); Technical Mathematics ($n_{T1} = 5$, $n_{T2} = 4$); and Management, Economics & Data Sciences ($n_{T1} = 3$, $n_{T2} = 2$).

Measures

Participants completed a paper-based questionnaire at both time points. All three basic psychological needs were operationalised based on established scales (Longo et al., 2016; example items and further information in Table 1) and rated on a 6-point Likert scale (1 = strongly disagree to 6 = strongly agree). Thereby, participants were asked to refer to the exercise classes in their answers. Participants provided their informed consent, allowing their data to be coordinated with their exam results and used for scientific analysis.

Statistical Analysis

A series of linear mixed models was employed in a hierarchical, step-by-step procedure to analyse the longitudinal effects on exam performance, explicitly accounting for the non-independence of repeated measurements nested within individuals. Exam performance was z-standardised across the three different courses to account for the variation in content and assessments. The baseline model established the random effects structure, including a random intercept for both individuals (persons) and study programmes to assess the variance partitioning. While a large amount of the total variance in exam performance was attributable to differences between individuals (intraclass correlation, ICC = .46), the study programmes indicated very low differences (ICC = .00). Consequently, the random intercept for study programme was dropped in all subsequent models.

Building on this, the analysis followed a hierarchical, three-step nested procedure. Model 1 (main effects) established the overall fixed effects structure for RQ1, testing the main effects of the three basic needs, time, and course on exam performance. Model 2 (two-way interactions) incorporated two-way interactions between time and the basic needs to address RQ2. Model 3 (three-way interactions) added the three-way interactions of course and time with basic needs to address RQ3, investigating if the time-varying effect of basic psychological needs satisfaction differs across courses.

Variable	Items	α_{T1}	α_{T2}	Example Item
Competence	3	.70	.69	I feel I am very good at the things I do
Autonomy	3	.68	.77	I feel I'm given a lot of freedom in deciding how I do things
Relatedness	3	.52	.62	I feel the people I interact with really care about me

Table 1: Overview of the variables surveyed, with item counts and Cronbach's alphas.

RESULTS

A summary of random components and model fit statistics of the linear mixed models analyses is given in Table 2. The fixed and random effects of the analysed models explain between 43 % and 58 % of the total variance in exam performance, indicating that the majority of variance exists between individuals. The fixed effects of Model 1 (time, course, basic needs) explain 10 % of the total variance, the same as the additional introduction of the time-basic needs interactions (Model 2). The fixed effects of the final, most complex model (Model 3), which incorporates all course-time-basic needs interactions, explain 17 % of the total variance in exam performance.

	Baseline Model	Model 1	Model 2	Model 3
Variance (Person)	0.39	0.35	0.35	0.41
Standard Deviation (Person)	0.63	0.59	0.59	0.64
Intraclass Correlation (ICC)	.43	.41	.41	.49
Variance (Residual)	0.52	0.51	0.51	0.42
Marginal R^2	.00	.10	.10	.17
Conditional R^2	.43	.46	.47	.58

Table 2: Summary of random components and model fit statistics for nested linear mixed models predicting exam performance.

The concrete results of the linear mixed model analyses are given in Table 3. Addressing RQ1, only relatedness showed a significant main effect on overall exam performance (across both time points and all courses). This indicates that, on average, students who reported higher feelings of belonging and connection performed better on exams. Conversely, the main effects for competence and autonomy were not statistically significant.

Regarding RQ2, the two-way interactions between time and basic needs in Model 2 showed no significant effect, indicating that the effects of basic needs on exam performance did not change over the course of the semester.

Finally, concerning RQ3, the three-way interaction terms revealed significant and distinct course-specific motivational trajectories. First, the two-way time-competence

interaction was almost significant in Model 3 (β = 0.28, p = .053) and indicates that a higher overall perceived competence is beneficial for improving performance from midterm to final exam, as shown in Figure 1. The three-way interaction of course, time,

	Model 1		Model 2		Model 3	
	β	SE	β	SE	β	SE
Intercept	-0.00	0.08	-0.00	0.09	0.02	0.10
Course 1 (Engineers)[a]	0.26	0.23	0.26	0.23	0.05	0.28
Course 2 (Business)[a]	-0.00	0.23	-0.01	0.23	-0.17	0.27
Time[b]	-0.02	0.12	-0.03	0.12	0.14	0.15
Competence	0.13	0.09	0.14	0.09	0.05	0.10
Autonomy	-0.02	0.09	-0.02	0.09	-0.19	0.11
Relatedness	0.29**	0.09	0.30**	0.09	0.23*	0.11
Time × Competence			0.15	0.14	0.28#	0.14
Time × Autonomy			-0.02	0.15	-0.34#	0.18
Time × Relatedness			0.04	0.15	-0.08	0.18
Course 1 × Competence					0.50#	0.27
Course 2 × Competence					0.40	0.27
Course 1 × Autonomy					0.23	0.30
Course 2 × Autonomy					0.40	0.29
Course 1 × Relatedness					-0.21	0.31
Course 2 × Relatedness					0.08	0.29
Course 1 × Time × Competence					-0.82*	0.39
Course 2 × Time × Competence					-0.86*	0.38
Course 1 × Time × Autonomy					0.61	0.49
Course 2 × Time × Autonomy					1.48**	0.46
Course 1 × Time × Relatedness					0.16	0.48
Course 2 × Time × Relatedness					-0.03	0.46

Table 3: Comparison of fixed effects (β) and standard errors (SE) across the nested linear mixed models predicting exam performance. [a] Compared to Course 3 (math majors and preservice teachers), [b] T2 compared to T1.

and competence was statistically significant and negative for both the course for engineering (Course 1: β = -0.82, p = .037) and business students (Course 2: β = -0.86,

p = .028) compared to the course for math majors and preservice mathematics teachers (Course 3). This indicates that the role of competence in performance improvement over time was higher in the course for math majors and preservice mathematics teachers than in the other two courses.

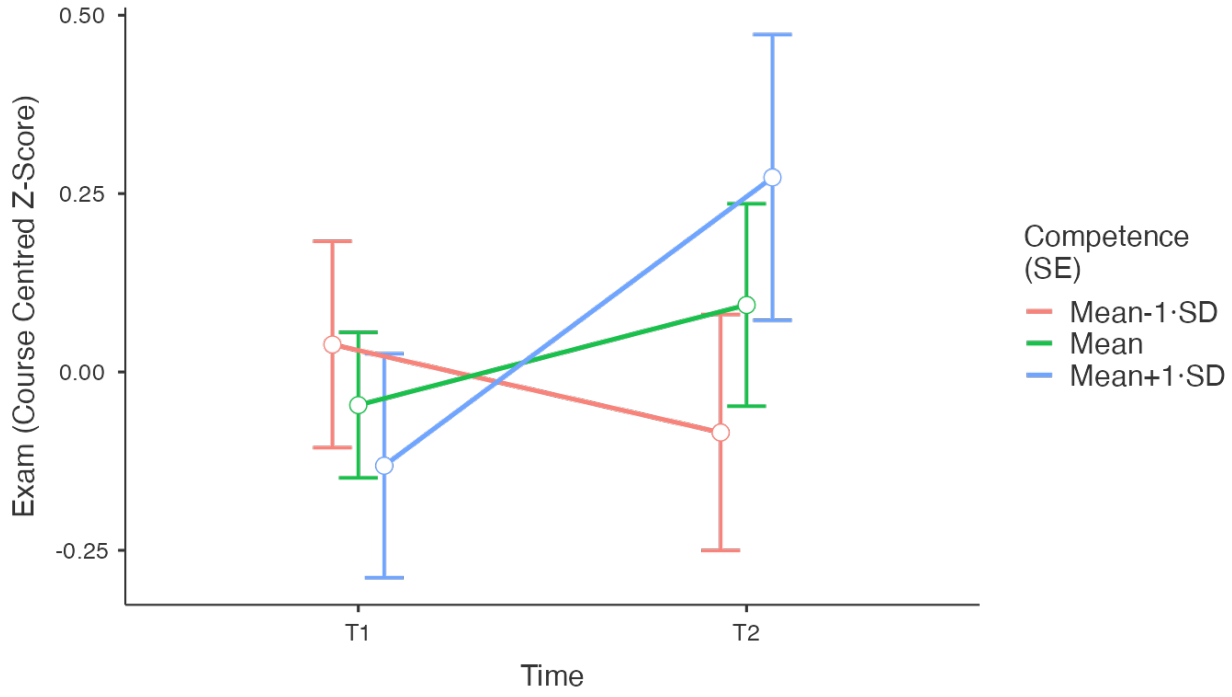

Figure 1: Visualisation of the time-competence interaction effect on exam performance

Second, the two-way time-autonomy interaction was almost significant in Model 3 ($\boldsymbol{\beta}$ = -0.34, p = .061) and indicates that a higher overall perceived autonomy relatively

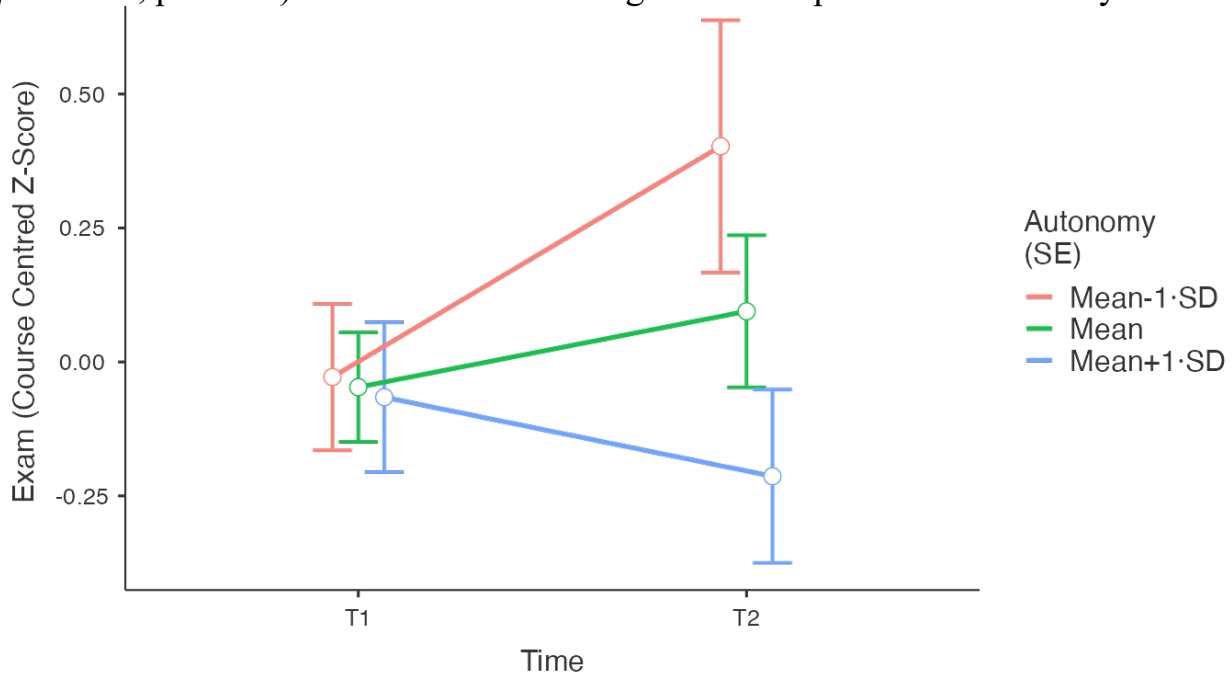

Figure 2: Visualisation of the time-autonomy interaction effect on exam performance

diminished performance from midterm to final exam, as shown in Figure 2. The three-way interaction of course, time, and autonomy was statistically significant and positive for the course for business students (Course 2: $\beta = 1.48$, $p = .002$) compared to the other two courses. This indicates that, unlike in the two other courses (and the whole sample), with higher autonomy, the exam performance improved over time in the business course.

DISCUSSION

In light of the challenges of learning mathematics at university, our paper analysed the potential of weekly exercise tutorials to increase exam performance by satisfying students' psychological needs – competence, autonomy, and relatedness (Ryan & Deci, 2020). Addressing RQ1, the findings highlight the positive main effect of relatedness on overall exam performance. This is surprising, as other studies rather emphasise the importance of perceived competence and autonomy (Wang et al., 2022). The effect of relatedness may be particularly pronounced here because a missing sense of belonging and a feeling of exclusion have been identified as common challenges in university mathematics education (Gildehaus et al., 2024; Lahdenperä & Nieminen, 2020; Solomon & Croft, 2016). This suggests that the tutorials have the potential to improve exam performance by fostering a sense of belonging and connection, thereby mitigating the inherent difficulty of the content.

The analyses for RQ2 and RQ3 indicate that the influence of basic needs is dynamic and course-dependent. While overall, perceived competence showed rather positive and perceived autonomy showed rather negative effects on students' improvement in exam performance over the course of the semester, the size and potentially also the direction of these effects appear to differ in different course settings, as indicated by the three-way interactions. In general, a negative link between autonomy and performance may suggest that in the highly demanding context of university mathematics, high perceived autonomy without sufficient instructional structure can lead to cognitive overload. For novice students, an overreliance on self-regulation in a complex domain may thus prove overwhelming, ultimately hindering exam success (Ryan & Deci, 2020). Future research is needed to better investigate this phenomenon as well as possible characteristics in course settings, different study groups, or other variables on which these effects depend.

In summary, the results underscore that basic needs satisfaction is a necessary, but not sufficient, condition for exam performance. While relatedness provides stable support, the predictive weight of competence wanes in the face of rising disciplinary demands, and the function of autonomy appears highly dependent on the student's primary field of study. These findings reinforce the need for a differentiated approach to motivational support, tailored both to the timing within the semester and the specific student population.

Limitations for this study include the single university sample, which restricts generalizability, and the use of only two time points (T1 and T2), which fails to capture continuous motivational fluctuations within the weekly tutorial cycle. Furthermore, the reliance on self-report measures of basic needs introduces potential bias. Nevertheless, the findings offer implications for pedagogical interventions, especially with regard to creating an inclusive social climate in exercise tutorials that facilitates the use of learning groups for effective cognitive and emotional regulation (Gildehaus et al., 2025). Regarding competence and autonomy, future research is desirable to analyse their effects in more detail and depth, to be able to support students' mathematics at university in exercise tutorials satisfying all basic needs.

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"He ended up being against the stereotype of a White Man. Not sexist, not racist." A Narrative Exploration of Mathematical Creativity and Identity

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Classrooms are social environments laden with accumulated histories, expectations, and assumptions. Instructor's identities and mathematical experiences have a profound impact on students' comfort in the classroom, which in turn impact their mathematical identities and mathematical creativity. As the person of power in the classroom, instructors can either perpetuate prior traumatic experiences or overwrite them with new salient fulfilling ones. This study examines how a White male professor assuages students' prior negative experiences by recognizing his privilege, showcasing his cultural awareness and appreciation, cultivating empathy for people on the fringe, and accepting multiple approaches and multiple answers.

Keywords: Mathematical Creativity, Mathematical Identity, Whiteness.

INTRODUCTION

The quote in the title comes from JH, a Mexican American student in Dr. Watson's Fall 2021 undergraduate Calculus 1 course. (All names are participant-chosen pseudonyms and conversational fillers such as "um", "like", "so", "I guess", or "you know" in quotes are removed for readability).

> Initially it was hard to be creative…I didn't know who [Dr. Watson] was and I expected him to be like every other teacher who is a White male and tell me I'm wrong and tell me to stop in the middle of my thought process and kind of like an old person. Very naggy and scornful, I guess. It was hard at first because I generally thought that my teacher was not going to necessarily resonate with me,…[or] try to understand me or me as a student, at least. I felt like I could not be creative, initially. But then in that same lecture where I thought I couldn't be creative I realized, I could be because I got to know my professor as a teacher and I feel like being creative in his class is actually easy. And as for how his gender or sex or race plays into it, I don't think it plays into it anymore. It used to before because I was intimidated by the fact that he was a White male because it's what you expect and that's what I got. And he ended up being against the stereotype of a White man. Not sexist, not racist, not like a typical professor. Just very positive, encouraging, and nice. Yes, so his sex and his race really no longer play into how I view my creativity or how I go about my creativity. (Interview)

The field of mathematics is a documented White male space (Battey & Leyva, 2016; Leyva, 2018). In mathematics, typically students of color must mold themselves into the White-male dominated system (McGee & Martin, 2011; Oppland-Cordell & Martin, 2011; Stinson, 2008) and "play the game" (Gutierrez, 2009) of academia to succeed. In the course explored in this report, Dr. Watson has removed the burden of

student assimilation, in effect beginning the process of "chang[ing] the game" (Gutierrez, 2009). This paper examines how a White male professor fostered a creative classroom climate by positioning diversity within an equitable, shared learning space. This paper dually discusses identity and mathematical creativity. Through a phenonmenological perspective with a narrative analysis, we present 1) Dr. Watson's identity utilizing mathematical autobiographies (MAB), interview transcripts, and other written texts 2) his students' data sources to display the impact of his identity on their mathematical creativity. As such our research question is, *How does a White male professor create a classroom atmosphere to foster mathematical creativity?*

THEORETICAL FRAMING

Our work is guided by a phenomenological perspective that provides a lens for examining how individuals make sense of their social and psychological worlds, prioritizing their lived perspectives (Abakpa et al., 2017). The phenomenological approach centers the meanings Dr. Watson assigns to his mathematical journey and the meanings students assign to their experiences in mathematical creativity. To focus on participants' mathematical self-stories within these experiences, we utilized narrative analysis from narrative inquiry methodology. Narrative analysis is a methodology that explores "intrapersonal and interpersonal participant experiences and actions to understand the human condition through story, which is justified in and of itself as a legitimate way of knowing" (Saldaña, 2009, p. 109). This methodology aligns with the phenomenological perspective, such that

> People shape their daily lives by stories of who they and others are and as they interpret their past in terms of these stories. Story, in the current idiom, is a portal through which a person enters the world and by which their experience of the world is interpreted and made personally meaningful. Narrative inquiry, the study of experience as story, then, is first and foremost a way of thinking about experience. Narrative inquiry as a methodology entails a view of the phenomenon. (Connelly & Clandinin, 2006, p. 375)

In this case the phenomenon is students' mathematical creativity. Together, the phenomenological perspective and narrative inquiry methodology clarify how teachers construct meaning in their pedagogical learning (Patterson, 2018), particularly as they work to foster students' mathematical creativity while navigating their own multifaceted identities. That is, given our phenomenology perspective, a narrative analysis is therefore fitting since we honor people's stories by examining the broader social, cultural, familial, and institutional stories that shape their lived experiences.

METHODS

The data presented here comes from a larger project that explored the impacts of fostering and valuing creativity on students and instructors in a Calculus I course (single-variable functions, their limits, derivatives, applications, and basic integrals). Calculus 1 instructors from the United States were invited to apply to participate in a

semester-long professional development (PD) focusing on mathematical creativity. Once accepted as REACT (Redesigning, Evaluating, and Assessing Creativity Tasks) Calculus Fellows, faculty participated in weekly online discussions using the Creativity-in-Progress Reflection on Problem Solving (Karakok et al., 2020) to 1) implement and assess the two non-routine tasks (Creativity-Fostering Tasks) provided by the research group and 2) create, implement, and assess four non-routine tasks. Prior to attending the weekly PD, fellows completed an "Entrance Ticket" with questions reflecting on the previous week or anticipating the upcoming week.

Although there were 3 cohorts of instructors (Spring 2019, Spring 2020, Fall 2021), only cohort 3 fellows authored a math autobiography pre-intervention and assigned their students to write their MAB during the first or second week of the semester. MABs asked participants 1) to reflect on the roles that their various identities (e.g., gender, race, language, culture, religion, sexuality etc. and their intersectionalities) play(ed) in their math learning story and 2) how their attitude toward mathematics has changed over the years, why they like it or don't like it. Post-intervention, student participants completed a "next chapter "(post-MAB) to include at least two different situations from the course that shaped their various identities. While the MABs were collected as part of the class, the post-MABs were submitted directly to the research group via Google Forms and were distributed to the instructors after grade submission.

Participants

Participants in this study came from a major public university in the Great Lakes region in the United States. Dr. Watson was 1 of 5 REACT Fellows from Cohort 3. All of his 100 Calculus 1 students were invited to join the research study. Sixty-three students consented and 58 completed both pre- and post-autobiographies. Their self-reported demographic breakdowns collected from surveys and MABs are as follows: 9 Asian males, 6 Asian females, 2 Half Asian/Half White females, 1 Unknown gender Asian, 1 Unknown gender Part Asian, 1 nonbinary Asian, 1 Black/White/Hispanic female, 3 Hispanic females/Latinas, 2 Hispanic/Latino males, 1 female from Spain, 7 White males, 24 White females. Eight of these students agreed to be interviewed. This paper will focus on Dr. Watson and the 8 interviewed students, whose demographics are detailed later.

Data Collection

Consistent with our social phenomenological stance, we treat both student and faculty participants as the primary authorities on their own emotional and experiential lives. Self-reported accounts therefore align with our research aims and offer adequate methodological grounding for examining how creativity-oriented instruction is experienced with respect to their gendered and racialized mathematical experiences.

Instructor data sources are snapshots from a) *pre-intervention*: REACT Fellow application and MAB, b) *during intervention*: entrance tickets, and c) *post-intervention*: interview. Student data sources for this paper come from a) *pre-*

intervention: MAB, and b) *post-intervention*: next chapter of their mathematical autobiographies, and interviews.

Data Analysis

Grounded in a phenomenological perspective, we take participants' accounts as credible and meaningful representations of their experiences. Thus, to tell a story about the classroom environment, and Dr. Watson's and the students' identities, a narrative analysis was used. Following Clandinin's (2013) framework, we attended to temporality, sociality, and place. Temporality highlights how experiences unfold across past, present, and future; we traced moments in Dr. Watson's narrative that connected earlier influences to current practices or future hopes. Sociality concerns the cultural, institutional, and relational contexts shaping interpretation; we coded for references to family, students, colleagues, teachers, cultural norms, and other social forces. Place draws attention to the physical settings—such as classrooms, homes, or other meaningful locations—that anchor experience. *Booster* and *speed bump* (Cawley et al., 2025) were also used to describe positive/supportive and difficult/challenging moments, respectively. While Cawley et al. (2025) apply booster only to "consistently negative" MABs, we expand its use for any type of MAB.

RESULTS

Dr. Watson's Identity

Dr. Watson provides insight into his childhood, his identity as a mathematician, his identity as a researcher, and his identity as an instructor. He describes his family as "rural poor," sometimes with only toast for dinner. To spare her child from realizing the meal was insufficient, his mother would present it as a special treat. Dr. Watson's family moved to 7 different houses by the time he was 13. Due to the "drifting spirit…in his bloodstream" (MAB) his Romani grandmother imparted upon his father and then down to him, he sought solace and comfort in the consistency of mathematics; he knew "there would be new kids on the playground, but they'd probably be doing the same mathematics" (MAB).

By third grade, he experienced his first booster when his teacher showed him algebra He felt as if she was letting him in on a "deep secret" and viewed this new math as "special". His next booster came from high school teacher, Mr. H. Dr. Watson describes Mr. H as "enthusiastically joyful about teaching math," with an infectious excitement that spread to his students. He felt that Mr. H made math "seem like a secret joke between himself and his students." In college he experienced his first speed bump: it was the first time he couldn't solve mathematics problems on his own. Once he started collaborating with peers, he did well. However, he viewed the need for outside help as an indication that he wasn't as "good" in math as he previously believed.

After giving a talk at a local conference, he experienced another booster in college that shifted his mathematical identity from mathematician to "mathematical preacher" (MAB), "mathematical prophet," or "mathematical evangelist" (Interview). He considered himself a "mediocre mathematics student, but a good communicator" (MAB). The importance of communicating mathematics is a strong theme in Dr. Watson's data sources. In his interview, he equates conceptual understanding to the ability to communicate a concept to another. In his MAB he says, he "loves giving talks," sharing his enthusiasm for math. He is parsing the two main responsibilities of a university professor: to be an expert in the content and in the pedagogical knowledge needed to disseminate that content. He distances himself from the former stating "I'm decent at math, but not great," "I don't have the mathematical firepower" and embraces the latter by infusing his class "with dramatic flair," to hold "a room spellbound."

During his first career as a high school teacher, he experienced both a booster and a speed bump. The booster was realizing math had a history through his master's thesis. Incorporating math history becomes an important part of his future pedagogical practices; it is mentioned in his application, in his entrance tickets, and in many of his students' post-MABs and interviews. The speedbump came when administrative load increased and time with students decreased. Upon consulting with long-time mentor Mr. H, he changed career paths, so this event could be considered a booster.

His pursuit of Ph.D. in mathematics education brought a significant speed bump. He describes education courses taught by professors who seemed to hate math. He became angry that he was

> attacked as a white male who benefitted from a long history of using mathematics as a way to develop hegemony over others. I was told math was a "white" way of knowing, and that I was perpetuating systemic racism. I remembered the toast we used to eat for dinner, and wondered how this could be true. When I tried to defend myself, I was accused of victim-blaming. (MAB)

Despite his desire to quit graduate school, upon support from his wife, he persisted. As he continued his studies, a major booster occurred.

> I came to see truth in the ideas I'd been too quick to reject. I never really felt I had a seat at the table when it came to discussions of power and privilege, but I learned how to be kind, and how to be more tolerant, how to stand up for people who came under attack by the mathematical establishment. (MAB)

This is a crucial turning point in his racial identity, that combined with his mathematical identity, seems to influence his instructor identity. He understands what it's like to be in the fringe: "I sit at the back of the room at the weekly colloquium," "I am not a good mathematician, at least not by the standards set by the other members of the math faculty," "some people in my department view my [math

history] work with a bemused smile, and others don't think of it as real mathematics." He harnesses these feelings of exclusion into advocacy for others.

Students' Identities and their interplay with Dr. Watson's

Across the post-MABs and the 8 interviews, students overwhelmingly have a very positive perception of Dr. Watson, frequently using "love," "fun," "favorite," "confident," "enjoy," "highlight," "interesting" and "best" to describe him or the class. Students' reflections below outline their identities and how (or whether) Dr. Watson's race and gender shaped their mathematical and creative self-perceptions.

Kap, nonbinary Asian, stated that they resisted the stereotype "Asians are good at math" by insisting they hated math, they sucked at it, and it was not for them. They described feeling "stifled" by expectations about who they were "supposed" to be. They had to unlearn this defiance to reach a neutral stance of mathematics. At the same time, Kap desired to be a "credit to my race and to show that…not being gender conforming, doesn't necessarily mean that I'm not also a focused and driven person" (Interview). Kap did not see their instructor's or classmates' identities influencing their ideas about creativity and emphasized their course felt diverse across the gender spectrum, race, socioeconomic status, and sexual orientation.

Anna, an international female student from Vietnam, contrasted her rigid schooling in Vietnam to a more flexible system in the United States. She acknowledged her privilege to afford resources in Vietnam for extra tutoring in English and afford international study in the U.S. She said she was affected by the stereotype that language study is for girls and natural sciences are for boys. She tied this to economic inequity noting that wealth is tied to a career in natural sciences. In her experience, people in her country "hammer people into thinking…you are predisposed to learning something…and you cannot change" (Interview). Because of these stereotypes, she was "scared to do math," because it used to "terrify me to death," and she would "cry…because it scared me so much." She repeatedly called these stereotypes "harmful," and "damaging." When speaking about the role of Dr. Watson's identity on her ideas about creativity or mathematics, she posits that although she believes he is a White man who also came from "a very ingrained and internalized perception of math as very masculine," that his perception was "neutralized" by his studies in math history. Specifically, she mentioned that he taught them an Indian scholar discovered a particular calculus concept before Newton. She described a highly impactful act by Dr. Watson—the only person in the U.S.—to spell her name correctly with proper accent placement as "heartwarming," "outstanding," and "a positive surprise."

Anastasia Vine, an Indian American woman, also commented on the negative effects of the stereotype that "you're brown, so you're smart" (MAB). She resented people who relied on her to do all the work and tied expectations for mathematical success to economic success within her community. Though she felt more racial discrimination than gender discrimination, she noted the intersectionality of her race and gender referring to it as "double duty." Her peers' perceptions that girls have better

handwriting, are more thorough in their explanations, and more driven to do well gave them more reason to impose all the work on her. When asked about the role of gender or race in her ideas about creativity, she said that her immigrant parents from India taught her strategies that her school teachers rejected, "`Oh honey, that's not the way we teach you, so don't do that." She felt that not accepting her parents' way was a hinderance to her creativity. Because White educators were her norm, Dr. Watson's identity did not influence her perceptions; however, she wished to have a teacher of color for the opportunity to relate to someone with a shared background.

Yoda, an Asian man, noted in both his interview and MAB the emphasis of math in his childhood via a story about completing multiplication tables at dinner using his multiplication-themed placement. He reported that his rule-based Asian parents contributed to "a lack of creativity because there's always kind of one way to do something" (Interview). Like Anastasia Vine, White teachers were Yoda's norm, so he didn't see any impact of Dr. Watson's identity on his.

Though Fleur identified as Black, White, and Hispanic, she focused exclusively on her Black identity in her Interview and MAB. There was a parallel in these data sources as she mentioned two different influential Black female peers:

> [The math genius was] the only person who understood the struggle of having to have your mom do your hair late at night with multiple hair products that all smelled a little funny…In this specific situation, being a young black girl and having another friend from the same background as me was immensely beneficial, not only for my character development, but for my mathematical development. (MAB)

> [F]or creativity, I know this girl in my class who was also Black and we've talked before about Calc, and…how to study for the exams together…I didn't really know her that well. It was just easier for me to connect with her and come up with a study plan with her just because we both recognized each other's backgrounds. (Interview)

Fleur reports stereotypes about women's affinity for English rather than math. She also noted the stereotype that "Black people aren't smart at all." This played out in group settings where White boys would exclude her in problem distribution and second-guessed her answers. It made her feel stupid and that she wasn't good at math. As a woman in Science, Technology, Engineering, Mathematics (STEM), she tried to showcase creativity during groupwork to prove she should be taken seriously and treated equally. Like her peers, Dr. Watson's White male identity was the norm. Her first instructor of color was in college. She preferred instructors and peers of color because she felt more comfortable with people who look like her. She disliked being the only person of color, because she felt stereotypes more likely to surface.

Similar to her peers, JH who grew up in a predominantly White area, stated that women are discouraged from mathematics and people question why she isn't an English major. She called her intersectionality as a Mexican woman a "double whammy," similar to Anastasia Vine. She yelled at a White male who assumed "that as a girl and as a Mexican…I have to be bad at math." She also invoked the same

Asian stereotype all the Asians interviewed noted, "When I told [another student] I was Mexican and not Indian, they're like, 'Oh so, then why are you good at math?'" She said that minorities:

> choose not to show their creativity because a) they don't see a lot of people like them in their class and b) the people who are not like them in their class are constantly telling them right and wrong. (Interview)

Fleur expressed the sentiment in a) and Anastasia Vine echoes point b). She finds solace in her all-female group referring to herself and a peer as "brown girls in STEM."

Kakashi, a Columbian American woman, also grew up in a predominantly White neighborhood. She was more "Hispanic-passing" with a stronger accent than the rest of her family so she was placed into a different track for second language learners, despite strong academic performance. When her 5th grade teacher asked for her removal from that track, she excelled in mathematics. Like her peers, she recounted racism faced as a Hispanic woman in math and her lifelong efforts to defy the stereotypes. She expressed reservations and discomfort interacting with White males due to poor treatment. However, she spoke very favorably about Dr. Watson, attributing her comfort to his positive attitude, and his value of collaboration and students' opinions.

Spartan acknowledged her privilege as a White woman and she didn't take her opportunities for granted. She doesn't attribute her race or gender to her creativity. When asked about the impact of Dr. Watson's identity on hers, she surmised that he also was aware of and took advantage of his opportunities. She considers herself lucky that Dr. Watson has studied in different countries, experienced many different cultures, and considers him cultured and knowledgeable about his "stance in society." She contrasted him from other White men claiming that they "have not emphasized that amount of care…and thoughtfulness and recognition to their privilege."

DISCUSSION

Across the eight narratives, students describe how racial, gendered, and cultural stereotypes shaped their relationships with mathematics, often constraining confidence and creativity. Collectively, the accounts show that students enter mathematics classrooms already navigating intersecting stereotypes, and that instructors can either reinforce or disrupt these experiences through their pedagogical choices. The excepts above portray Dr. Watson as a White male with a deep understanding of inclusion and exclusion. He empathizes with people on the fringe because he has been placed there. His rhetoric about both his influential teachers includes the word "secret" which by definition implies insiders and outsiders; in both cases, he was included by them. In contrast, his position "in the back of the room at weekly mathematics colloquium," places him at the fringe. This sensitivity shapes how he interprets students' trajectories. In his Interview and Entrance Ticket, he details a remarkable development of a student—a self-described math hater

(confirmed in her MAB)—based on her actions in his previous Precalculus class and the current class. She began the semester "back of the room super quiet", but gradually moved closer until she's sat in the front. She also presented a problem unapologetically without prefacing that it might be wrong, becoming one of the most engaged students—even leading a group discussion on one of his Creativity-Fostering Tasks. He said in his Interview, "what has happened for her, she's gone from way out—she was on the fringe, she was outside the fringe…now, she's much more central." This event was not only monumental to Dr. Watson, but to the student as indicated in her post-MAB, "There is one moment I'm really proud of, when I went in front of the class to show my work. I was so scared but I did it."

Anastasia Vine recalled her teacher's dismissal of culturally-rooted strategies: "your creativity is kinda hindered because most education systems didn't care for how you did different methods of work." Yoda's said allowing only one method stifles creativity. Kakashi agrees, lamenting that in her White schooling, "creativity kinda lacks. People were always being taught the same thing over and over and over again." Dr. Watson is a witness:

> my experience tells me that most of my students think of mathematics as a set of rules and procedures which have existed since the dawn of time. Such a view of mathematics leaves little room to see the creative aspects of our discipline. (REACT Application)

In his interview, Dr. Watson discussed at length an activity that requires "making connections or multiple approaches/answers" (Savić et al., 2025). In this activity, there are an infinitely many valid responses and no single correct one. By affirming every contribution, he aims to "level the playing field," directly countering experiences like those described by Anastasia Vine and Kakashi. Kakashi further highlighted Dr. Watson's reaction to a South Korean student's unique method:

> He was just very excited about it…I thought it was so charming that he was so excited about how she came up with her answer…it was so different from anything else I have ever seen about how to solve that problem, it made me feel "okay, may be if I was taught in school a way that is completely different but still gets the same answer, then may be my way is also going to be accepted by my professor." It was also just nice. (Interview)

It is obvious in this vignette that Dr. Watson embodies the same excitement as Mr. H. His creativity-fostering teaching action "prompt[s] and encourage[s] different approaches or divergent thinking," (Savić, 2025).

Students attributed his math history knowledge to his cultural understanding and felt it enhanced their creative performance. Indeed, Shao et al. (2019) proclaim that exposure to foreign cultures can improve one's creativity by expanding their cultural assumptions to draw, blend, and reinterpret ideas from a more diverse arsenal.

It's important to recognize that classrooms are social environments laden with accumulated histories, expectations, and assumptions. Instructor's identities and mathematical experiences have a profound impact on students' comfort in the

classroom, which in turn impact their mathematical identities and mathematical creativity. As the person of power in the classroom, instructors can either perpetuate prior traumatic experiences or overwrite them with new salient fulfilling ones. Dr. Watson paves a path for authoring new editions to students' stories by recognizing privilege, developing cultural awareness and appreciation, cultivating empathy for people on the fringe, and accepting multiple approaches and multiple answers.

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Postsecondary Privileging of Pure over Applied Mathematics and Sense of Belonging for "Nontraditional" Mathematics Students

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Existing research on students studying postsecondary mathematics has typically focused on "traditional" students who attend more prestigious universities and has overlooked students who begin their mathematics studies at lower-status institutions. As a result, little is known about factors that influence persistence of more "non-traditional mathematics majors. In this study, we investigate the experiences of these "nontraditional" majors at lower-prestige institutions, exploring their experiences through in-depth interviews that we analyse with thematic analysis through a Critical Bifocality lens. We explore potential mismatches between students' motivations for majoring in mathematics and the kinds of mathematics that are privileged within the postsecondary mathematics department, both inside and outside of math courses.

Keywords: retention, belonging, nontraditional mathematics majors, Critical Bifocality, thematic analysis

The number of students graduating with a postsecondary degree in mathematics has been declining in both the U.S. and worldwide over the last several decades, with even greater declines noted in institutions that serve students from the most disadvantaged backgrounds (Hall et al., 2022; Musu-Gillette et al., 2015; Rowell et al. 2023; Seymour & Hewitt, 1997; Seymour et al., 2019). However, to date, there are very few studies on the factors that influence the persistence and retention of mathematics majors, particularly those from more marginalized backgrounds.

Instead, most research has studied STEM (science, technology, engineering and mathematics) or "math and science majors" more generally (see Seymour et al., 2019 for an overview). Some of focuses on the influence of high school mathematics achievement (e.g. Wang, 2013; Trusty, 2002) or particular first- and second-year mathematics *courses* on student persistence in STEM majors (e.g. Ellis et al., 2016), but none focus explicitly on postsecondary students pursuing mathematics degrees. Thus, little is known about which factors impact recruitment and retention of mathematics majors specifically (Hubbard, 2023).

Moreover, existing research on STEM retention has been almost exclusively conducted with students at more prestigious universities. Little is known about mathematics and STEM majors at lower-prestige institutions, even though these institutions are significantly more likely to be attended by lower-SES and first-generation students and other marginalized groups (Smith et al., 2013; Campbell et al, 2021).

To address this gap, this study investigates the needs of mathematics majors attending two-year colleges within a large public urban university system in the U.S. Two-year

colleges are more affordable and lower-prestige public colleges in the U.S. that students can attend as a more affordable path to complete the first two years of a bachelor's degree. Almost half of all mathematics majors in a given year in the U.S. are enrolled in two-year colleges (NCES, 2020), and roughly half of all Bachelor's degree graduates in the U.S. started at two-year colleges (Foley et al., 2021). However, no studies have investigated two-year mathematics majors directly. Thus, two-year colleges are a critical yet understudied part of the math major pipeline.

In this study, we explore two-year students' motivations for majoring in mathematics, and the extent to which students see their reasons for being interested in mathematics as in alignment or misalignment with mathematics department structures and culture, both inside and outside the classroom.

The data from this report come from eight in-depth interviews conducted with students in the mathematics degree at two-year colleges in the largest U.S. urban university system, where half the population is first-generation college students and roughly 40% have household incomes below $20,000/year. These data are part of a larger mixed-methods project that seeks to identify factors impacting two-year college mathematics recruitment and retention. In answering the research questions above, we seek to contextualize the field's understanding of the motivations of math majors for pursuing mathematics, and to provide hypotheses about mechanisms by which such motivations might interact with larger structural factors within mathematics departments.

LITERATURE REVIEW

Tinto's model of student departure (1987) is arguably the most influential existing model of postsecondary student retention. Tinto posited that family background, pre-college schooling, and individual student attributes influence student persistence via two "integration" variables, which in turn impact a student's decision whether to drop out: 1) academic integration, often measured by course grades; and 2) social integration, often measured by frequency of interaction with other students and faculty. However, subsequent literature is mixed regarding the relevance of Tinto's model for two-year college and other "non-traditional" students (Bean & Metzner, 1985; Ashar & Skenes, 1993; Borglum & Kubala, 2000; Karp et al., 2010; Stuart et al., 2014).

Bean and Metzner subsequently developed a retention model (1985) specifically for "non-traditional" adult learners that focused on an institution's academic utility rather than social environment. They argued that social integration is significantly less influential for such students, as they typically enter higher education with their own support systems and social networks. This model contains three main categories: environmental, academic, and background variables, and these variables are posited to influence academic and psychological outcomes, which in turn determine a student's decision to persist. While this model may be more relevant for two-year mathematics majors, it still is not designed to examine factors relevant specifically to mathematics.

A larger body of literature on STEM majors provides some preliminary information about what may be relevant to the retention and persistence of mathematics majors,

identifying factors such as grades and disposition towards mathematics in high school (Musu-Gillette et al., 2015; Shaw & Barbuti, 2010; Wang, 2013; Trusty, 2002; Evans et al., 2020), perceptions of mathematics self-efficacy or self-concept in postsecondary mathematics courses (Ellis et al., 2016; Wang, 2013), sense of belonging and social integration (Rainey et al., 2018; Tinto, 1987; Rodriguez, & Blaney, 2021; Good et al., 2012); engagement with undergraduate research opportunities (Russel et al., 2007; Eagan et al., 2013; Seymour et al., 2004; Hunter et al., 2007); and mentorship (Apriceno, et al., 2020; Atkins et al., 2020; Chelberg & Bosman, 2019).

In general, each of these factors has been associated with increased retention and/or persistence in STEM degree attainment. Research in the area of sense of belonging notes how both institutional cultures and student-to-student interactions can influence students' perception that they fit in or belong within their chosen STEM major. The underrepresentation of women and students of color in STEM, in particular, means that improving sense of belonging for such students requires active investigation into the institutional and pedagogical environments in which students learn (Rainey et al., 2018; Rodriguez & Blaney, 2021; Good et al., 2012).

Research on participation in undergraduate research experiences (UREs) and engaging with faculty mentorship has consistently reported positive impacts on STEM major retention and subsequent interest in STEM careers (Seymour et al., 2004; Eagan et al., 2013; Apriceno et al., 2020; Atkins et al., 2020; Chelberg & Bosman, 2019; Russell et al., 2007). Participation in UREs and involvement with mentorship are also likely to contribute to students' sense of belonging. For example, Hunter et al. (2007) reported that UREs fostered a sense of belonging to students' desired scientific discipline and helped them to "[feel] like a scientist," (p. 53). However, given that almost all research on STEM majors has so far been conducted at four-year institutions (institutions that award bachelor's degrees), it remains an open question the extent to which two-year mathematics majors may perceive such resources as useful or accessible within their institutional context. The current study aims to address this gap.

CONCEPTUAL FRAMEWORK

To structure our investigation of potential mismatch between two-year college math majors' motivations for pursuing mathematics and potential mismatch between those motivations and mathematics department structures and culture, we turned to Critical Bifocality as it has been conceptualized and operationalized for use in mathematics education research (Chen & Horn, 2022). Traditionally, Critical Bifocality is a theoretical perspective that posits that investigations into social processes of marginalization or oppression necessitate a bifocal approach—focusing on the relationship between individual agency and structural features of environment. Two-year college students are more likely than four-year college students to belong to groups historically marginalized in higher education (e.g., Goldrick-Rab, 2007), and two-year colleges (and the students who attend them) are often subject to stigma (Ocean et al., 2025). This makes Critical Bifocality relevant to math majors at two-

year colleges in the U.S.

Chen and Horn (2022) provide four principles that characterize research in mathematics education that take such a Critical Bifocality approach. Although all four principles were instrumental in informing the larger project, here we focus on two principles that were most relevant for informing the analysis discussed in this report. They are, "(1) analyzing interactions between structures and lives; while (2) recognizing the pervasiveness of oppressive structures," (p. 802). Here, we report findings that speak to potential mismatches between pervasive institutional structures and students' individual motivations for studying mathematics.

METHOD

In summer 2024, we invited current two-year mathematics majors to participate in a paid one-hour interview about their experiences. Eight agreed to participate and varied on a range of characteristics (see Table 1).

Participant ID	Age	Gender	Last mathematics course(s) completed prior to interview	Enrollment intensity	History of transfer
1001	44	Man	Calculus I	part-time	yes
1002	29	Man	Calculus III, abstract algebra, differential equations	full-time	yes
1003	24	Woman	Calculus I	full-time	yes
1004	26	Woman	Precalculus	part-time	yes
1005	22	Man	Calculus I	full-time	yes
1006	25	Man	Abstract algebra	part-time	yes
1007	21	Man	Calculus III, linear algebra, abstract algebra	full-time	no
1008	23	Man	Calculus I	full-time	no

Table 1. Descriptive Characteristics of Interview Participants

Each interview was guided by a semi-structured protocol that included questions about: rationale behind major choice; course-taking and placement; perceptions of quality of instruction in college mathematics courses; influences of external life factors such as work, family, and health/disability; and influences of institutional culture, structures and resources. The data for the current analysis is drawn from questions regarding students' motivations to major in mathematics and students' use and perceptions of institutional features commonly noted as influencing persistence in STEM majors in the broader literature (undergraduate research opportunities, mentorship, and community involvement).

Analysis

Each interview was recorded and transcribed, then coded in several stages using thematic content analysis (Braun & Clarke, 2006). First, five researchers read all transcripts and met to generate emergent themes. For example, at this stage researchers agreed that there was an important theme regarding the notion of "real world applicability" and students' motivation to major in mathematics; however, the exact definition of "real world" was not yet precisely specified. Then, two researchers took the generative initial themes, reviewed and refined them, and developed a more nuanced set of subcodes. For example, some students were motivated to major in in mathematics because they perceived that a mathematics degree would help them advance in their current/former job. Others noted a motivation based on a broader appreciation for the relevance of mathematics in explaining phenomenon outside the classroom. This was carried out in several cyclical rounds until consensus was reached between the two coders about subcodes and definitions of the overarching themes. Then the final list of themes was shared with a third researcher for further discussion and refinement, leading to final thematic analysis results presented here.

RESULTS

One of the key themes that emerged from student interviews was that two-year mathematics majors in the sample were largely interested in mathematics for its utility—often preferring applied to pure mathematics. Common across seven of the eight participants' perspectives was the belief that a mathematics major would be useful to them because it would help them advance in their current career (1001, 1003, 1004, 1005) or allow them to be successful when they eventually transfer to a bachelor's degree or graduate program (1002, 1005, 1007). In contrast, one participant (1006) reporting majoring in mathematics because they intended to become a mathematician. Most interviewees expressed an appreciation and interest in aspects of mathematics that were applied or "practical" compared to purely theoretical aspects.

However, this interest in applied mathematics was frequently perceived as a misfit with mathematics department structures and culture, where pure mathematics was seen as the "norm". Two subcodes emerged under this theme: one related to the ways that students perceived this mismatch within their mathematics courses, and the second related to the ways this mismatch was perceived to play out in extracurricular mathematics activities such as math club, research opportunities, and internships.

Misfit Between Student Motivations and Course Structure/Culture

Several students described how their interest in applied mathematics appeared not to fit well with existing course structures, which they saw as focusing more on pure mathematics. For instance, when describing what an ideal experience would be like within the math major, 1005 stated that most of what they would learn would be about, "not just solving maths, they will be able to teach you how to apply this mathematics into the work field...Because like I said, learning how to solve [for] x and y is not enough...I want to know how to use this stuff out there [in the workforce]." Here 1005 contrasts purely theoretical aspects ("just solving maths") with practical aspects that

would support their transition into the workforce.

Although career relevance is a motivation that has been reported by retention literature focusing on non-traditional students (e.g. Bean & Metzner, 1985), participants occasionally provided evidence that they viewed this as contrasting with the motivations of a "typical" mathematics major or of their mathematics professors. This was well-illustrated by 1002, who was set to graduate with their associate's degree (a two-year degree that can be credited towards the first two years of a bachelor's degree) in mathematics the next term. When asked what kind of mathematical activities they enjoyed, they responded, "There are people who love math ... I am not one of them, for sure...The people who love math, who are deep into the most abstract imaginary concepts that are applicable maybe in like 500 years or something… that's just not me." Although 1002 went on to say they do like mathematics, they saw their form of liking as outside of the "norm" for the motivations of a typical mathematics major because it did not privilege "pure" mathematics ways of thinking.

1007, who was also set to graduate with their associate's degree in mathematics that term, expressed similar views. They stated, "I guess I'm not really like those kind of mathematicians where they...see this simple thing and then they'll just devote a couple years to that thing. And in the end, it's just for curiosity...I actually wanted to see something, how I can use math in the real world." Here, 1007 draws a contrast between themself and "those kinds of mathematicians" who pursue mathematics for its own sake. Overall, our findings support existing literature while also providing additional nuance related to mathematics majors in particular. That is, students appeared to interpret their belonging in mathematics within a departmental culture that they perceived to privilege interest in theoretical mathematics over interest in mathematics for the sake of its practice-oriented applications. This appeared to subsequently weaken these students' sense of belonging in the major. The extent to which such a departmental culture may contribute to students leaving the major is an open question, and one that we intend to investigate in subsequent years of this project.

Misfit Between Student Motivations and Extracurricular Mathematics Activities Offered by the Mathematics Department

In addition to experiencing misfit within their courses, a number of students also discussed how the extracurricular mathematics offerings fit poorly with their particular mathematical interests, and this often related to their feelings of (not) belonging in the major. Among the seven who reported majoring in mathematics because of an interest in its practicality or use in applied fields, it was common for participants to report a feeling of misalignment with institutional resources such as internships, or departmental resources such as social math clubs and undergraduate research projects. For instance, both 1007 and 1003 reported having an initial interest in joining a social math club, but each mentioned a hesitation in participating related to perceived misfit between their interest in applied mathematics vs. a perceived focus on pure mathematics. For 1003, the computer science club felt "more accessible" than the math

club, and so they chose to "attack [math] from the lens of computer science" instead. Likewise, 1007 expressed reservations about the mathematics professor leading the math club having graduated from an elite postsecondary institution with a mathematics degree. 1007 described themselves as being more easily able to communicate with a math club facilitator, who, "might not be the most intelligent or like most successful who graduated from, I don't know, MIT, Caltech, NYU, but the way they communicate with students, the way they engage with students are the best...I guess those kind of professors at math clubs, I think those would be really enjoyable." In these excerpts we see misalignment between students' motivations for studying mathematics and structural features of their mathematics departments related to extracurricular mathematics activities that can be important to retention and persistence in the major.

A rare counterexample to the experiences of most of the two-year math majors is 1006, who was motivated to study math to become a mathematician. 1006 utilized a variety of departmental resources, including the math club and a summer research experience for undergraduates (REU). 1006 reported engaging in these opportunities due to the connection they felt to the other students and faculty members that participated. They perceived these individuals as those who, like themself, "also want to talk about [math] problems for the sake of just talking about the problems instead of just going to get the grade...you find people like that sometimes in your classes as well. So, Math Team, certain classmates in my classes who also actually enjoy what they're doing make me feel like I belong." 1006 described math team as "where the mathematician math majors go" in contrast to math majors interested in other applied disciplines. In this way, 1006 saw their motivations as in alignment with departmental structures and resources, which further reinforced their perception of belongingness within the major. We see such findings as in alignment with other work demonstrating the potential for mathematics departments to reinforce exclusionary assumptions not only about who is welcome in mathematical spaces, but also about what kinds of mathematics are welcomed into those spaces (Rodd & Bartholomew, 2006; Battey & Leyva, 2016).

IMPLICATIONS AND CONCLUSION

The purpose of the larger project for which this data was gathered was to generate hypotheses regarding aspects of current retention literature that may not capture relevant aspects of the experiences of two-year college math majors. We found that two-year mathematics majors often expressed interest in mathematics due to its practicality. They described the importance of this practicality in several ways, including the utility of mathematics for career aspirations or future degrees that utilize mathematics, and the practical aspects of mathematics courses themselves that involve problem-solving and application. In particular, students overwhelmingly perceived this interest in practicality as in conflict with a more dominant culture of mathematics that focused on mathematics for its own sake. They described this as an issue both in their postsecondary mathematics courses and in extracurricular mathematics activities, all of which were perceived as being primarily oriented around pure mathematics. Preferring applied over pure mathematics was seen as a mark of not belonging in

mathematics. These findings underscore a possible mechanism by which the culture of postsecondary mathematics may be critical to influencing "nontraditional" mathematics majors' feeling of non-belonging in the major and the wider mathematical community. We see the cases presented here as attending to Critical Bifocality by illustrating a relationship between individual experience and mathematics department structural features that may be uniquely amplified within the context of postsecondary institutions attended by more "nontraditional" students. One line of research stemming from this work that we intend to investigate in the future concerns the alternative communities with which these mathematics majors *do* find belonging if not within the mathematics department itself. Such research is imperative if we are to reconceptualize mathematics major retention outside of the typical four-year institutional context.

ACKNOWLEDGEMENTS

This research was supported by the U.S. National Science Foundation (#2300725). Opinions reflect those of the authors and not necessarily those of the granting agency.

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Creating bilingual resources to teach magnitudes through a master's course based on Questioning the World Paradigm

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Abstract. This poster presents a master's-level course designed to develop Multilingual Instructional Resources (MIR) for teaching magnitudes in an indigenous primary school. The course is structured as a Study and Research Path (SRP) within the Questioning the World Paradigm, grounded in the Anthropological Theory of the Didactic. The poster focuses on the design of the SRP and its initial didactic milieu, the evolution of this milieu through the media-milieu dialectic, and selected examples of MIR. The presentation combines schematic representations, task evolution, and key results. The study addresses the research question: How does participation in an SRP-based master's course influence student-researchers' ability to design MIR that foster plurilingual and pluricultural competence?

Keywords: multilingual mathematics education, multilingual instructional resources, plurilingual and pluricultural competence, ATD, study and research path.

CONTEXT AND AIM

Multilingual mathematics education is increasingly relevant, particularly in indigenous contexts (Barwell et al., 2007). However, mathematics teaching remains largely monolingual. In Mexico, despite linguistic diversity, instructional materials for science and mathematics in indigenous languages are scarce. This study, drawing on previous research (Romo-Vázquez et al., 2025), addresses this gap through the education of researchers focused on MIR design. The guiding question is: How can MIR support the development of plurilingual and pluricultural competence?

THEORETICAL FRAMEWORK

The study draws on the Anthropological Theory of the Didactic, particularly the Questioning the World Paradigm, which structures the course as an inquiry process organized around a generating question. Learning develops through interactions between questions, resources, and tools within a didactic milieu. The media-milieu dialectic is used to analyse how external resources are transformed into internalized knowledge. MIR are defined as resources integrating multiple languages to support mathematical activity and plurilingual and pluricultural competence. This grounding informs both the design of the MIR and the criteria used for analyzing their evolution.

METHODOLOGY AND COURSE DESIGN

This study involves a master's program in mathematics education (Cinvestav) and an indigenous bilingual primary school (Otomi-Spanish). The course, implemented in the third semester of four, involved seven student-researchers and was structured as an

SRP including: study of theoretical frameworks, analysis of existing MIR, expert lectures and workshops, and collaboration with indigenous teachers. Students developed successive versions of MIR, from initial designs to classroom-ready resources. Data include these versions, student logbooks, and documentation of collaborative work. Analysis focuses on changes in task design, tracking linguistic and cultural integration, and attention to mathematical concepts, systematically linking the evolution of the didactic milieu to observed outcomes.

RESULTS: EVOLUTION OF THE DIDACTIC MILIEU

The didactic milieu evolved progressively. First, theoretical work enabled a shift from translation-based approaches to meaning-oriented integration of languages. Second, analysis of a more developed MIR supported the identification of the design principles, such as clarifying mathematical objectives and avoiding superficial plurilingual and pluricultural competence. Third, workshops highlighted language as a tool for conceptualization, revealing lexical ambiguities and cross-linguistic contrasts. Finally, collaboration with indigenous teachers led to contextual adaptations, improved linguistic accuracy, and increased cultural awareness.

TWO EXAMPLES OF MIR

Two examples illustrate this evolution. In a MIR on regular polygons, initial tasks focused on naming figures in Spanish and Otomi. Iterative redesign led to tasks emphasizing mathematical properties and linking linguistic expression with geometric structures. In a MIR on perimeter using Tangram, redesign integrated Otomi lexicon for measuring, comparing, and ordering. These examples show how MIR functions both as products of the evolving milieu and as drivers of its transformation, revealing a shift from lexical translation to conceptually grounded multilingual task design.

CONCLUSIONS

A master's course structured as an SRP can support MIR development by integrating mathematical, linguistic, and cultural dimensions through iterative design and collective analysis. The media-milieu dialectic provides a lens to understand how resources evolve and how students-researchers appropriate theoretical tools. More broadly, this approach contributes to legitimizing indigenous languages as languages of mathematical knowledge, promoting diversity and inclusion in mathematics education.

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Students' Emotion Graphs during Proof Construction

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Keywords: proving, development, affect, emotion, qualitative.

Learning how to prove is a difficult transition for students (Stylianides & Stylianides, 2009). Transition-to-proof courses are a shift for students, to writing arguments and justifications. We seek to understand students' *emotions* during this shift, as a form of *affect* (Grootenboer et al., 2015). There is a need for tools through which students can easily communicate their affect and experience to researchers and teachers. The research question then is: How do students represent their emotions during proof construction attempts and what facets of their experience can we understand? I present students' *emotion graphs*, as their own drawn representations of their experience.

We take proving as a form of problem solving (Stylianides & Stylianides, 2009). *Affect* encompasses a wide range of constructs including attitudes, emotions, etc. Emotions are more amenable to influence by the environment and teacher and thus ultimately a vehicle for changing longer term affect (Grootenboer & Marshman, 2015). Emotions encode information about a situation to the individual (DeBellis & Goldin, 2006). For example, frustration is typically viewed as negative but it can play a positive role to signal to the student to try another solution path. Given the immediacy of valence (positive, negative) in considering emotions, we explore how students represent their emotions on a positive to negative scale over the duration of an experience.

METHOD

Participants were n=11 university students enrolled in a "transition-to-proof" course at an American university; this course is common at U.S. universities to teach students the tools of proving before they take higher level proof-based courses such as abstract algebra. The first half of the course focused on logic, and the second half introduced basic concepts in real analysis, linear algebra, and number theory. I conducted four semi-structured task-based interviews with each participant, over the duration of their course. In each interview, participants were given two statements to prove about basic number theory (even, odd, parity, etc.), had fifteen minutes to work on each and afterwards drew an *emotion graph* and debriefed their thinking. The validity of their proof was only discussed after the above. Data consists of video observations, observation notes, student work, and interview recordings.

RESULTS & DISCUSSION

We analyzed emotion graphs by first identifying the graph start and end behavior: which quadrant they were marked in and relative to each other. Using constant comparative techniques, we identified six graph shapes and what each represented about their experience, triangulated with interview data. The six graph types were: Positive, Negative, Flat, "Concave Up," "Concave Down," and Other. Descriptions of

the six graph types will be given in the poster. The most prevalent code was "Concave Up" (39% of 88 graphs), which depicted overcoming an impasse (Figure 1).

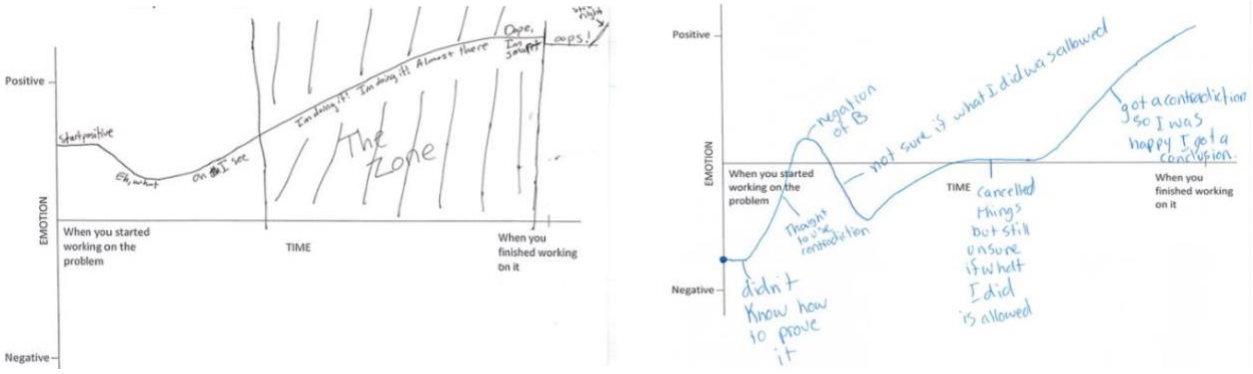

Figure 1: Emotion graphs from proof construction attempts coded as "Concave Up."

The next most prevalent codes were Positive (18%) graphs that were monotonically positive and Negative (11%) graphs that were monotonically negative.

These results extend work by Satyam et al. (2022) in providing a preliminary codebook for analysing tertiary students' emotion graphs, specifically in proof or problem solving based courses. The emotion graph types represent categories of student experience and perception of success with a proof, which instructors can use to target support. Additional studies that have used emotion graphing include Andrà & Scalvini (2023). This work shows how emotion graphs, by combining drawing and structure, have utility as research and pedagogical tools.

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Mathematical Identities of Women in Doctoral Mathematics: The Case of Ida

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Women in doctoral mathematics continue to be underrepresented internationally. This inequitable participation alerts the mathematical community to ongoing cultural factors that continue to marginalize women. This study furthers the literature about women in doctoral mathematics by leveraging a figured worlds perspective to investigate the mathematical identities of women pursuing PhDs in mathematics. Findings include discussion of one women's unique way of integrating her identities as a woman and as a mathematician. The results expand upon previous literature in which women worked to distance their identities as women from their identities as mathematicians.

Keywords: graduate mathematics, women, identity.

INTRODUCTION

Women in advanced mathematics continue to be underrepresented despite representational progress in primary, secondary, and undergraduate mathematics (NCSES, 2021; UNESCO, 2017). In addition to inequitable representation, women pursuing PhDs in mathematics have experienced overt prejudice, feelings of invisibility, insufficient or negative advising experiences, and a lack of belonging (Herzig, 2004). Furthermore, instances of women's academic success are often mitigated by accounts of struggle, bringing into question the *quality* of their experiences (Ataide Pinheiro, 2021).

Beyond representation, cultural factors are fundamental to women's experiences in mathematics. Women in mathematics, in the US and elsewhere, are tasked with navigating cultures that preclude the integration of their identities as women and mathematicians (Hottinger, 2016; Mendick, 2006; Mkhize, 2022). Gendered identities are produced and reinforced in mathematics classrooms, with mathematical ability often associated with masculinity (Mendick, 2006). This association with masculinity, paired with the idea that mathematical ability is innate, creates an impasse for women who are generally underrepresented in mathematics (Leyva et al., 2021). Women are faced with choosing to be a woman *or* a mathematician, often deciding to distance themselves from one identity or the other (Damarin, 2008; Hottinger, 2016; LaTona-Tequida, 2025).

Despite evidence of the impacts of cultural factors, the limited research investigating women in advanced mathematics is overwhelmingly focused on understanding individual factors influencing persistence and attrition (Herzig, 2002a, 2002b), with few studies focused on departmental and societal contexts that affect women (e.g., Ataide Pinheiro, 2021). At the undergraduate level, literature investigating women's

mathematical identities has revealed the varying ways women organize their mathematical selves in response to the masculine context of mathematics. This work has revealed women may choose to minimize their gender identities or opt for a hybridized version of their gender and mathematical identities (Rodd & Bartholomew, 2006; Solomon et al., 2011).

This study furthers the literature about women in doctoral mathematics by leveraging perspectives used at the undergraduate level to investigate the mathematical identities of women pursuing PhDs in mathematics. Using a figured worlds theoretical lens, I answer the research question – *How do the descriptions of doctoral women students' mathematical past, present, and future characterize their mathematical identities?*

THEORETICAL PERSPECTIVE

To answer the research question, I leverage the figured worlds framework (Holland et al., 1998). Figured worlds are defined as "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). That is, figured worlds are cultural contexts in which people expect and improvise appropriate roles for themselves and others.

Within the figured worlds framework, identity is defined as a socially constructed self-concept in which "People tell others who they are, but even more important, they tell themselves and then try to act as though they are who they say they are" (Holland et al., 1998, p. 3). This definition reflects the understanding that identities are socially dependent in that one must think of themselves as a certain kind of person and must also be *recognized* as that type of person.

The need for identity to be codified through recognition leads to what Holland et al. (1998) call *positional identities*. Positional identities capture how individuals see themselves and are seen by others in relation to power, status, and social hierarchies within a figured world. They reflect a person's relative standing in social interactions, and are often shaped by race, gender, class, education.

As individuals develop positional identities within figured worlds, they do not arrive as blank slates. Positional identities are also informed by one's *habitus* and *history-in-person* (Holland et al., 1998). Habitus refers to a set of deeply ingrained dispositions and ways of being that individuals develop through their social backgrounds within the family and home. More specific to personal experiences, history-in-person is described as "the sediment from past experiences upon which one improvises, using the cultural resources available, in response to the subject positions afforded one in the present" (p. 18). This captures how identities might evolve over time and how an individual's past experiences actively shape their present.

Within these worlds, individuals are not merely shaped by the experiences from their past; they also have the capacity to reshape their own identities through *self-authoring*. Self-authoring is the active process of negotiating, resisting, or transforming one's

identity within social structures. It is this process through which individuals can reflect on their positioning within a figured world and challenge or redefine dominant narratives that impose limiting identities. In sum, one's positional identity is shaped by one's upbringing, past experiences, and relevant cultural context. Yet, through self-authoring, positional identity is not entirely determined by these constructs.

METHODOLOGY

The approach of this study was a single-case study design. The rationale for this design is grounded in the assumptions for an unusual case, which identifies a unique or rare example to study in-depth (Yin, 2018). It is important to note that while generalizability is typically a desirable outcome of research results, this is not the primary aim of this study. I elaborate on the case selection in discussion of the participants.

Context and Participants

This study took place at Southwest University (SWU), a large public university in the southwest United States that is classified as a very high research activity institution [1]. The mathematics department at SWU has approximately 200 students in their PhD program, with about 15% of students identifying as women. There are approximately 60 faculty, a minority of which are women.

This work is part of a larger study in which the participants included five women doctoral students at SWU. The self-identified women were selected using a stratified purposeful sampling with respect to years' experience in the program (Patton, 1990). This included one woman enrolled in coursework, two women in early research, one woman in advanced research, and one woman taking a leave of absence after completing coursework. I also attempted to prioritize my selection based on maximum variation (Patton, 1990) with respect to race, ethnicity, or nationality. The larger study also necessitated the additional sampling criteria that participants be members in the local graduate chapter of the Association for Women in Mathematics.

In getting to know the participants, one woman, Ida, stood out in her descriptions of being a woman in mathematics. Ida is a heterosexual White woman from the United States who was in the final year of her PhD. She was unique among the participants in that she often engaged with her identity as a woman, including viewing her identity as a woman mathematician positively. Due to her unique interpretation of her experience as a woman in mathematics, I selected her as a single case to investigate.

Data Corpus

Participant data included a mathematics autobiography, one episodic narrative interview (Mueller, 2019), six weeks of diary entries (Cresswell & Poth, 2016), one future-focused diary entry, and one semi-structured interview per participant (Bernard, 1998). There was also one focus group interview with all five participants (Morgan, 1997). The data collection instruments were designed to capture the participants' math-life stories before, during, and after their PhD studies. They also included targeted questions specific to being a woman in mathematics. To understand how the

participants' identities were shaped with respect to their particular figured world of PhD mathematics I also asked about their graduate experiences with coursework, qualifying exams, and research. The data for this study includes all of Ida' participant data, including her contributions to the focus group.

Analysis

Preliminary analysis included transcription of interview data and memoing all participant data to capture initial ideas. I also completed descriptive accounts for each interview, again keeping track of emerging (Cresswell & Poth, 2016).

Within the case study design, I conducted a narrative inquiry, *restorying* recollections of the participant's life experience to create a coherent narrative (Connelly & Clandinin, 1990). Within the narrative inquiry, I utilized a three-dimensional inquiry space as an analytic tool (Cresswell & Poth, 2016). The three-dimensional inquiry space refers to interaction (encompassing identity), continuity (encompassing past, present, and future), and place (encompassing the context of a particular figured world). These dimensions were used to organize Ida's math-life story towards the narrative structure of past, present, and future.

Throughout this analysis I focused on passages that referred to descriptions of participation, positioning, mediating cultural artifacts, particularly those instances which were presented as personally meaningful or related to their identities as women. For each stage of the stories (e.g., past, present, future) I worked to connect the Ida's mathematical identity to the place, or mathematical figured world, at that point in time.

FINDINGS

In Ida's restorying, I describe her mathematical identity through her mathematical past, present, and future. Ida's math-life story is one of self-reported mathematical brilliance. Her interpretation of her position in mathematics tended toward a positive perspective but she did not shy away from topics related to her being a woman in mathematics and a member of a marginalized group. Furthermore, I argue it is her sense of being talented in the discipline that supports her access to her identity as a woman in mathematical contexts.

Past

Ida's description of her mathematical past is characterized by a love of the discipline and a self-described innate mathematical ability. Her early years included exposure to advanced mathematics and tensions around mathematics and gender as she transitioned to post-secondary education. As a child, she reported an immersion in advanced mathematics through her relationship with her father, an applied mathematician:

> I constantly saw him doing math at home, and he began to teach me mathematics for fun. These are some of my fondest memories, and it was how we bonded… By age 7, he was teaching me real analysis concepts like the epsilon-delta definition of a derivative. I thought about math all of the time. (Ida, year 6, Math Autobiography)

Ida's self-described "exceptional mathematical mind" is apparent early on as she recalls learning real analysis before the age of ten. As part of this genius label, she also recounted that students at her high school nicknamed her "base case" when they were learning inductions, signifying an acknowledgement of her mathematical ability by those around her. Relatedly, she reflected that she "was exempt from a lot of stereotyping about women in math" because of her reputation as a genius. These experiences reflect a habitus and history-in-person of mathematical privilege, with exposure to advanced mathematical thinking at an early age and recognition as mathematically gifted in the figured world of high-school.

As Ida transitioned to her undergraduate studies at Ivy League University (ILU) [2], there were many new components to Ida's math-life. The figured world of ILU brought in new players, symbols, and values. There was the structure of advising, the emphasis on grades, TA positions, mentorship, and awards. As Ida recounted her undergraduate studies, she discussed her lack of confidence at the time. She explained why feeling like an imposter made sense in a place like ILU "I think that being a mathematician at a top university is something that is rare and difficult, and I think that imposter syndrome is very natural in that situation" (Ida, year 6, Interview 1). She went on to describe how her academic success related to her sense of doubt:

> I was a top student at [ILU]. Realistically, what makes me so special, it seems against the odds that any particular person would have some sort of special mathematical mind…that would warrant the types of things that were happening. (Ida, year 6, Interview 1)

Ida added to this that she thought the pressure of studying mathematics at ILU "makes imposter syndrome really natural, whether you're a woman or a man. (Ida, year 6, Interview 1). From Ida' perspective her success shouldn't have been possible, not because she was a woman but because she was studying math at ILU. Despite also acknowledging that these pressures might have "hit a little harder for the women," Ida's rationale for her self-doubt is rooted in her beliefs about the relationship between mathematical genius and studying mathematics at a high level.

In seeming contrast to an a-gendered perspective on imposter syndrome, Ida fervently described the pressure she felt to prove herself in comparison to men:

> I always felt that I had to be unequivocally better than all of the men in order to be taken seriously…I always had to have perfect scores, have the best grade in the class, have everything super, super correct because I always felt like if I missed five points on the exam, if I was a 95 instead of a hundred, then that would be the reason that someone would write me off. (Ida, year 6, Interview 1)

This excerpt gives a sense of the immense pressure Ida felt specifically about being a woman in mathematics. It seems that the added pressure Ida describes is related to her being *recognized* as a legitimate mathematical person and as such might explain why her gender is a critical component when discussing her peers when compared to her internal experience of self-doubt. Ida's undergraduate experiences demonstrate a history-in-person that extends from her habitus, reflecting a sense of confidence and

self-identified exceptionalism. However, as she navigates her identity as a woman mathematician at a prestigious university it is the recognition of others, not her internal beliefs about her ability, that motivates her need to prove herself.

Overall, Ida's mathematical past rooted in a habitus in which her relationship with her father served as an early inculcation to mathematical thinking. Furthermore, innate mathematical ability appeared to be central to Ida's self-authoring of her mathematical identity; this underlying belief mitigated her self-doubt and allowed her to avoid undermining characterizations of women in mathematics.

Present

Ida's doctoral studies are characterized by more academic success at SWU. With respect to coursework, qualifying exams, and research her experience generally gives the impression of ease and comfort. Ida's love for mathematics is highlighted in her descriptions of her graduate studies. She expressed enthusiasm and positivity when discussing both coursework and qualifying exams. For example:

> I just honestly love it. I'm so grateful for it. I pick up the book that I used for that real analysis course all the time… I did such a breadth of learning, and I think it stretched my mind to be quicker and to think outside the box more. I just found the coursework incredibly valuable. (Ida, year 6, Interview 2)

She also showed her enthusiasm for qualifying exams in the following reflection:

> I did really love the quals and I did really love preparing for the quals and really, really getting that depth of knowledge and also bonding with my cohort over that. (Ida, year 6, Interview 2)

Beyond this nostalgic recollection Ida's description of her qualifying exam experience seemed uneventful. She recalled "…I did all of the prep work…and then I took my quals at the end of the year and then I passed them and that was kind of it." (Ida, year 6, Interview 2). This uneventfulness suggests a lack of struggle with the exams.

When asked about her transition to independent research, Ida proclaimed, "I love doing independent research" and that she wouldn't change anything about her advancement to candidacy. Ida's attitude about beginning work as an independent researcher appeared jovial and there was a sense, like other milestones, that it came with ease.

Ida's discussion of being a woman in mathematics during her time at SWU echoed some of her earlier sentiments about gender bias and a need to prove herself. However, she also discussed the freedom of not having a woman in math. When asked how being a woman has impacted her mathematical experiences in general, she responded:

> I think that the instant assumption first off, is not going to be that I'm a mathematician at all. And secondly…that I'm sort of guilty until proven innocent. So, I don't want to confirm any ideas that someone might have that I might be less than a great mathematician…I feel like I need to be quietly correct all the time. (Ida, year 6, Interview 1)

This understanding that people will make assumptions about her as a mathematician because she is a woman, relates back to her statements from her undergraduate studies in which she had to be nearly perfect. Ida further discussed the benefits of being correct "I have a lot of good at math privilege, even though I don't have male privilege" (Ida, year 6, Interview 1). She went on to discuss this idea in more detail:

> I don't know if it's true that if I was middle of the pack, then I would be treated as if I was no good…I think it would be more likely. But I have experienced a sense that if you are good enough, that can transcend the gender stuff. (Ida, year 6, Interview 1)

Ida appeared to call out aspects of her mathematical identity – innate mathematical ability – and that contributes to having an advantageous position in the figured world of PhD mathematics. Her theory, that being exceptional can transform your experience as a woman, speaks to the inherent positioning of women as illegitimate in mathematics, denying them access to mathematical identities. However, Ida appeared to leverage her mathematical privilege while expressing her identity as a woman.

Despite these somewhat bleak ideas about what it means to be a woman in mathematics; Ida found a silver lining in automatically being positioned as outside of mathematics:

> But as far as female mathematicians go… there's not any sort of consistent pattern…In some ways, that's very freeing because I'm just making my way…In some ways. It leaves a lot of questions. When should I have children? Unclear. (Ida, year 6 Interview 1)

Ida's response indicates a sense of freedom in not being easily categorized into a certain mathematical stereotype. However, as she discussed the unknown of being a woman in mathematics a tension arose. Ida's previous testimonies of being good enough at math to avoid the implications of gender are absconded as questions about when to have children compromise her identity as a mathematician.

Ida's self-authoring as a woman mathematician also becomes evident when describing a conflict with her advisor. Ida's described her advisor, a woman, as an "old school feminist." While navigating their conflict Ida described what she viewed as part of their discordance:

> Maybe I need to act a little bit more like a caricature of a rude man in order to have her be more respectful of me…considering I am one of the most feminine women in the department, and my advisor seems to hate feminine women. (Ida, year 6, Diary Entry)

This excerpt demonstrates Ida's identity as a woman through the contrast in how she described being expected to act by her advisor. Ida makes no effort to distance herself from her identity as a woman while she is simultaneously being disrespected for it. This, combined with her sense of freedom from stereotyping, signifies Ida's intersecting identities as a woman and a "mathematical genius." Ida's self-authoring of her mathematical present demonstrates a mathematical identity of confidence and highlights the integration of her identity as a mathematician and as a woman.

Future

Looking toward her next steps, Ida described her greatest hope as centred around her recently secured postdoctoral position. She described the opportunity as a job "in a top research group…that really sets me up for a good career." When asked more directly about her future math-life she said she was excited to start new projects in part because the techniques she developed in PhD could be applied to an array of problems. She greatly emphasized mathematics in her description of her future:

> When I do math, I feel really fulfilled. I can just get up in the morning and be in this flow state for three hours and fulfil this need of mine for my brain to go crazy…and be in this math space. That makes me really happy. (Ida, year 6, Interview 1)

This revery exemplifies Ida's close and positive relationship with mathematics. Additionally, she talked about the prospect of working with other people. She explained "as a woman" she thinks a lot about "whether I like the group, whether I find them to be collaborative, what types of stuff they do, what types of outreach they do, all of this sort of stuff." She also made a note that she shouldn't focus on these features but as a woman she does.

With respect to her family and her future in mathematics, Ida detailed her plans after her postdoc, stating "it's always a question in my mind of if we start having kids, if I'll want to continue to work or not, which I feel like is completely taboo to say." This statement reveals a tension in which Ida cannot fully integrate her identities as a mathematician and a woman. In this instance she is constrained by the systems in place in academia that would make it challenging for her to return to the academy after leaving to be with her future children.

CONCLUSION AND DISCUSSION

The findings from this study extend the literature on women pursuing doctoral degrees in mathematics by detailing the ways in which one woman was able to integrate her identity as a mathematician and her identity as a woman. Through the lens of the figured worlds framework, Ida's identity becomes more crystalline. Regarding her habitus, Ida's dad normalized advanced mathematics at an early age. Additionally, her history-in-person reveals a thread of confidence as a "mathematical genius." Furthermore, Ida's account of innate mathematical ability was taken up in her self-authoring, positioning her as a legitimate mathematician. This legitimacy is evident in her PhD studies where it was solidified in her alignment with the beliefs and values of those in positions of authority. Ida expressed a consistently clear claim to being mathematically gifted and it is this sense of innate mathematical genius as part of her history-in-person and continued self-authoring that may contribute to Ida's access to her identities as a woman and a mathematician simultaneously. However, there remained instances of tension, particularly around childrearing, in which Ida's positional identity as a legitimate mathematician was compromised.

The ways in which women and mathematics are understood to be mutually exclusive has created an opportunity in this study to investigate how women respond to cultural narratives of who counts as a mathematician. Ida's willingness to engage with her identity as a woman mathematician was unique among participants in the larger study and deters from what has been frequently identified in the literature (Hottinger, 2016; Rodd & Bartholomew, 2006; Solomon et al., 2011). It appears that her mathematical identity, her identity as a woman, and the co-occurrence of the two was fortified by a lifetime of belief in her innate mathematical ability.

NOTES

1. Very high research activity is designation by the Carnegie Classification of Institutions of Higher Education that described universities in the United States that on average in a single year, spend at least $50 million on research and development and award at least 70 research doctorates.
2. Ivy League Universities are a group of long-established schools in the eastern US having high academic and social prestige.

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